Instructor's Solutions Manual

Third Edition

Fundamentals of ProbabilitY With Stochastic Processes

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Contents

Chapter 1

Axioms of Probability

1.2 SAMPLE SPACE AND EVENTS

1. For $1 \leq i, j \leq 3$, by (i, j) we mean that Vann's card number is i, and Paul's card number is j. Clearly, $A = \{(1, 2), (1, 3), (2, 3)\}$ and $B = \{(2, 1), (3, 1), (3, 2)\}$.

(a) Since $A \cap B = \emptyset$, the events A and B are mutually exclusive.

(b) None of (1, 1), (2, 2), (3, 3) belongs to $A \cup B$. Hence $A \cup B$ not being the sample space shows that A and B are not complements of one another.

2. $S = \{RRR, RRB, RBR, RBB, BRR, BRB, BBR, BBB\}$.

3. $\{x: 0 < x < 20\}$; $\{1, 2, 3, \ldots, 19\}$.

4. Denote the dictionaries by d_1, d_2; the third book by a. The answers are $\{d_1d_2a, d_1ad_2, d_2d_1a, d_2ad_1, ad_1d_2, ad_2d_1\}$ and $\{d_1d_2a, ad_1d_2\}$.

5. EF: One 1 and one even.

E^cF: One 1 and one odd.

E^cF^c: Both even or both belong to $\{3, 5\}$.

6. $S = \{QQ, QN, QP, QD, DN, DP, NP, NN, PP\}$. **(a)** $\{QP\}$; **(b)** $\{DN, DP, NN\}$; **(c)** $\emptyset$.

7. $S = \{x: 7 \leq x \leq 9\frac{1}{6}\}$; $\{x: 7 \leq x \leq 7\frac{1}{4}\} \cup \{x: 7\frac{3}{4} \leq x \leq 8\frac{1}{4}\} \cup \{x: 8\frac{3}{4} \leq x \leq 9\frac{1}{6}\}$.

8. $E \cup F \cup G = G$: If E or F occurs, then G occurs.

$EFG = G$: If G occurs, then E and F occur.

9. For $1 \leq i \leq 3$, $1 \leq j \leq 3$, by a_ib_j we mean passenger a gets off at hotel i and passenger b gets off at hotel j. The answers are $\{a_ib_j: 1 \leq i \leq 3,\ 1 \leq j \leq 3\}$ and $\{a_1b_1, a_2b_2, a_3b_3\}$, respectively.

10. **(a)** $(E \cup F)(F \cup G) = (F \cup E)(F \cup G) = F \cup EG$.

(b) Using part (a), we have

$$(E \cup F)(E^c \cup F)(E \cup F^c) = (F \cup EE^c)(E \cup F^c) = F(E \cup F^c) = FE \cup FF^c = FE.$$

11. **(a)** AB^cC^c; **(b)** $A \cup B \cup C$; **(c)** $A^cB^cC^c$; **(d)** $ABC^c \cup AB^cC \cup A^cBC$;
(e) $AB^cC^c \cup A^cB^cC \cup A^cBC^c$; **(f)** $(A - B) \cup (B - A) = (A \cup B) - AB$.

12. If $B = \emptyset$, the relation is obvious. If the relation is true for every event A, then it is true for S, the sample space, as well. Thus

$$S = (B \cap S^c) \cup (B^c \cap S) = \emptyset \cup B^c = B^c,$$

showing that $B = \emptyset$.

13. Parts (a) and (d) are obviously true; part (c) is true by DeMorgan's law; part (b) is false: throw a four-sided die; let $F = \{1, 2, 3\}$, $G = \{2, 3, 4\}$, $E = \{1, 4\}$.

14. **(a)** $\bigcup_{n=1}^{\infty} A_n$; **(b)** $\bigcup_{n=1}^{37} A_n$.

15. Straightforward.

16. Straightforward.

17. Straightforward.

18. Let a_1, a_2, and a_3 be the first, the second, and the third volumes of the dictionary. Let a_4, a_5, a_6, and a_7 be the remaining books. Let $A = \{a_1, a_2, \ldots, a_7\}$; the answers are

$$S = \left\{x_1x_2x_3x_4x_5x_6x_7 \colon x_i \in A,\ 1 \le i \le 7,\ \text{and } x_i \ne x_j \text{ if } i \ne j\right\}$$

and

$$\left\{x_1x_2x_3x_4x_5x_6x_7 \in S \colon x_ix_{i+1}x_{i+2} = a_1a_2a_3 \text{ for some } i,\ 1 \le i \le 5\right\},$$

respectively.

19. $\bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} A_n$.

20. Let $B_1 = A_1$, $B_2 = A_2 - A_1$, $B_3 = A_3 - (A_1 \cup A_2), \ldots, B_n = A_n - \bigcup_{i=1}^{n-1} A_i, \ldots$.

1.4 BASIC THEOREMS

1. No; $P(\text{sum } 11) = 2/36$ while $P(\text{sum } 12) = 1/36$.

2. $0.33 + 0.07 = 0.40$.

3. Let E be the event that an earthquake will damage the structure next year. Let H be the event that a hurricane will damage the structure next year. We are given that $P(E) = 0.015$, $P(H) = 0.025$, and $P(EH) = 0.0073$. Since

$$P(E \cup H) = P(E) + P(H) - P(EH) = 0.015 + 0.025 - 0.0073 = 0.0327,$$

the probability that next year the structure will be damaged by an earthquake and/or a hurricane is 0.0327. The probability that it is not damaged by any of the two natural disasters is 0.9673.

4. Let A be the event of a randomly selected driver having an accident during the next 12 months. Let B be the event that the person is male. By Theorem 1.7, the desired probability is

$$P(A) = P(AB) + P(AB^c) = 0.12 + 0.06 = 0.18.$$

5. Let A be the event that a randomly selected investor invests in traditional annuities. Let B be the event that he or she invests in the stock market. Then $P(A) = 0.75$, $P(B) = 0.45$, and $P(A \cup B) = 0.85$. Since,

$$P(AB) = P(A) + P(B) - P(A \cup B) = 0.75 + 0.45 - 0.85 = 0.35,$$

35% invest in both stock market and traditional annuities.

6. The probability that the first horse wins is 2/7. The probability that the second horse wins is 3/10. Since the events that the first horse wins and the second horse wins are mutually exclusive, the probability that either the first horse or the second horse will win is

$$\frac{2}{7} + \frac{3}{10} = \frac{41}{70}.$$

7. In point of fact Rockford was right the first time. The reporter is assuming that both autopsies are performed by a given doctor. The probability that both autopsies are performed by the same doctor–whichever doctor it may be–is 1/2. Let AB represent the case in which Dr. A performs the first autopsy and Dr. B performs the second autopsy, with similar representations for other cases. Then the sample space is $S = \{AA, AB, BA, BB\}$. The event that both autopsies are performed by the same doctor is $\{AA, BB\}$. Clearly, the probability of this event is 2/4=1/2.

8. Let m be the probability that Marty will be hired. Then $m + (m + 0.2) + m = 1$ which gives $m = 8/30$; so the answer is $8/30 + 2/10 = 7/15$.

9. Let s be the probability that the patient selected at random suffers from schizophrenia. Then $s + s/3 + s/2 + s/10 = 1$ which gives $s = 15/29$.

10. $P(A \cup B) \leq 1$ implies that $P(A) + P(B) - P(AB) \leq 1$.

11. **(a)** $2/52 + 2/52 = 1/13$; **(b)** $12/52 + 26/52 - 6/53 = 8/13$; **(c)** $1 - (16/52) = 9/13$.

12. **(a)** False; toss a die and let $A = \{1, 2\}$, $B = \{2, 3\}$, and $C = \{1, 3\}$.

(b) False; toss a die and let $A = \{1, 2, 3, 4\}$, $B = \{1, 2, 3, 4, 5\}$, $C = \{1, 2, 3, 4, 5, 6\}$.

13. A simple Venn diagram shows that the answers are 65% and 10%, respectively.

14. Applying Theorem 1.6 twice, we have

$$\begin{aligned} P(A \cup B \cup C) &= P(A \cup B) + P(C) - P\big((A \cup B)C\big) \\ &= P(A) + P(B) - P(AB) + P(C) - P(AC \cup BC) \\ &= P(A) + P(B) - P(AB) + P(C) - P(AC) - P(BC) + P(ABC) \\ &= P(A) + P(B) + P(C) - P(AB) - P(AC) - P(BC) + P(ABC). \end{aligned}$$

15. Using Theorem 1.5, we have that the desired probability is

$$\begin{aligned} &P(AB - ABC) + P(AC - ABC) + P(BC - ABC) \\ &\quad = P(AB) - P(ABC) + P(AC) - P(ABC) + P(BC) - P(ABC) \\ &\quad = P(AB) + P(AC) + P(BC) - 3P(ABC). \end{aligned}$$

16. $7/11$.

17. $\sum_{i=1}^{n} p_{ij}$.

18. Let M and F denote the events that the randomly selected student earned an A on the midterm exam and an A on the final exam, respectively. Then

$$P(MF) = P(M) + P(F) - P(M \cup F),$$

where $P(M) = 17/33$, $P(F) = 14/33$, and by DeMorgan's law,

$$P(M \cup F) = 1 - P(M^c F^c) = 1 - \frac{11}{33} = \frac{22}{33}.$$

Therefore,

$$P(MF) = \frac{17}{33} + \frac{14}{33} - \frac{22}{33} = \frac{3}{11}.$$

19. A Venn diagram shows that the answers are $1/8$, $5/24$, and $5/24$, respectively.

20. The equation has real roots if and only if $b^2 \geq 4c$. From the 36 possible outcomes for (b, c), in the following 19 cases we have that $b^2 \geq 4c$: (2, 1), (3, 1), (3, 2), (4, 1), ..., (4, 4), (5, 1), ..., (5, 6), (6, 1), ..., (6, 6). Therefore, the answer is $19/36$.

21. The only prime divisors of 63 are 3 and 7. Thus the number selected is relatively prime to 63 if and only if it is neither divisible by 3 nor by 7. Let A and B be the events that the outcome is divisible by 3 and 7, respectively. The desired quantity is

$$P(A^cB^c) = 1 - P(A \cup B) = 1 - P(A) - P(B) + P(AB)$$
$$= 1 - \frac{21}{63} - \frac{9}{63} + \frac{3}{63} = \frac{4}{7}.$$

22. Let T and F be the events that the number selected is divisible by 3 and 5, respectively.

(a) The desired quantity is the probability of the event TF^c:

$$P(TF^c) = P(T) - P(TF) = \frac{333}{1000} - \frac{66}{1000} = \frac{267}{1000}.$$

(b) The desired quantity is the probability of the event T^cF^c:

$$P(T^cF^c) = 1 - P(T \cup F) = 1 - P(T) - P(F) + P(TF)$$
$$= 1 - \frac{333}{1000} - \frac{200}{1000} + \frac{66}{1000} = \frac{533}{1000}.$$

23. (Draw a Venn diagram.) From the data we have that 55% passed all three, 5% passed calculus and physics but not chemistry, and 20% passed calculus and chemistry but not physics. So at least $(55+5+20)\% = 80\%$ must have passed calculus. This number is greater than the given 78% for all of the students who passed calculus. Therefore, the data is incorrect.

24. By symmetry the answer is $1/4$.

25. Let A, B, and C be the events that the number selected is divisible by 4, 5, and 7, respectively. We are interested in $P(AB^cC^c)$. Now $AB^cC^c = A - A(B \cup C)$ and $A(B \cup C) \subseteq A$. So by Theorem 1.5,

$$P(AB^cC^c) = P(A) - P\big(A(B \cup C)\big) = P(A) - P(AB \cup AC)$$
$$= P(A) - P(AB) - P(AC) + P(ABC)$$
$$= \frac{250}{1000} - \frac{50}{1000} - \frac{35}{1000} + \frac{7}{1000} = \frac{172}{1000}.$$

26. A Venn diagram shows that the answer is 0.36.

27. Let A be the event that the first number selected is greater than the second; let B be the event that the second number selected is greater than the first; and let C be the event that the two numbers selected are equal. Then $P(A) + P(B) + P(C) = 1$, $P(A) = P(B)$, and $P(C) = 1/100$. These give $P(A) = 99/200$.

28. Let $B_1 = A_1$, and for $n \geq 2$, $B_n = A_n - \bigcup_{i=1}^{n-1} A_i$. Then $\{B_1, B_2, \dots\}$ is a sequence of mutually exclusive events and $\bigcup_{i=1}^{\infty} A_i = \bigcup_{i=1}^{\infty} B_i$. Hence

$$P\Big(\bigcup_{n=1}^{\infty} A_n\Big) = P\Big(\bigcup_{n=1}^{\infty} B_n\Big) = \sum_{n=1}^{\infty} P(B_n) \le \sum_{n=1}^{\infty} P(A_n),$$

since $B_n \subseteq A_n$, $n \ge 1$.

29. By Boole's inequality (Exercise 28),

$$P\Big(\bigcap_{n=1}^{\infty} A_n\Big) = 1 - P\Big(\bigcup_{n=1}^{\infty} A_n^c\Big) \ge 1 - \sum_{n=1}^{\infty} P(A_n^c).$$

30. She is wrong! Consider the next 50 flights. For $1 \le i \le 50$, let A_i be the event that the ith mission will be completed without mishap. Then $\bigcap_{i=1}^{50} A_i$ is the event that all of the next 50 missions will be completed successfully. We will show that $P\Big(\bigcap_{i=1}^{50} A_i\Big) > 0$. This proves that Mia is wrong. Note that the probability of the simultaneous occurrence of any number of A_i^c 's is nonzero. Furthermore, consider any set E consisting of n ($n \le 50$) of the A_i^c's. It is reasonable to assume that the probability of the simultaneous occurrence of the events of E is *strictly* less than the probability of the simultaneous occurrence of the events of any subset of E. Using these facts, it is straightforward to conclude from the inclusion–exclusion principle that,

$$P\Big(\bigcup_{i=1}^{50} A_i^c\Big) < \sum_{i=1}^{50} P(A_i^c) = \sum_{i=1}^{50} \frac{1}{50} = 1.$$

Thus, by DeMorgan's law,

$$P\Big(\bigcap_{i=1}^{50} A_i\Big) = 1 - P\Big(\bigcup_{i=1}^{50} A_i^c\Big) > 1 - 1 = 0.$$

31. Q satisfies Axioms 1 and 2, but not necessarily Axiom 3. So it is not, in general, a probability on S. Let $S = \{1, 2, 3, \}$. Let $P\big(\{1\}\big) = P\big(\{2\}\big) = P\big(\{3\}\big) = 1/3$. Then $Q\big(\{1\}\big) = Q\big(\{2\}\big) = 1/9$, whereas $Q\big(\{1, 2\}\big) = P\big(\{1, 2\}\big)^2 = 4/9$. Therefore,

$$Q\big(\{1, 2, \}\big) \ne Q\big(\{1\}\big) + Q\big(\{2\}\big).$$

R is not a probability on S because it does not satisfy Axiom 2; that is, $R(S) \ne 1$.

32. Let BRB mean that a blue hat is placed on the first player's head, a red hat on the second player's head, and a blue hat on the third player's head, with similar representations for other cases. The sample space is

$$S = \{BBB, BRB, BBR, BRR, RRR, RRB, RBR, RBB\}.$$

This shows that the probability that two of the players will have hats of the same color and the third player's hat will be of the opposite color is $6/8 = 3/4$. The following improvement, based on this observation, explained by Sara Robinson in Tuesday, April 10, 2001 issue of the *New York Times,* is due to Professor Elwyn Berlekamp of the University of California at Berkeley.

> Three-fourths of the time, two of the players will have hats of the same color and the third player's hat will be the opposite color. The group can win every time this happens by using the following strategy: Once the game starts, each player looks at the other two players' hats. If the two hats are different colors, he [or she] passes. If they are the same color, the player guesses his [or her] own hat is the opposite color. This way, every time the hat colors are distributed two and one, one player will guess correctly and the others will pass, and the group will win the game. When all the hats are the same color, however, all three players will guess incorrectly and the group will lose.

1.7 RANDOM SELECTION OF POINTS FROM INTERVALS

1. $\dfrac{30-10}{30-0}=\dfrac{2}{3}$.

2. $\dfrac{0.0635-0.04}{0.12-0.04}=0.294$.

3. **(a)** False; in the experiment of choosing a point at random from the interval $(0, 1)$, let $A=(0,1)-\{1/2\}$. A is not the sample space but $P(A)=1$.
(b) False; in the same experiment $P\big(\{1/2\}\big)=0$ while $\{\frac{1}{2}\}\neq\emptyset$.

4. $P(A\cup B)\geq P(A)=1$, so $P(A\cup B)=1$. This gives

$$P(AB)=P(A)+P(B)-P(A\cup B)=1+1-1=1.$$

5. The answer is

$$P\big(\{1,2,\dots,1999\}\big)=\sum_{i=1}^{1999}P\big(\{i\}\big)=\sum_{i=1}^{1999}0=0.$$

6. For $i=0,1,2,\dots,9$, the probability that i appears as the first digit of the decimal representation of the selected point is the probability that the point falls into the interval $\Big[\dfrac{i}{10},\dfrac{i+1}{10}\Big)$. Therefore, it equals

$$\frac{\dfrac{i+1}{10}-\dfrac{i}{10}}{1-0}=\frac{1}{10}.$$

This shows that all numerals are equally likely to appear as the first digit of the decimal representation of the selected point.

7. No, it is not. Let $S = \{w_1, w_2, \dots\}$. Suppose that for some $p > 0$, $P\big(\{w_i\}\big) = p$, $i = 1, 2, \dots$. Then, by Axioms 2 and 3, $\sum_{i=1}^{\infty} p = 1$. This is impossible.

8. Use induction. For $n = 1$, the theorem is trivial. Exercise 4 proves the theorem for $n = 2$. Suppose that the theorem is true for n. We show it for $n + 1$,

$$\begin{aligned} P(A_1A_2\cdots A_nA_{n+1}) &= P(A_1A_2\cdots A_n) + P(A_{n+1}) - P(A_1A_2\cdots A_n \cup A_{n+1}) \\ &= 1 + 1 - 1 = 1, \end{aligned}$$

where $P(A_1A_2\cdots A_n) = 1$ is true by the induction hypothesis, and

$$P(A_1A_2\cdots A_n \cup A_{n+1}) \geq P(A_{n+1}) = 1,$$

implies that $P(A_1A_2\cdots A_n \cup A_{n+1}) = 1$.

9. **(a)** Clearly, $\dfrac{1}{2} \in \displaystyle\bigcap_{n=1}^{\infty} \Big(\frac{1}{2} - \frac{1}{2n}, \frac{1}{2} + \frac{1}{2n}\Big)$. If $x \in \displaystyle\bigcap_{n=1}^{\infty} \Big(\frac{1}{2} - \frac{1}{2n}, \frac{1}{2} + \frac{1}{2n}\Big)$, then, for all $n \geq 1$,

$$\frac{1}{2} - \frac{1}{2n} < x < \frac{1}{2} + \frac{1}{2n}.$$

Letting $n \to \infty$, we obtain $1/2 \leq x \leq 1/2$; thus $x = 1/2$.

(b) Let A_n be the event that the point selected at random is in $\Big(\dfrac{1}{2} - \dfrac{1}{2n}, \dfrac{1}{2} + \dfrac{1}{2n}\Big)$; then

$$A_1 \supseteq A_2 \supseteq A_3 \supseteq \cdots \supseteq A_n \supseteq A_{n+1} \supseteq \cdots .$$

Since $P(A_n) = \dfrac{1}{n}$, by the continuity property of the probability function,

$$P\big(\{1/2\}\big) = \lim_{n\to\infty} P(A_n) = 0.$$

10. The set of rational numbers is countable. Let $\mathbf{Q} = \{r_1, r_2, r_3, \dots\}$ be the set of rational numbers in $(0, 1)$. Then

$$P(\mathbf{Q}) = P\big(\{r_1, r_2, r_3, \dots\}\big) = \sum_{i=1}^{\infty} P\big(\{r_i\}\big) = 0.$$

Let $\mathbf{I}$ be the set of irrational numbers in $(0, 1)$; then

$$P(\mathbf{I}) = P(\mathbf{Q}^c) = 1 - P(\mathbf{Q}) = 1.$$

11. For $i = 0, 1, 2, \dots, 9$, the probability that i appears as the nth digit of the decimal representation of the selected point is the probability that the point falls into the following subset of $(0, 1)$:

$$\bigcup_{m=0}^{10^{n-1}-1} \Big[\frac{10m + i}{10^n}, \frac{10m + i + 1}{10^n}\Big).$$

Since the intervals in this union are mutually exclusive, the probability that the point falls into this subset is

$$\sum_{m=0}^{10^{n-1}-1} \frac{\dfrac{10m+i+1}{10^n} - \dfrac{10m+i}{10^n}}{1-0} = 10^{n-1} \cdot \frac{1}{10^n} = \frac{1}{10}.$$

This shows that all numerals are equally likely to appear as the nth digit of the decimal representation of the selected point.

12. $P(B_m) \leq \sum_{n=m}^{\infty} P(A_n)$. Since $\sum_{n=1}^{\infty} P(A_n)$ converges,

$$\lim_{m\to\infty} P(B_m) \leq \lim_{m\to\infty} \sum_{n=m}^{\infty} P(A_n) = 0.$$

This gives $\lim_{m\to\infty} P(B_m) = 0$. Therefore,

$$B_1 \supseteq B_2 \supseteq B_3 \supseteq \cdots \supseteq B_m \supseteq B_{m+1} \supseteq \cdots$$

implies that

$$P\Big(\bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} A_n\Big) = P\Big(\bigcap_{m=1}^{\infty} B_m\Big) = \lim_{m\to\infty} P(B_m) = 0.$$

13. In the experiment of choosing a random point from (0, 1), let $E_t = (0,1) - \{t\}$, for $0 < t < 1$. Then $P(E_t) = 1$ for all t, while

$$P\Big(\bigcap_{t\in(0,1)} E_t\Big) = P(\emptyset) = 0.$$

14. Clearly $r_n \in (\alpha_n, \beta_n)$. By the geometric series theorem,

$$\sum_{n=1}^{\infty}(\beta_n - \alpha_n) = \sum_{n=1}^{\infty} \frac{\varepsilon}{2^{n+1}} = \varepsilon \frac{\dfrac{1}{4}}{1-\dfrac{1}{2}} = \frac{\varepsilon}{2} < \varepsilon.$$

REVIEW PROBLEMS FOR CHAPTER 1

1. $\dfrac{3.25-2}{4.3-2} = 0.54.$

2. We have that

$$S = \Big\{\big(\emptyset, \{1\}\big), \big(\emptyset, \{2\}\big), \big(\emptyset, \{1,2\}\big), \big(\{1\}, \{2\}\big), \big(\{1\}, \{1,2\}\big), \big(\{2\}, \{1,2\}\big)\Big\}.$$

The desired events are

(a) $\big\{(\emptyset, \{1\}), (\emptyset, \{2\}), (\emptyset, \{1, 2\}), (\{1\}, \{2\})\big\}$; **(b)** $\big\{(\emptyset, \{1, 2\}), (\{1\}, \{2\})\big\}$;

(c) $\big\{(\emptyset, \{1\}), (\emptyset, \{2\}), (\emptyset, \{1, 2\}), (\{1\}, \{1, 2\}), (\{2\}, \{1, 2\})\big\}$.

3. Since $A \subseteq B$, we have that $B^c \subseteq A^c$. This implies that (a) is false but (b) is true.

4. In the experiment of tossing a die let $A = \{1, 3, 5\}$ and $B = \{5\}$; then both (a) and (b) are false.

5. We may define a sample space S as follows.

$$S = \big\{x_1x_2 \cdots x_n : n \geq 1,\ x_i \in \{\text{H,T}\};\ x_i \neq x_{i+1},\ 1 \leq i \leq n-2;\ x_{n-1} = x_n\big\}.$$

6. A venn diagram shows that 18 are neither male nor for surgery.

7. We have that $ABC \subseteq BC$, so $P(ABC) \leq P(BC)$ and hence $P(BC) - P(ABC) \geq 0$. This and the following give the result.

$$\begin{aligned} P(A \cup B \cup C) &= P(A) + P(B) + P(C) - \big[P(AB) + P(AC) + P(BC) - P(ABC)\big] \\ &\leq P(A) + P(B) + P(C). \end{aligned}$$

8. If $P(AB) = P(AC) = P(BC) = 0$, then $P(ABC) = 0$ since $ABC \subseteq AB$. These imply that

$$\begin{aligned} P(A \cup B \cup C) &= P(A) + P(B) + P(C) - P(AB) - P(AC) - P(BC) + P(ABC) \\ &= P(A) + P(B) + P(C). \end{aligned}$$

Now suppose that

$$P(A \cup B \cup C) = P(A) + P(B) + P(C).$$

This relation implies that

$$P(AB) + P(BC) + \big[P(AC) - P(ABC)\big] = 0. \tag{1}$$

Since $P(AC) - P(ABC) \geq 0$ we have that the sum of three nonnegative quantities is 0; so each of them is 0. That is,

$$P(AB) = 0, \quad P(BC) = 0, \quad P(AC) = P(ABC). \tag{2}$$

Now rewriting (1) as

$$P(AB) + P(AC) + \big[P(BC) - P(ABC)\big] = 0,$$

the same argument implies that

$$P(AB) = 0, \quad P(AC) = 0, \quad P(BC) = P(ABC). \tag{3}$$

Comparing (2) and (3) we have

$$P(AB) = P(AC) = P(BC) = 0.$$

9. Let W be the event that a randomly selected person from this community drinks or serves white wine. Let R be the event that she or he drinks or serves red wine. We are given that $P(W) = 0.40$, $P(R) = 0.50$, and $P(W \cup R) = 0.70$. Since

$$P(WR) = P(W) + P(R) - P(W \cup R) = 0.40 + 0.50 - 0.70 = 0.20,$$

20% percent drink or serve both red and white wine.

10. No, it is not right. The probability that the second student chooses the tire the first student chose is 1/4.

11. By De Morgan's second law,

$$P(A^c B^c) = 1 - P\big((A^c B^c)^c\big) = 1 - P(A \cup B) = 1 - P(A) - P(B) + P(AB).$$

12. By Theorem 1.5 and the fact that $A - B$ and $B - A$ are mutually exclusive,

$$\begin{aligned} P\big((A - B) \cup (B - A)\big) &= P(A - B) + P(B - A) = P(A - AB) + P(B - AB) \\ &= P(A) - P(AB) + P(B) - P(AB) = P(A) + P(B) - 2P(AB). \end{aligned}$$

13. Denote a box of books by a_i, if it is received from publisher i, $i = 1, 2, 3$. The sample space is

$$S = \big\{x_1x_2x_3x_4x_5x_6 : \text{ two of the } x_i\text{'s are } a_1, \text{ two of them are } a_2, \text{ and the remaining two are } a_3\big\}.$$

The desired event is $E = \big\{x_1x_2x_3x_4x_5x_6 \in S : x_5 = x_6\big\}$.

14. Let E, F, G, and H be the events that the next baby born in this town has blood type O, A, B, and AB, respectively. Then

$$P(E) = P(F), \quad P(G) = \frac{1}{10}P(F), \quad P(G) = 2P(H).$$

These imply

$$P(E) = P(F) = 20P(H).$$

Therefore, from

$$P(E) + P(F) + P(G) + P(H) = 1,$$

we get

$$20P(H) + 20P(H) + 2P(H) + P(H) = 1,$$

which gives $P(H) = 1/43$.

15. Let F, S, and N be the events that the number selected is divisible by 4, 7, and 9, respectively. We are interested in $P(F^c S^c N^c)$ which is equal to $1 - P(F \cup S \cup N)$ by DeMorgan's law.

Now

$$P(F \cup S \cup N) = P(F) + P(S) + P(N) - P(FS) - P(FN) - P(SN) + P(FSN)$$
$$= \frac{250}{1000} + \frac{142}{1000} + \frac{111}{1000} - \frac{35}{1000} - \frac{27}{1000} - \frac{15}{1000} + \frac{3}{1000} = 0.429.$$

So the desired probability is 0.571.

16. The number is relatively prime to 150 if is not divisible by 2, 3, or 5. Let A, B, and C be the events that the number selected is divisible by 2, 3, and 5, respectively. We are interested in $P(A^cB^cC^c) = 1 - P(A \cup B \cup C)$. Now

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(AB) - P(AC) - P(BC) + P(ABC)$$
$$= \frac{75}{150} + \frac{50}{150} + \frac{30}{150} - \frac{25}{150} - \frac{15}{150} - \frac{10}{150} + \frac{5}{150} = \frac{11}{15}.$$

Therefore, the answer is $1 - \dfrac{11}{15} = \dfrac{4}{15}$.

17. **(a)** $U_i^c D_i^c$; **(b)** $U_1 U_2 \cdots U_n$; **(c)** $(U_1^c D_1^c) \cup (U_2^c D_2^c) \cup \cdots \cup (U_n^c D_n^c)$;

(d) $(U_1 D_2 U_3^c D_3^c) \cup (U_1 U_2^c D_2^c D_3) \cup (D_1 U_2 U_3^c D_3^c) \cup (D_1 U_2^c D_2^c U_3)$
$\cup (D_1^c U_1^c D_2 U_3) \cup (D_1^c U_1^c U_2 D_2) \cup (D_1^c U_1^c D_2^c U_2^c D_3^c U_3^c)$;

(e) $D_1^c D_2^c \cdots D_n^c$.

18. $\dfrac{199 - 96}{199 - 0} = \dfrac{103}{199}$.

19. We must have $b^2 < 4ac$. There are $6 \times 6 \times 6 = 216$ possible outcomes for a, b, and c. For cases in which $a < c$, $a > c$, and $a = c$, it can be checked that there are 73, 73, and 27 cases in which $b^2 < 4ac$, respectively. Therefore, the desired probability is

$$\frac{73 + 73 + 27}{216} = \frac{173}{216}.$$

Chapter 2

Combinatorial Methods

2.2 COUNTING PRINCIPLES

1. The total number of six-digit numbers is $9 \times 10 \times 10 \times 10 \times 10 \times 10 = 9 \times 10^5$ since the first digit cannot be 0. The number of six-digit numbers without the digit five is $8 \times 9 \times 9 \times 9 \times 9 \times 9 = 8 \times 9^5$. Hence there are $9 \times 10^5 - 8 \times 9^5 = 427,608$ six-digit numbers that contain the digit five.

2. **(a)** $5^5 = 3125$. **(b)** $5^3 = 125$.

3. There are $26 \times 26 \times 26 = 17,576$ distinct sets of initials. Hence in any town with more than 17,576 inhabitants, there are at least two persons with the same initials. The answer to the question is therefore yes.

4. $4^{15} = 1,073,741,824$.

5. $\dfrac{2}{2^{23}} = \dfrac{1}{2^{22}} \approx 0.00000024$.

6. **(a)** $52^5 = 380,204,032$. **(b)** $52 \times 51 \times 50 \times 49 \times 48 = 311,875,200$.

7. $6/36 = 1/6$.

8. **(a)** $\dfrac{4 \times 3 \times 2 \times 2}{12 \times 8 \times 8 \times 4} = \dfrac{1}{64}$. **(b)** $1 - \dfrac{8 \times 5 \times 6 \times 2}{12 \times 8 \times 8 \times 4} = \dfrac{27}{32}$.

9. $\dfrac{1}{4^{15}} \approx 0.00000000093$.

10. $26 \times 25 \times 24 \times 10 \times 9 \times 8 = 11,232,000$.

11. There are $26^3 \times 10^2 = 1,757,600$ such codes; so the answer is positive.

12. 2^{nm}.

13. $(2+1)(3+1)(2+1) = 36$. (See the solution to Exercise 24.)

14. There are $(2^6 - 1)2^3 = 504$ possible sandwiches. So the claim is true.

15. **(a)** $5^4 = 625$. **(b)** $5^4 - 5 \times 4 \times 3 \times 2 = 505$.

16. $2^{12} = 4096$.

17. $1 - \dfrac{48 \times 48 \times 48 \times 48}{52 \times 52 \times 52 \times 52} = 0.274$.

18. $10 \times 9 \times 8 \times 7 = 5040$. **(a)** $9 \times 9 \times 8 \times 7 = 4536$; **(b)** $5040 - 1 \times 1 \times 8 \times 7 = 4984$.

19. $1 - \dfrac{(N-1)^n}{N^n}$.

20. By Example 2.6, the probability is 0.507 that among Jenny and the next 22 people she meets randomly there are two with the same birthday. However, it is quite possible that one of these two persons is not Jenny. Let n be the minimum number of people Jenny must meet so that the chances are better than even that someone shares her birthday. To find n, let A denote the event that among the next n people Jenny meets randomly someone's birthday is the same as Jenny's. We have

$$P(A) = 1 - P(A^c) = 1 - \frac{364^n}{365^n}.$$

To have $P(A) > 1/2$, we must find the smallest n for which

$$1 - \frac{364^n}{365^n} > \frac{1}{2},$$

or

$$\frac{364^n}{365^n} < \frac{1}{2}.$$

This gives

$$n > \frac{\log \dfrac{1}{2}}{\log \dfrac{364}{365}} = 252.652.$$

Therefore, for the desired probability to be greater than 0.5, n must be 253. To some this might seem counterintuitive.

21. Draw a tree diagram for the situation in which the salesperson goes from I to B first. In this situation, you will find that in 7 out of 23 cases, she will end up staying at island I. By symmetry, if she goes from I to H, D, or F first, in each of these situations in 7 out of 23 cases she will end up staying at island I. So there are $4 \times 23 = 92$ cases altogether and in $4 \times 7 = 28$ of them the salesperson will end up staying at island I. Since $28/92 = 0.3043$, the answer is 30.43%. Note that the probability that the salesperson will end up staying at island I is *not* 0.3043 because not all of the cases are equiprobable.

22. He is at 0 first, next he goes to 1 or -1. If at 1, then he goes to 0 or 2. If at -1, then he goes to 0 or -2, and so on. Draw a tree diagram. You will find that after walking 4 blocks, he is at one of the points 4, 2, 0, -2, or -4. There are 16 possible cases altogether. Of these 6 end up at 0, none at 1, and none at -1. Therefore, the answer to (a) is 6/16 and the answer to (b) is 0.

23. We can think of a number less than 1,000,000 as a six-digit number by allowing it to start with 0 or 0's. With this convention, it should be clear that there are 9^6 such numbers without the digit five. Hence the desired probability is $1 - (9^6/10^6) = 0.469$.

24. Divisors of N are of the form $p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}$, where $e_i = 0, 1, 2, \ldots, n_i$, $1 \leq i \leq k$. Therefore, the answer is $(n_1 + 1)(n_2 + 1) \cdots (n_k + 1)$.

25. There are 6^4 possibilities altogether. In 5^4 of these possibilities there is no 3. In 5^3 of these possibilities only the first die lands 3. In 5^3 of these possibilities only the second die lands 3, and so on. Therefore, the answer is

$$\frac{5^4 + 4 \times 5^3}{6^4} = 0.868.$$

26. Any subset of the set {salami, turkey, bologna, corned beef, ham, Swiss cheese, American cheese} except the empty set can form a reasonable sandwich. There are $2^7 - 1$ possibilities. To every sandwich a subset of the set {lettuce, tomato, mayonnaise} can also be added. Since there are 3 possibilities for bread, the final answer is $(2^7 - 1) \times 2^3 \times 3 = 3048$ and the advertisement is true.

27. $\dfrac{11 \times 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4}{11^8} = 0.031.$

28. For $i = 1, 2, 3$, let A_i be the event that no one departs at stop i. The desired quantity is $P(A_1^c A_2^c A_3^c) = 1 - P(A_1 \cup A_2 \cup A_3)$. Now

$$\begin{aligned} P(A_1 \cup A_2 \cup A_3) &= P(A_1) + P(A_2) + P(A_3) \\ &\quad - P(A_1A_2) - P(A_1A_3) - P(A_2A_3) + P(A_1A_2A_3) \\ &= \frac{2^6}{3^6} + \frac{2^6}{3^6} + \frac{2^6}{3^6} - \frac{1}{3^6} - \frac{1}{3^6} - \frac{1}{3^6} + 0 = \frac{7}{27}. \end{aligned}$$

Therefore, the desired probability is $1 - (7/27) = 20/27$.

29. For $0 \leq i \leq 9$, the sum of the first two digits is i in $(i+1)$ ways. Therefore, there are $(i+1)^2$ numbers in the given set with the sum of the first two digits equal to the sum of the last two digits and equal to i. For $i = 10$, there are 9^2 numbers in the given set with the sum of the first two digits equal to the sum of the last two digits and equal to 10. For $i = 11$, the corresponding numbers are 8^2 and so on. Therefore, there are altogether

$$1^2 + 2^2 + \cdots + 10^2 + 9^2 + 8^2 + \cdots + 1^2 = 670$$

numbers with the desired probability and hence the answer is $670/10^4 = 0.067$.

30. Let A be the event that the number selected contains at least one 0. Let B be the event that it contains at least one 1 and C be the event that it contains at least one 2. The desired quantity is $P(ABC) = 1 - P(A^c \cup B^c \cup C^c)$, where

$$\begin{aligned} P(A^c \cup B^c \cup C^c) &= P(A^c) + P(B^c) + P(C^c) \\ &\quad - P(A^cB^c) - P(A^cC^c) - P(B^cC^c) + P(A^cB^cC^c) \\ &= \frac{9^r}{9\times 10^{r-1}} + \frac{8\times 9^{r-1}}{9\times 10^{r-1}} + \frac{8\times 9^{r-1}}{9\times 10^{r-1}} - \frac{8^r}{9\times 10^{r-1}} - \frac{8^r}{9\times 10^{r-1}} \\ &\quad - \frac{7\times 8^{r-1}}{9\times 10^{r-1}} + \frac{7^r}{9\times 10^{r-1}}. \end{aligned}$$

2.3 PERMUTATIONS

1. The answer is $\dfrac{1}{4!} = \dfrac{1}{24} \approx 0.0417$.

2. $3! = 6$.

3. $\dfrac{8!}{3!\,5!} = 56$.

4. The probability that John will arrive right after Jim is $7!/8!$ (consider Jim and John as one arrival). Therefore, the answer is $1 - (7!/8!) = 0.875$.

Another Solution: If Jim is the last person, John will not arrive after Jim. Therefore, the remaining seven can arrive in $7!$ ways. If Jim is not the last person, the total number of possibilities in which John will not arrive right after Jim is $7 \times 6 \times 6!$. So the answer is

$$\frac{7! + 7\times 6\times 6!}{8!} = 0.875.$$

5. **(a)** $3^{12} = 531,441$. **(b)** $\dfrac{12!}{6!\,6!} = 924$. **(c)** $\dfrac{12!}{3!\,4!\,5!} = 27,720$.

6. ${}_6P_2 = 30$.

7. $\dfrac{20!}{4!\,3!\,5!\,8!} = 3,491,888,400$.

8. $\dfrac{(5\times 4\times 7)\times(4\times 3\times 6)\times(3\times 2\times 5)}{3!} = 50,400$.

9. There are 8! schedule possibilities. By symmetry, in 8!/2 of them Dr. Richman's lecture precedes Dr. Chollet's and in 8!/2 ways Dr. Richman's lecture precedes Dr. Chollet's. So the answer is $8!/2 = 20,160$.

10. $\dfrac{11!}{3!\,2!\,3!\,3!} = 92,400.$

11. $1 - (6!/6^6) = 0.985.$

12. **(a)** $\dfrac{11!}{4!\,4!\,2!} = 34,650.$

(b) Treating all P's as one entity, the answer is $\dfrac{10!}{4!\,4!} = 6300.$

(c) Treating all I's as one entity, the answer is $\dfrac{8!}{4!\,2!} = 840.$

(d) Treating all P's as one entity, and all I's as another entity, the answer is $\dfrac{7!}{4!} = 210.$

(e) By (a) and (c), The answer is $840/34650 = 0.024$.

13. $\left(\dfrac{8!}{2!\,3!\,3!}\right)\Big/6^8 = 0.000333.$

14. $\left(\dfrac{9!}{3!\,3!\,3!}\right)\Big/52^9 = 6.043 \times 10^{-13}.$

15. $\dfrac{m!}{(n+m)!}.$

16. Each girl and each boy has the same chance of occupying the 13th chair. So the answer is $12/20 = 0.6$. This can also be seen from $\dfrac{12 \times 19!}{20!} = \dfrac{12}{20} = 0.6.$

17. $\dfrac{12!}{12^{12}} = 0.000054.$

18. Look at the five math books as one entity. The answer is $\dfrac{5! \times 18!}{22!} = 0.00068.$

19. $1 - \dfrac{{}_9P_7}{9^7} = 0.962.$

20. $\dfrac{2 \times 5! \times 5!}{10!} = 0.0079.$

21. $n!/n^n$.

22. $1-(6!/6^6)=0.985.$

23. Suppose that A and B are not on speaking terms. ${}_{134}P_4$ committees can be formed in which neither A serves nor B; $4\times{}_{134}P_3$ committees can be formed in which A serves and B does not. The same numbers of committees can be formed in which B serves and A does not. Therefore, the answer is ${}_{134}P_4+2(4\times{}_{134}P_3)=326,998,056.$

24. **(a)** m^n. **(b)** ${}_mP_n$. **(c)** $n!$.

25. $\left(3\cdot\dfrac{8!}{2!\,3!\,2!\,1!}\right)\Big/6^8=0.003.$

26. **(a)** $$\frac{20!}{39\times37\times35\times\cdots\times5\times3\times1}=7.61\times10^{-6}.$$

(b) $$\frac{1}{39\times37\times35\times\cdots\times5\times3\times1}=3.13\times10^{-24}.$$

27. Thirty people can sit in 30! ways at a round table. But for each way, if they rotate 30 times (everybody move one chair to the left at a time) no new situations will be created. Thus in $30!/30=29!$ ways 15 married couples can sit at a round table. Think of each married couple as one entity and note that in $15!/15=14!$ ways 15 such entities can sit at a round table. We have that the 15 couples can sit at a round table in $(2!)^{15}\cdot14!$ different ways because if the couples of each entity change positions between themselves, a new situation will be created. So the desired probability is

$$\frac{14!(2!)^{15}}{29!}=3.23\times10^{-16}.$$

The answer to the second part is

$$\frac{24!(2!)^5}{29!}=2.25\times10^{-6}.$$

28. In 13! ways the balls can be drawn one after another. The number of those in which the first white appears in the second or in the fourth or in the sixth or in the eighth draw is calculated as follows. (These are Jack's turns.)

$$8\times5\times11!+8\times7\times6\times5\times9!+8\times7\times6\times5\times4\times5\times7!$$
$$+\,8\times7\times6\times5\times4\times3\times2\times5\times5!=2,399,846,400.$$

Therefore, the answer is $2,399,846,400/13!=0.385.$

2.4 COMBINATIONS

1. $\binom{20}{6} = 38,760.$

2. $\sum_{i=51}^{100} \binom{100}{i} = 583,379,627,841,332,604,080,945,354,060 \approx 5.8 \times 10^{29}.$

3. $\binom{20}{6}\binom{25}{6} = 6,864,396,000.$

4. $\dfrac{\binom{12}{3}\binom{40}{2}}{\binom{52}{5}} = 0.066.$

5. $\binom{N-1}{n-1} \Big/ \binom{N}{n} = \dfrac{n}{N}.$

6. $\binom{5}{3}\binom{2}{2} = 10.$

7. $\binom{8}{3}\binom{5}{2}\binom{3}{3} = 560.$

8. $\binom{18}{6} + \binom{18}{4} = 21,624.$

9. $\binom{10}{5} \Big/ \binom{12}{7} = 0.318.$

10. The coefficient of 2^3x^9 in the expansion of $(2+x)^{12}$ is $\binom{12}{9}$. Therefore, the coefficient of x^9 is $2^3\binom{12}{9} = 1760.$

11. The coefficient of $(2x)^3(-4y)^4$ in the expansion of $(2x - 4y)^7$ is $\binom{7}{4}$. Thus the coefficient of x^3y^2 in this expansion is $2^3(-4)^4\binom{7}{4} = 71,680.$

12. $\binom{9}{3}\left[\binom{6}{4} + 2\binom{6}{3}\right] = 4620.$

13. **(a)** $\binom{10}{5}\Big/2^{10} = 0.246$; **(b)** $\sum_{i=5}^{10}\binom{10}{i}\Big/2^{10} = 0.623$.

14. If their minimum is larger than 5, they are all from the set $\{6, 7, 8, \ldots, 20\}$. Hence the answer is $\binom{15}{5}\Big/\binom{20}{5} = 0.194$.

15. **(a)** $\dfrac{\binom{6}{2}\binom{28}{4}}{\binom{34}{6}} = 0.228$; **(b)** $\dfrac{\binom{6}{6}+\binom{6}{6}+\binom{10}{6}+\binom{12}{6}}{\binom{34}{6}} = 0.00084$.

16. $\dfrac{\binom{50}{5}\binom{150}{45}}{\binom{200}{50}} = 0.00206$.

17. $\sum_{i=0}^{n} 2^i\binom{n}{i} = \sum_{i=0}^{n}\binom{n}{i}2^i 1^{n-i} = (2+1)^n = 3^n$.

$$\sum_{i=0}^{n} x^i\binom{n}{i} = \sum_{i=0}^{n}\binom{n}{i}x^i 1^{n-i} = (x+1)^n.$$

18. $\left[\binom{6}{2}5^4\right]\Big/6^6 = 0.201$.

19. $2^{12}\Big/\binom{24}{12} = 0.00151$.

20. **Royal Flush:** $\dfrac{4}{\binom{52}{5}} = 0.0000015$.

Straight flush: $\dfrac{36}{\binom{52}{5}} = 0.000014$.

Four of a kind: $\dfrac{13 \times 12\binom{4}{1}}{\binom{52}{5}} = 0.00024$.

Full house: $$\frac{13\binom{4}{3} \cdot 12\binom{4}{2}}{\binom{52}{5}} = 0.0014.$$

Flush: $$\frac{4\binom{13}{5} - 40}{\binom{52}{5}} = 0.002.$$

Straight: $$\frac{10(4)^5 - 40}{\binom{52}{5}} = 0.0039.$$

Three of a kind: $$\frac{13\binom{4}{3} \cdot \binom{12}{2}4^2}{\binom{52}{5}} = 0.021.$$

Two pairs: $$\frac{\binom{13}{2}\binom{4}{2}\binom{4}{2} \cdot 11\binom{4}{1}}{\binom{52}{5}} = 0.048.$$

One pair: $$\frac{13\binom{4}{2} \cdot \binom{12}{3}4^3}{\binom{52}{5}} = 0.42.$$

None of the above: $1-$ the sum of all of the above cases $= 0.5034445$.

21. The desired probability is

$$\frac{\binom{12}{6}\binom{12}{6}}{\binom{24}{12}} = 0.3157.$$

22. The answer is the solution of the equation $\binom{x}{3} = 20$. This equation is equivalent to $x(x-1)(x-2) = 120$ and its solution is $x = 6$.

23. There are $9 \times 10^3 = 9000$ four-digit numbers. From every 4-combination of the set $\{0, 1, \ldots, 9\}$, exactly one four-digit number can be constructed in which its ones place is less than its tens place, its tens place is less than its hundreds place, and its hundreds place is less than its thousands place. Therefore, the number of such four-digit numbers is $\binom{10}{4} = 210$. Hence the desired probability is 0.023333.

24.

$$\begin{aligned}(x+y+z)^2 &= \sum_{n_1+n_2+n_3=2} \frac{n!}{n_1!\,n_2!\,n_3!}\, x^{n_1} y^{n_2} z^{n_3} \\ &= \frac{2!}{2!\,0!\,0!}\, x^2y^0z^0 + \frac{2!}{0!\,2!\,0!}\, x^0y^2z^0 + \frac{2!}{0!\,0!\,2!}\, x^0y^0z^2 \\ &\quad + \frac{2!}{1!\,1!\,0!}\, x^1y^1z^0 + \frac{2!}{1!\,0!\,1!}\, x^1y^0z^1 + \frac{2!}{0!\,1!\,1!}\, x^0y^1z^1 \\ &= x^2 + y^2 + z^2 + 2xy + 2xz + 2yz.\end{aligned}$$

25. The coefficient of $(2x)^2(-y)^3(3z)^2$ in the expansion of $(2x - y + 3z)^7$ is $\dfrac{7!}{2!\,3!\,2!}$. Thus the coefficient of $x^2y^3z^2$ in this expansion is $2^2(-1)^3(3)^2\dfrac{7!}{2!\,3!\,2!} = -7560$.

26. The coefficient of $(2x)^3(-y)^7(3)^3$ in the expansion of $(2x - y + 3)^{13}$ is $\dfrac{13!}{3!\,7!\,3!}$. Therefore, the coefficient of x^3y^7 in this expansion is $2^3(-1)^7(3)^3\dfrac{13!}{3!\,7!\,3!} = -7,413,120$.

27. In $\dfrac{52!}{13!\,13!\,13!\,13!} = \dfrac{52!}{(13!)^4}$ ways 52 cards can be dealt among four people. Hence the sample space contains $52!/(13!)^4$ points. Now in 4! ways the four different suits can be distributed among the players; thus the desired probability is $4!/[52!/(13!)^4] \approx 4.47 \times 10^{-28}$.

28. The theorem is valid for $k = 2$; it is the binomial expansion. Suppose that it is true for all integers $\leq k - 1$. We show it for k. By the binomial expansion,

$$\begin{aligned}(x_1 + x_2 + \cdots + x_k)^n &= \sum_{n_1=0}^{n} \binom{n}{n_1} x_1^{n_1}(x_2 + \cdots + x_k)^{n-n_1} \\ &= \sum_{n_1=0}^{n} \binom{n}{n_1} x_1^{n_1} \sum_{n_2+n_3+\cdots+n_k=n-n_1} \frac{(n-n_1)!}{n_2!\,n_3!\,\cdots\,n_k!}\, x_2^{n_2}x_3^{n_3}\cdots x_k^{n_k} \\ &= \sum_{n_1+n_2+\cdots+n_k=n} \binom{n}{n_1} \frac{(n-n_1)!}{n_2!\,n_3!\,\cdots\,n_k!}\, x_1^{n_1}x_2^{n_2}\cdots x_k^{n_k}\end{aligned}$$

$$= \sum_{n_1+n_2+\cdots+n_k=n} \frac{n!}{n_1!\, n_2! \,\cdots\, n_k!} x_1^{n_1} x_2^{n_2} \cdots x_k^{n_k}.$$

29. We must have 8 steps. Since the distance from M to L is ten 5-centimeter intervals and the first step is made at M, there are 9 spots left at which the remaining 7 steps can be made. So the answer is $\binom{9}{7} = 36$.

30. **(a)** $\dfrac{\binom{2}{1}\binom{98}{49} + \binom{98}{48}}{\binom{100}{50}} = 0.753$; **(b)** $2^{50} \Big/ \binom{100}{50} = 1.16 \times 10^{-14}$.

31. **(a)** It must be clear that

$$\begin{aligned} n_1 &= \binom{n}{2} \\ n_2 &= \binom{n_1}{2} + nn_1 \\ n_3 &= \binom{n_2}{2} + n_2(n + n_1) \\ n_4 &= \binom{n_3}{2} + n_3(n + n_1 + n_2) \\ &\vdots \\ n_k &= \binom{n_{k-1}}{2} + n_{k-1}(n + n_1 + \cdots + n_{k-1}). \end{aligned}$$

(b) For $n = 25,000$, successive calculations of n_k's yield,

$$\begin{aligned} n_1 &= 312, 487, 500, \\ n_2 &= 48, 832, 030, 859, 381, 250, \\ n_3 &= 1, 192, 283, 634, 186, 401, 370, 231, 933, 886, 715, 625, \\ n_4 &= 710, 770, 132, 174, 366, 339, 321, 713, 883, 042, 336, 781, 236, \\ &\quad\; 550, 151, 462, 446, 793, 456, 831, 056, 250. \end{aligned}$$

For $n = 25,000$, the total number of all possible hybrids in the first four generations, $n_1+n_2+n_3+n_4$, is 710,770,132,174,366,339,321,713,883,042,337,973,520,184,337, 863,865,857,421,889,665,625. This number is approximately 710×10^{63}.

32. For $n = 1$, we have the trivial identity

$$x + y = \binom{1}{0} x^0 y^{1-0} + \binom{1}{1} x^1 y^{1-1}.$$

Assume that

$$(x+y)^{n-1} = \sum_{i=0}^{n-1} \binom{n-1}{i} x^i y^{n-1-i}.$$

This gives

$$\begin{aligned}
(x+y)^n &= (x+y)\sum_{i=0}^{n-1} \binom{n-1}{i} x^i y^{n-1-i} \\
&= \sum_{i=0}^{n-1} \binom{n-1}{i} x^{i+1} y^{n-1-i} + \sum_{i=0}^{n-1} \binom{n-1}{i} x^i y^{n-i} \\
&= \sum_{i=1}^{n} \binom{n-1}{i-1} x^i y^{n-i} + \sum_{i=0}^{n-1} \binom{n-1}{i} x^i y^{n-i} \\
&= x^n + \sum_{i=1}^{n-1} \left[\binom{n-1}{i-1} + \binom{n-1}{i}\right] x^i y^{n-i} + y^n \\
&= x^n + \sum_{i=1}^{n-1} \binom{n}{i} x^i y^{n-i} + y^n = \sum_{i=0}^{n} \binom{n}{i} x^i y^{n-i}.
\end{aligned}$$

33. The desired probability is computed as follows.

$$\binom{12}{6}\left[\binom{30}{2}\binom{28}{2}\binom{26}{2}\binom{24}{2}\binom{22}{2}\binom{20}{2}\binom{18}{3}\binom{15}{3}\binom{12}{3}\binom{9}{3}\binom{6}{3}\binom{3}{3}\right]\Big/ 12^{30} \approx 0.000346.$$

34. **(a)** $\dfrac{\binom{10}{6}2^6}{\binom{20}{6}} = 0.347$; **(b)** $\dfrac{\binom{10}{1}\binom{9}{4}2^4}{\binom{20}{6}} = 0.520$;

(c) $\dfrac{\binom{10}{2}\binom{8}{2}2^2}{\binom{20}{6}} = 0.130$; **(d)** $\dfrac{\binom{10}{3}}{\binom{20}{6}} = 0.0031.$

35. $\dfrac{\binom{26}{13}\binom{26}{13}}{\binom{52}{26}} = 0.218.$

36. Let a 6-element combination of a set of integers be denoted by $\{a_1, a_2, \ldots, a_6\}$, where $a_1 < a_2 < \cdots < a_6$. It can be easily verified that the function $h\colon \mathcal{B} \to \mathcal{A}$ defined by

$$h\big(\{a_1, a_2, \ldots, a_6\}\big) = \{a_1, a_2 + 1, \ldots, a_6 + 5\}$$

is one-to-one and onto. Therefore, there is a one-to-one correspondence between $\mathcal{B}$ and $\mathcal{A}$. This shows that the number of elements in $\mathcal{A}$ is $\binom{44}{6}$. Thus the probability that no consecutive integers are selected among the winning numbers is $\binom{44}{6}\Big/\binom{49}{6} \approx 0.505$. This implies that the probability of at least two consecutive integers among the winning numbers is approximately $1 - 0.505 = 0.495$. Given that there are 47 integers between 1 and 49, this high probability might be counter-intuitive. Even without knowledge of expected value, a keen student might observe that, on the average, there should be $(49 - 1)/7 = 6.86$ numbers between each a_i and a_{i+1}, $1 \leq i \leq 5$. Thus he or she might erroneously think that it is unlikely to obtain consecutive integers frequently.

37. **(a)** Let E_i be the event that car i remains unoccupied. The desired probability is

$$P(E_1^c E_2^c \cdots E_n^c) = 1 - P(E_1 \cup E_2 \cup \cdots \cup E_n).$$

Clearly,

$$P(E_i) = \frac{(n-1)^m}{n^m}, \quad 1 \leq i \leq n;$$

$$P(E_i E_j) = \frac{(n-2)^m}{n^m}, \quad 1 \leq i, j \leq n,\ i \neq j;$$

$$P(E_i E_j E_k) = \frac{(n-3)^m}{n^m}, \quad 1 \leq i, j, k \leq n,\ i \neq j \neq k;$$

and so on. Therefore, by the inclusion-exclusion principle,

$$P(E_1 \cup E_2 \cup \cdots \cup E_n) = \sum_{i=1}^{n} (-1)^{i-1} \binom{n}{i} \frac{(n-i)^m}{n^m}.$$

So

$$P(E_1^c E_2^c \cdots E_n^c) = 1 - \sum_{i=1}^{n} (-1)^{i-1} \binom{n}{i} \frac{(n-i)^m}{n^m} = \sum_{i=0}^{n} (-1)^{i} \binom{n}{i} \frac{(n-i)^m}{n^m}$$

$$= \frac{1}{n^m} \sum_{i=0}^{n} (-1)^i \binom{n}{i} (n-i)^m.$$

(b) Let F be the event that cars $1, 2, \ldots, n-r$ are all occupied and the remaining cars are unoccupied. The desired probability is $\binom{n}{r} P(F)$. Now by part (a), the number of ways m

passengers can be distributed among $n-r$ cars, no car remaining unoccupied is

$$\sum_{i=0}^{n-r}(-1)^i\binom{n-r}{i}(n-r-i)^m.$$

So

$$P(F)=\frac{1}{n^m}\sum_{i=0}^{n-r}(-1)^i\binom{n-r}{i}(n-r-i)^m$$

and hence the desired probability is

$$\frac{1}{n^m}\binom{n}{r}\sum_{i=0}^{n-r}(-1)^i\binom{n-r}{i}(n-r-i)^m.$$

38. Let the n indistinguishable balls be represented by n identical oranges and the n distinguishable cells be represented by n persons. We should count the number of different ways that the n oranges can be divided among the n persons, and the number of different ways in which exactly one person does not get an orange. The answer to the latter part is $n(n-1)$ since in this case one person does not get an orange, one person gets exactly two oranges, and the remaining persons each get exactly one orange. There are n choices for the person who does not get an orange and $n-1$ choices for the person who gets exactly two oranges; $n(n-1)$ choices altogether. To count the number of different ways that the n oranges can be divided among the n persons, add $n-1$ identical apples to the oranges and note that by Theorem 2.4, the total number of permutations of these $n-1$ apples and n oranges is $\dfrac{(2n-1)!}{n!\,(n-1)!}$. (We can arrange $n-1$ identical apples and n identical oranges in a row in $(2n-1)!/\big[n!\,(n-1)!\big]$ ways.) Now each one of these $\dfrac{(2n-1)!}{n!\,(n-1)!}=\dbinom{2n-1}{n}$ permutations corresponds to a way of dividing the n oranges among the n persons and vice versa. Give all of the oranges preceding the first apple to the first person, the oranges between the first and the second apples to the second person, the oranges between the second and the third apples to the third person and so on. Therefore, if, for example, an apple appears in the beginning of the permutation, the first person does not get an orange, and if two apples are at the end of the permutations, the $(n-1)$st and the nth persons get no oranges. Thus the answer is $n(n-1)\Big/\dbinom{2n-1}{n}$.

39. The left side of the identity is the binomial expansion of $(1-1)^n=0$.

40. Using the hint, we have

$$\begin{aligned}\binom{n}{0}+\binom{n+1}{1}+\binom{n+2}{2}+\cdots+\binom{n+r}{r}\\ &=\binom{n}{0}+\binom{n+2}{1}-\binom{n+1}{0}+\binom{n+3}{2}-\binom{n+2}{1}\\ &\quad+\binom{n+4}{3}-\binom{n+3}{2}+\cdots+\binom{n+r+1}{r}-\binom{n+r}{r-1}\\ &=\binom{n}{0}-\binom{n+1}{0}+\binom{n+r+1}{r}=\binom{n+r+1}{r}.\end{aligned}$$

41. The identity expresses that to choose r balls from n red and m blue balls, we must choose either r red balls, 0 blue balls or $r-1$ red balls, one blue ball or $r-2$ red balls, two blue balls or $\cdots$ 0 red balls, r blue balls.

42. Note that $\dfrac{1}{i+1}\dbinom{n}{i}=\dfrac{1}{n+1}\dbinom{n+1}{i+1}$. Hence

$$\text{The given sum }=\frac{1}{n+1}\left[\binom{n+1}{1}+\binom{n+1}{2}+\cdots+\binom{n+1}{n+1}\right]=\frac{1}{n+1}(2^{n+1}-1).$$

43. $\left[\dbinom{5}{2}3^3\right]\Big/4^5=0.264.$

44. **(a)** $$P_N=\frac{\dbinom{t}{m}\dbinom{N-t}{n-m}}{\dbinom{N}{n}}.$$

(b) From part (a), we have

$$\frac{P_N}{P_{N-1}}=\frac{(N-t)(N-n)}{N(N-t-n+m)}.$$

This implies $P_N>P_{N-1}$ if and only if $(N-t)(N-n)>N(N-t-n+m)$ or, equivalently, if and only if $N\le nt/m$. So P_N is increasing if and only if $N\le nt/m$. This shows that the maximum of P_N is at $[nt/m]$, where by $[nt/m]$ we mean the greatest integer $\le nt/m$.

45. The sample space consists of $(n+1)^4$ elements. Let the elements of the sample be denoted by x_1, x_2, x_3, and x_4. To count the number of samples (x_1, x_2, x_3, x_4) for which $x_1+x_2=x_3+x_4$, let $y_3=n-x_3$ and $y_4=n-x_4$. Then y_3 and y_4 are also random elements from the set $\{0, 1, 2, \ldots, n\}$. The number of cases in which $x_1+x_2=x_3+x_4$ is identical to the number of cases in which $x_1+x_2+y_3+y_4=2n$. By Example 2.23, the number of nonnegative integer solutions to this equation is $\binom{2n+3}{3}$. However, this also counts the solutions in which one of x_1, x_2, y_3, and y_4 is greater than n. Because of the restrictions $0 \le x_1, x_2, y_3, y_4 \le n$, we must subtract, from this number, the total number of the solutions in which one of x_1, x_2, y_3, and y_4 is greater than n. Such solutions are obtained by finding all nonnegative integer solutions of the equation $x_1 + x_2 + y_3 + y_4 = n - 1$, and then adding $n + 1$ to exactly one of x_1, x_2, y_3, and y_4. Their count is 4 times the number of nonnegative integer solutions of $x_1 + x_2 + y_3 + y_4 = n - 1$; that is, $4\binom{n+2}{3}$. Therefore, the desired probability is

$$\frac{\binom{2n+3}{3} - 4\binom{n+2}{3}}{(n+1)^4} = \frac{2n^2 + 4n + 3}{3(n+1)^3}.$$

46. **(a)** The $n - m$ unqualified applicants are "ringers." The experiment is not affected by their inclusion, so that the probability of any one of the qualified applicants being selected is the same as it would be if there were only qualified applicants. That is, $1/m$. This is because in a random arrangement of m qualified applicants, the probability that a given applicant is the first one is $1/m$.

(b) Let A be the event that a given qualified applicant is hired. We will show that $P(A) = 1/m$. Let E_i be the event that the given qualified applicant is the ith applicant interviewed, *and* he or she is the first qualified applicant to be interviewed. Clearly,

$$P(A) = \sum_{i=1}^{n-m+1} P(E_i),$$

where

$$P(E_i) = \frac{{}_{n-m}P_{i-1} \cdot 1 \cdot (n-i)!}{n!}.$$

Therefore,

$$\begin{aligned}
P(A) &= \sum_{i=1}^{n-m+1} \frac{{}_{n-m}P_{i-1} \cdot (n-i)!}{n!} \\
&= \sum_{i=1}^{n-m+1} \frac{\dfrac{(n-m)!}{(n-m-i+1)!}\,(n-i)!}{n!} \\
&= \sum_{i=1}^{n-m+1} \frac{1}{m!} \cdot \frac{1}{\dfrac{n!}{m!\,(n-m)!}} \cdot \frac{(n-i)!}{(n-m-i+1)!\,(m-1)!}\,(m-1)! \\
&= \sum_{i=1}^{n-m+1} \frac{1}{m} \cdot \frac{1}{\binom{n}{m}} \binom{n-i}{m-1}
\end{aligned}$$

$$= \frac{1}{m} \cdot \frac{1}{\binom{n}{m}} \sum_{i=1}^{n-m+1} \binom{n-i}{m-1}. \tag{4}$$

To calculate $\sum_{i=1}^{n-m+1} \binom{n-i}{m-1}$, note that $\binom{n-i}{m-1}$ is the coefficient of x^{m-1} in the expansion of $(1+x)^{n-i}$. Therefore, $\sum_{i=1}^{n-m+1} \binom{n-i}{m-1}$ is the coefficient of x^{m-1} in the expansion of

$$\sum_{i=1}^{n-m+1} (1+x)^{n-i} = \frac{(1+x)^n - (1+x)^{m-1}}{x}.$$

This shows that $\sum_{i=1}^{n-m+1} \binom{n-i}{m-1}$ is the coefficient of x^m in the expansion of $(1+x)^n - (1+x)^{m-1}$, which is $\binom{n}{m}$. So (4) implies that

$$P(A) = \frac{1}{m} \cdot \frac{1}{\binom{n}{m}} \cdot \binom{n}{m} = \frac{1}{m}.$$

47. Clearly, $N = 6^{10}$, $N(A_i) = 5^{10}$, $N(A_iA_j) = 4^{10}$, $i \neq j$, and so on. So S_1 has $\binom{6}{1}$ equal terms, S_2 has $\binom{6}{2}$ equal terms, and so on. Therefore, the solution is

$$6^{10} - \binom{6}{1}5^{10} + \binom{6}{2}4^{10} - \binom{6}{3}3^{10} + \binom{6}{4}2^{10} - \binom{6}{5}1^{10} + \binom{6}{6}0^{10} = 16,435,440.$$

48. $|A_0| = \frac{1}{2}\binom{n}{3}\binom{n-3}{3}$, $\quad |A_1| = \frac{1}{2}\binom{n}{3}\binom{3}{1}\binom{n-3}{2}$, $\quad |A_2| = \frac{1}{2}\binom{n}{3}\binom{3}{2}\binom{n-3}{1}$.

The answer is

$$\frac{|A_0|}{|A_0| + |A_1| + |A_2|} = \frac{(n-4)(n-5)}{n^2+2}.$$

49. The coefficient of x^n in $(1+x)^{2n}$ is $\binom{2n}{n}$. Its coefficient in $(1+x)^n(1+x)^n$ is

$$\binom{n}{0}\binom{n}{n} + \binom{n}{1}\binom{n}{n-1} + \binom{n}{2}\binom{n}{n-2} + \cdots + \binom{n}{n}\binom{n}{0}$$
$$= \binom{n}{0}^2 + \binom{n}{1}^2 + \binom{n}{2}^2 + \cdots + \binom{n}{n}^2,$$

since $\binom{n}{i} = \binom{n}{n-1}$, $0 \le i \le n$.

50. Consider a particular set of k letters. Let M be the number of possibilities in which only these k letters are addressed correctly. The desired probability is the quantity $\binom{n}{k}M \Big/ n!$. All we got to do is to find M. To do so, note that the remaining $n-k$ letters are all addressed incorrectly. For these $n-k$ letters, there are $n-k$ addresses. But the addresses are written on the envelopes at random. The probability that none is addressed correctly on one hand is $M/(n-k)!$, and on the other hand, by Example 2.24, is

$$1 - \sum_{i=1}^{n-k} \frac{(-1)^{i-1}}{i!} = \sum_{i=2}^{n} \frac{(-1)^{i-1}}{i!}.$$

So M satisfies

$$\frac{M}{(n-k)!} = \sum_{i=2}^{n} \frac{(-1)^{i-1}}{i!},$$

and hence

$$M = (n-k)! \sum_{i=2}^{n} \frac{(-1)^{i-1}}{i!}.$$

The final answer is

$$\frac{\binom{n}{k}M}{n!} = \frac{\binom{n}{k}(n-k)! \displaystyle\sum_{i=2}^{n} \frac{(-1)^{i-1}}{i!}}{n!} = \frac{1}{k!} \sum_{i=2}^{n} \frac{(-1)^{i-1}}{i!}.$$

51. The set of all sequences of H's and T's of length i with no successive H's are obtained either by adding a T to the tails of all such sequences of length $i-1$, or a TH to the tails of all such sequences of length $i-2$. Therefore,

$$x_i = x_{i-1} + x_{i-2}, \quad i \ge 2.$$

Clearly, $x_1 = 2$ and $x_3 = 3$. For consistency, we define $x_0 = 1$. From the theory of recurrence relations we know that the solution of $x_i = x_{i-1} + x_{i-2}$ is of the form $x_i = Ar_1^i + Br_2^i$, where r_1 and r_2 are the solutions of $r^2 = r + 1$. Therefore, $r_1 = \dfrac{1+\sqrt{5}}{2}$ and $r_2 = \dfrac{1-\sqrt{5}}{2}$ and so

$$x_i = A\Big(\frac{1+\sqrt{5}}{2}\Big)^i + B\Big(\frac{1-\sqrt{5}}{2}\Big)^i.$$

Using the initial conditions $x_0 = 1$ and $x_2 = 2$, we obtain $A = \dfrac{5+3\sqrt{5}}{10}$ and $B = \dfrac{5-3\sqrt{5}}{10}$.

Hence the answer is

$$\frac{x_n}{2^n} = \frac{1}{2^n}\Big[\Big(\frac{5+3\sqrt{5}}{10}\Big)\Big(\frac{1+\sqrt{5}}{2}\Big)^n + \Big(\frac{5-3\sqrt{5}}{10}\Big)\Big(\frac{1-\sqrt{5}}{2}\Big)^n\Big]$$
$$= \frac{1}{10 \times 2^{2n}}\Big[\big(5+3\sqrt{5}\big)\big(1+\sqrt{5}\big)^n + \big(5-3\sqrt{5}\big)\big(1-\sqrt{5}\big)^n\Big].$$

52. For this exercise, a solution is given by Abramson and Moser in the October 1970 issue of the *American Mathematical Monthly.*

2.5 STIRLING's FORMULA

1. **(a)** $\displaystyle \binom{2n}{n}\frac{1}{2^{2n}} = \frac{(2n)!}{n!\,n!}\,\frac{1}{2^{2n}} \sim \frac{\sqrt{4\pi n}\,(2n)^{2n}\,e^{-2n}}{(2\pi n)\,n^{2n}\,e^{-2n}\,2^{2n}} \sim \frac{1}{\sqrt{\pi n}}.$

(b) $\displaystyle \frac{\big[(2n)!\big]^3}{(4n)!\,(n!)^2} \sim \frac{\big[\sqrt{4\pi n}\,(2n)^{2n}\,e^{-2n}\big]^3}{\sqrt{8\pi n}\,(4n)^{4n}\,e^{-4n}\,(2\pi n)\,n^{2n}\,e^{-2n}} = \frac{\sqrt{2}}{4^n}.$

REVIEW PROBLEMS FOR CHAPTER 2

1. The desired quantity is equal to the number of subsets of all seven varieties of fruit minus 1 (the empty set); so it is $2^7 - 1 = 127$.

2. The number of choices Virginia has is equal to the number of subsets of $\{1, 2, 5, 10, 20\}$ minus 1 (for empty set). So the answer is $2^5 - 1 = 31$.

3. $(6 \times 5 \times 4 \times 3)/6^4 = 0.278$.

4. $10\Big/\dbinom{10}{2} = 0.222$.

5. $\dfrac{9!}{3!\,2!\,2!\,2!} = 7560$.

6. $5!/5 = 4! = 24$.

7. $3! \cdot 4! \cdot 4! \cdot 4! = 82,944$.

8. $1 - \dfrac{\dbinom{23}{6}}{\dbinom{30}{6}} = 0.83$.

9. Since the refrigerators are identical, the answer is 1.

10. $6! = 720$.

11. (Draw a tree diagram.) In 18 out of 52 possible cases the tournament ends because John wins 4 games without winning 3 in a row. So the answer is 34.62%.

12. Yes, it is because the probability of what happened is $1/7^2 = 0.02$.

13. $9^8 = 43,046,721$.

14. **(a)** $26 \times 25 \times 24 \times 23 \times 22 \times 21 = 165,765,600$;

(b) $26 \times 25 \times 24 \times 23 \times 22 \times 5 = 39,468,000$;

(c) $\binom{5}{2}26\binom{3}{1}25\binom{2}{1}24\binom{1}{1}23 = 21,528,000$.

15. $$\frac{\binom{6}{3} + \binom{6}{1} + \binom{6}{1} + \binom{6}{1}\binom{2}{1}\binom{2}{1}}{\binom{10}{3}} = 0.467.$$

Another Solution: $$\frac{\binom{6}{3} + \binom{6}{1}\binom{4}{2}}{\binom{10}{3}} = 0.467.$$

16. $$\frac{8 \times 4 \times {}_6P_4}{{}_8P_6} = 0.571.$$

17. $$1 - \frac{27^8}{28^8} = 0.252.$$

18. $$\frac{(3!/3)(5!)^3}{15!/15} = 0.000396.$$

19. $3^{12} = 531,441$.

20. $$\frac{\binom{4}{1}\binom{48}{12}\binom{3}{1}\binom{36}{12}\binom{2}{1}\binom{24}{12}\binom{1}{1}\binom{12}{12}}{\dfrac{52!}{13!\,13!\,13!\,13!}} = 0.1055.$$

21. Let A_1, A_2, A_3, and A_4 be the events that there is no professor, no associate professor, no assistant professor, and no instructor in the committee, respectively. The desired probability is

$$P(A_1^c A_2^c A_3^c A_4^c) = 1 - P(A_1 \cup A_2 \cup A_3 \cup A_4),$$

where $P(A_1 \cup A_2 \cup A_3 \cup A_4)$ is calculated using the inclusion-exclusion principle:

$$\begin{aligned} P(A_1 \cup A_2 \cup A_3 \cup A_4) &= P(A_1) + P(A_2) + P(A_3) + P(A_4) \\ &\quad - P(A_1A_2) - P(A_1A_3) - P(A_1A_4) - P(A_2A_3) - P(A_2A_4) - P(A_3A_4) \\ &\quad + P(A_1A_2A_3) + P(A_1A_3A_4) + P(A_1A_2A_4) + P(A_2A_3A_4) - P(A_1A_2A_3A_4) \\ &= \left[1\Big/\binom{34}{6}\right]\left[\binom{28}{6} + \binom{28}{6} + \binom{24}{6} + \binom{22}{6} - \binom{22}{6} - \binom{18}{6} - \binom{16}{6} - \binom{18}{6} \right. \\ &\quad \left. - \binom{16}{6} - \binom{12}{6} + \binom{12}{6} + \binom{6}{6} + \binom{10}{6} + \binom{6}{6} - 0\right] = 0.621. \end{aligned}$$

Therefore, the desired probability equals $1 - 0.621 = 0.379$.

22. $\dfrac{(15!)^2}{30!/(2!)^{15}} = 0.0002112.$

23. $(N - n + 1)\Big/\dbinom{N}{n}.$

24. **(a)** $\dfrac{\binom{4}{2}\binom{48}{24}}{\binom{52}{26}} = 0.390;$ **(b)** $\dfrac{\binom{40}{1}}{\binom{52}{13}} = 6.299 \times 10^{-11};$

(c) $\dfrac{\binom{13}{5}\binom{39}{8}\binom{8}{8}\binom{31}{5}}{\binom{52}{13}\binom{39}{13}} = 0.00000261.$

25. $12!/(3!)^4 = 369,600.$

26. There is a one-to-one correspondence between all cases in which the eighth outcome obtained is not a repetition and all cases in which the first outcome obtained will not be repeated. The answer is

$$\frac{6 \times 5 \times 5 \times 5 \times 5 \times 5 \times 5 \times 5}{6 \times 6 \times 6 \times 6 \times 6 \times 6 \times 6 \times 6} = \left(\frac{5}{6}\right)^7 = 0.279.$$

27. There are $9 \times 10^3 = 9,000$ four-digit numbers. To count the number of desired four-digit numbers, note that if 0 is to be one of the digits, then the thousands place of the number must be

0, but this cannot be the case since the first digit of an n-digit number is nonzero. Keeping this in mind, it must be clear that from every 4-combination of the set $\{1, 2, \dots, 9\}$, exactly one four-digit number can be constructed in which its ones place is greater than its tens place, its tens place is greater than it hundreds place, and its hundreds place is greater than its thousands place. Therefore, the number of such four-digit numbers is $\binom{9}{4} = 126$. Hence the desired probability is $= 0.014$.

28. Since the sum of the digits of 100,000 is 1, we ignore 100,000 and assume that all of the numbers have five digits by placing 0's in front of those with less than five digits. The following process establishes a one-to-one correspondence between such numbers, $d_1d_2d_3d_4d_5$, $\sum_{i=1}^{5} d_i = 8$, and placement of 8 identical objects into 5 distinguishable cells: Put d_1 of the objects into the first cell, d_2 of the objects into the second cell, d_3 into the third cell, and so on. Since this can be done in $\binom{8+5-1}{5-1} = \binom{12}{8} = 495$ ways, the number of integers from the set $\{1, 2, 3, \dots, 100000\}$ in which the sum of the digits is 8 is 495. Hence the desired probability is $495/100,000 = 0.00495$.

Chapter 3

Conditional Probability and Independence

3.1 CONDITIONAL PROBABILITY

1. $P(W \mid U) = \dfrac{P(UW)}{P(U)} = \dfrac{0.15}{0.25} = 0.60.$

2. Let E be the event that in the blood of the randomly selected soldier A antigen is found. Let F be the event that the blood type of the soldier is A. We have

$$P(F \mid E) = \frac{P(FE)}{P(E)} = \frac{0.41}{0.41 + 0.04} = 0.911.$$

3. $\dfrac{0.20}{0.32} = 0.625.$

4. The reduced sample space is $\{(1, 4), (2, 3), (3, 2), (4, 1), (4, 6), (5, 5), (6, 4)\}$; therefore, the desired probability is $1/7$.

5. $\dfrac{30 - 20}{30 - 15} = \dfrac{2}{3}.$

6. Both of the inequalities are equivalent to $P(AB) > P(A)P(B)$.

7. $\dfrac{1/3}{(1/3) + (1/2)} = \dfrac{2}{5}.$

8. $4/30 = 0.133.$

9. $$\frac{\dfrac{\binom{40}{2}\binom{65}{6}}{\binom{105}{8}}}{1-\displaystyle\sum_{i=0}^{2}\frac{\binom{40}{8-i}\binom{65}{i}}{\binom{105}{8}}}=0.239.$$

10. $$P(\alpha=i\mid\beta=0)=\begin{cases}1/19 & \text{if } i=0\\ 2/19 & \text{if } i=1,2,3,\ldots,9\\ 0 & \text{if } i=10,11,12,\ldots,18.\end{cases}$$

11. Let b^*gb mean that the oldest child of the family is a boy, the second oldest is a girl, the youngest is a boy, and the boy found in the family is the oldest child, with similar representations for other cases. The reduced sample space is

$$S=\big\{ggb^*,\, gb^*g,\, b^*gg,\, b^*bg,\, bb^*g,\, gb^*b,\, gbb^*,\, bgb^*,\, b^*gb,\, b^*bb,\, bb^*b,\, bbb^*\big\}.$$

Note that the outcomes of the sample space are not equiprobable. We have that

$$\begin{aligned}&P\big(\{ggb^*\}\big)=P\big(\{gb^*g\}\big)=P\big(\{b^*gg\}\big)=1/7\\&P\big(\{b^*bg\}\big)=P\big(\{bb^*g\}\big)=1/14\\&P\big(\{gb^*b\}\big)=P\big(\{gbb^*\}\big)=1/14\\&P\big(\{bgb^*\}\big)=P\big(\{b^*gb\}\big)=1/14\\&P\big(\{b^*bb\}\big)=P\big(\{bb^*b\}\big)=P\big(\{bbb^*\}\big)=1/21.\end{aligned}$$

The solutions to (a), (b), (c) are as follows.

(a) $P\big(\{bb^*g\}\big)=1/14$;

(b) $P\big(\{bb^*g,\, gbb^*,\, bgb^*,\, bb^*b,\, bbb^*\}\big)=13/42$;

(c) $P\big(\{b^*bg,\, bb^*g,\, gb^*b,\, gbb^*,\, bgb^*,\, b^*gb\}\big)=3/7$.

12. $P(A)=1$ implies that $P(A\cup B)=1$. Hence, by

$$P(A\cup B)=P(A)+P(B)-P(AB),$$

we have that $P(B)=P(AB)$. Therefore,

$$P(B\mid A)=\frac{P(AB)}{P(A)}=\frac{P(B)}{1}=P(B).$$

13. $P(A \mid B) = \dfrac{P(AB)}{b}$, where

$$P(AB) = P(A) + P(B) - P(A \cup B) \geq P(A) + P(B) - 1 = a + b - 1.$$

14. **(a)** $P(AB) \geq 0$, $P(B) > 0$. Therefore, $P(A \mid B) = \dfrac{P(AB)}{P(B)} \geq 0$.

(b) $P(S \mid B) = \dfrac{P(SB)}{P(B)} = \dfrac{P(B)}{P(B)} = 1$.

(c)
$$P\Big(\bigcup_{i=1}^{\infty} A_i \;\Big|\; B\Big) = \frac{P\Big(\big(\bigcup_{i=1}^{\infty} A_i\big)B\Big)}{P(B)} = \frac{P\big(\bigcup_{i=1}^{\infty} A_i B\big)}{P(B)}$$
$$= \frac{\sum_{i=1}^{\infty} P(A_i B)}{P(B)} = \sum_{i=1}^{\infty} \frac{P(A_i B)}{P(B)} = \sum_{i=1}^{\infty} P(A_i \mid B).$$

Note that $P(\cup_{i=1}^{\infty} A_i B) = \sum_{i=1}^{\infty} P(A_i B)$, since mutual exclusiveness of A_i's imply that of $A_i B$'s; i.e., $A_i A_j = \emptyset$, $i \neq j$, implies that $(A_i B)(A_j B) = \emptyset$, $i \neq j$.

15. The given inequalities imply that $P(EF) \geq P(GF)$ and $P(EF^c) \geq P(GF^c)$. Thus

$$P(E) = P(EF) + P(EF^c) \geq P(GF) + P(GF^c) = P(G).$$

16. Reduce the sample space: Marlon chooses from six dramas and seven comedies two at random. What is the probability that they are both comedies? The answer is $\binom{7}{2} \Big/ \binom{13}{2} = 0.269$.

17. Reduce the sample space: There are 21 crayons of which three are red. Seven of these crayons are selected at random and given to Marty. What is the probability that three of them are red? The answer is $\binom{18}{4} \Big/ \binom{21}{7} = 0.0263$.

18. **(a)** The reduced sample space is $S = \{1, 3, 5, 7, 9, \ldots, 9999\}$. There are 5000 elements in S. Since the set $\{5, 7, 9, 11, 13, 15, \ldots, 9999\}$ includes exactly $4998/3 = 1666$ odd numbers that are divisible by three, the reduced sample space has 1667 odd numbers that are divisible by 3. So the answer is $1667/5000 = 0.3334$.

(b) Let O be the event that the number selected at random is odd. Let F be the event that it is divisible by 5 and T be the event that it is divisible by 3. The desired probability is calculated as follows.

$$P(F^c T^c \mid O) = 1 - P(F \cup T \mid O) = 1 - P(F \mid O) - P(T \mid O) + P(FT \mid O)$$
$$= 1 - \frac{1000}{5000} - \frac{1667}{5000} + \frac{333}{5000} = 0.5332.$$

19. Let A be the event that during this period he has hiked in Oregon Ridge Park at least once. Let B be the event that during this period he has hiked in this park at least twice. We have

$$P(B \mid A) = \frac{P(B)}{P(A)},$$

where

$$P(A) = 1 - \frac{5^{10}}{6^{10}} = 0.838$$

and

$$P(B) = 1 - \frac{5^{10}}{6^{10}} - \frac{10 \times 5^9}{6^{10}} = 0.515.$$

So the answer is $0.515/0.838 = 0.615$.

20. The numbers of 333 red and 583 blue chips are divisible by 3. Thus the reduced sample space has $333 + 583 = 916$ points. Of these numbers, $[1000/15] = 66$ belong to red balls and are divisible by 5 and $[1750/15] = 116$ belong to blue balls and are divisible by 5. Thus the desired probability is $182/916 = 0.199$.

21. Reduce the sample space: There are two types of animals in a laboratory, 15 type I and 13 type II. Six animals are selected at random; what is the probability that at least two of them are Type II? The answer is

$$1 - \frac{\dbinom{15}{6} + \dbinom{13}{1}\dbinom{15}{5}}{\dbinom{28}{6}} = 0.883.$$

22. Reduce the sample space: 30 students of which 12 are French and nine are Korean are divided randomly into two classes of 15 each. What is the probability that one of them has exactly four French and exactly three Korean students? The solution to this problem is

$$\frac{\dbinom{12}{4}\dbinom{9}{3}\dbinom{9}{8}}{\dbinom{30}{15}\dbinom{15}{15}} = 0.00241.$$

23. This sounds puzzling because apparently the only deduction from the name "Mary" is that one of the children is a girl. But the crucial difference between this and Example 3.2 is reflected in the implicit assumption that both girls cannot be Mary. That is, the same name cannot be used for two children in the same family. In fact, any other identifying feature that cannot be shared by both girls would do the trick.

3.2 LAW OF MULTIPLICATION

1. Let G be the event that Susan is guilty. Let L be the event that Robert will lie. The probability that Robert will commit perjury is

$$P(GL) = P(G)P(L \mid G) = (0.65)(0.25) = 0.1625.$$

2. The answer is

$$\frac{11}{14} \times \frac{10}{13} \times \frac{9}{12} \times \frac{8}{11} \times \frac{7}{10} \times \frac{6}{9} = 0.15.$$

3. By the law of multiplication, the answer is

$$\frac{52}{52} \times \frac{50}{51} \times \frac{48}{50} \times \frac{46}{49} \times \frac{44}{48} \times \frac{42}{47} = 0.72.$$

4. **(a)** $\dfrac{8}{20} \times \dfrac{7}{19} \times \dfrac{6}{18} \times \dfrac{5}{17} = 0.0144;$

(b) $\dfrac{8}{20} \times \dfrac{7}{19} \times \dfrac{12}{18} + \dfrac{8}{20} \times \dfrac{12}{19} \times \dfrac{7}{18} + \dfrac{12}{20} \times \dfrac{8}{19} \times \dfrac{7}{18} + \dfrac{8}{20} \times \dfrac{7}{19} \times \dfrac{6}{18} = 0.344.$

5. **(a)** $\dfrac{6}{11} \times \dfrac{5}{10} \times \dfrac{5}{9} \times \dfrac{4}{8} \times \dfrac{4}{7} \times \dfrac{3}{6} \times \dfrac{3}{5} \times \dfrac{2}{4} \times \dfrac{2}{3} \times \dfrac{1}{2} \times \dfrac{1}{1} = 0.00216.$

(b) $\dfrac{5}{11} \times \dfrac{4}{10} \times \dfrac{3}{9} \times \dfrac{2}{8} \times \dfrac{1}{7} = 0.00216.$

6. $\dfrac{3}{8} \times \dfrac{5}{10} \times \dfrac{5}{13} \times \dfrac{8}{15} + \dfrac{5}{8} \times \dfrac{3}{11} \times \dfrac{8}{13} \times \dfrac{5}{16} = 0.0712.$

7. Let A_i be the event that the ith person draws the "you lose" paper. Clearly,

$$P(A_1) = \frac{1}{200},$$

$$P(A_2) = P(A_1^c A_2) = P(A_1^c)P(A_2 \mid A_1^c) = \frac{199}{200} \cdot \frac{1}{199} = \frac{1}{200},$$

$$P(A_3) = P(A_1^c A_2^c A_3) = P(A_1^c)P(A_2^c \mid A_1^c)P(A_3 \mid A_1^c A_2^c) = \frac{199}{200} \cdot \frac{198}{199} \cdot \frac{1}{198} = \frac{1}{200},$$

and so on. Therefore, $P(A_i) = 1/200$ for $1 \le i \le 200$. This means that it makes no difference if you draw first, last or anywhere in the middle. Here is Marilyn Vos Savant's intuitive solution to this problem:

It makes no difference if you draw first, last, or anywhere in the middle. Look at it this way: Say the robbers make everyone draw at once. You'd agree that everyone has the same change of losing (one in 200), right? Taking turns just makes that same event happen in a slow and orderly fashion. Envision a raffle at a church with 200 people in attendance, each person buys a ticket. Some buy a ticket when they arrive, some during the event, and some just before the winner is drawn. It doesn't matter. At the party the end result is this: all 200 guests draw a slip of paper, and, regardless of when they look at the slips, the result will be identical: one will lose. You can't alter your chances by looking at your slip before anyone else does, or waiting until everyone else has looked at theirs.

8. Let B be the event that a randomly selected person from the population at large has poor credit report. Let I be the event that the person selected at random will improve his or her credit rating within the next three years. We have

$$P(B \mid I) = \frac{P(BI)}{P(I)} = \frac{P(I \mid B)P(B)}{P(I)} = \frac{(0.30)(0.18)}{0.75} = 0.072.$$

The desired probability is $1-0.072 = 0.928$. Therefore, 92.8% of the people who will improve their credit records within the next three years are the ones with good credit ratings.

9. For $1 \leq n \leq 39$, let E_n be the event that none of the first $n-1$ cards is a heart or the ace of spades. Let F_n be the event that the nth card drawn is the ace of spades. Then the event of "no heart before the ace of spades" is $\bigcup_{n=1}^{39} E_n F_n$. Clearly, $\{E_n F_n,\ 1 \leq n \leq 39\}$ forms a sequence of mutually exclusive events. Hence

$$P\Big(\bigcup_{n=1}^{39} E_n F_n\Big) = \sum_{n=1}^{39} P(E_n F_n) = \sum_{n=1}^{39} P(E_n)P(F_n \mid E_n)$$

$$= \sum_{n=1}^{39} \frac{\binom{38}{n-1}}{\binom{52}{n-1}} \times \frac{1}{53-n} = \frac{1}{14},$$

a result which is not unexpected.

10. $P(F)P(E \mid F) = \dfrac{\binom{13}{3}\binom{39}{6}}{\binom{52}{9}} \times \dfrac{10}{43} = 0.059.$

11. By the law of multiplication,

$$P(A_n) = \frac{2}{3} \times \frac{3}{4} \times \frac{4}{5} \times \cdots \times \frac{n+1}{n+2} = \frac{2}{n+2}.$$

Now since $A_1 \supseteq A_2 \supseteq A_3 \supseteq \cdots \supseteq A_n \supseteq A_{n+1} \supseteq \cdots$, by Theorem 1.8,

$$P\Big(\bigcap_{i=1}^{\infty} A_i\Big) = \lim_{n\to\infty} P(A_n) = 0.$$

3.3 LAW OF TOTAL PROBABILITY

1. $\frac{1}{2} \times 0.05 + \frac{1}{2} \times 0.0025 = 0.02625.$

2. $(0.16)(0.60) + (0.20)(0.40) = 0.176.$

3. $\frac{1}{3}(0.75) + \frac{1}{3}(0.68) + \frac{1}{3}(0.47) = 0.633.$

4. $\frac{12}{51} \times \frac{13}{52} + \frac{13}{51} \times \frac{39}{52} = \frac{1}{4}.$

5. $\frac{11}{50} \times \frac{\binom{13}{2}}{\binom{52}{2}} + \frac{12}{50} \times \frac{\binom{13}{1}\binom{39}{1}}{\binom{52}{2}} + \frac{13}{50} \times \frac{\binom{39}{2}}{\binom{52}{2}} = \frac{1}{4}.$

6. $(0.20)(0.40) + (0.35)(0.60) = 0.290.$

7. $(0.37)(0.80) + (0.63)(0.65) = 0.7055.$

8. $\frac{1}{6}(0.6) + \frac{1}{6}(0.5) + \frac{1}{6}(0.7) + \frac{1}{6}(0.9) + \frac{1}{6}(0.7) + \frac{1}{6}(0.8) = 0.7.$

9. $(0.50)(0.04) + (0.30)(0.02) + (0.20)(0.04) = 0.034.$

10. Let B be the event that the randomly selected child from the countryside is a boy. Let E be the event that the randomly selected child is the first child of the family and F be the event that he or she is the second child of the family. Clearly, $P(E) = 2/3$ and $P(F) = 1/3$. By the law of total probability,

$$P(B) = P(B \mid E)P(E) + P(B \mid F)P(F) = \frac{1}{2} \times \frac{2}{3} + \frac{1}{2} \times \frac{1}{3} = \frac{1}{2}.$$

Therefore, assuming that sex distributions are equally probable, in the Chinese countryside, the distribution of sexes will remain equal. Here is Marilyn Vos Savant's intuitive solution to this problem:

The distribution of sexes will remain roughly equal. That's because–no matter how many or how few children are born anywhere, anytime, with or without restriction–half will be boys and half will be girls: Only the act of conception (not the government!) determines their sex.

One can demonstrate this mathematically. (In this example, we'll assume that women with firstborn girls will always have a second child.) Let's say 100 women give birth, half to boys and half to girls. The half with boys must end their families. There are now 50 boys and 50 girls. The half with girls (50) give birth again, half to boys and half to girls. This adds 25 boys and 25 girls, so there are now 75 boys and 75 girls. Now all must end their families. So the result of the policy is that there will be fewer children in number, but the boy/girl ratio will not be affected.

11. The probability that the first person gets a gold coin is $3/5$. The probability that the second person gets a gold coin is

$$\frac{2}{4} \times \frac{3}{5} + \frac{3}{4} \times \frac{2}{5} = \frac{3}{5}.$$

The probability that the third person gets a gold coin is

$$\frac{3}{5} \times \frac{2}{4} \times \frac{1}{3} + \frac{3}{5} \times \frac{2}{4} \times \frac{2}{3} + \frac{2}{5} \times \frac{3}{4} \times \frac{2}{5} + \frac{2}{5} \times \frac{1}{4} \times \frac{3}{3} = \frac{3}{5},$$

and so on. Therefore, they are all equal.

12. ***A Probabilistic Solution:*** Let n be the number of adults in the town. Let x be the number of men in the town. Then $n - x$ is the number of women in the town. Since the number of married men and married women are equal, we have

$$x \cdot \frac{7}{9} = (n - x) \cdot \frac{3}{5}.$$

This relation implies that $x = (27/62)n$. Therefore, the probability that a randomly selected adult is male is $(27/62)n\big/n = 27/62$. The probability that a randomly selected adult is female is $1 - (27/62) = 35/62$. Let A be the event that a randomly selected adult is married. Let M be the event that the randomly selected adult is a man, and let W be the event that the randomly selected adult is a woman. By the law of total probability,

$$\begin{aligned} P(A) &= P(A \mid M)P(M) + P(A \mid W)P(W) \\ &= \frac{7}{9} \cdot \frac{27}{62} + \frac{3}{5} \cdot \frac{35}{62} = \frac{42}{62} = \frac{21}{31} \approx 0.677. \end{aligned}$$

Therefore, 21/31st of the adults are married.

An Arithmetical Solution: The *common numerator* of the two fractions is 21. Hence 21/27th of the men and 21/35th of the women are married. We find the common numerator because the number of married men and the number of married women are equal. This shows that of every $27 + 35 = 62$ adults, $21 + 21 = 42$ are married. Hence 42/62th = 21/31st of the adults in the town are married.

13. The answer is clearly 0.40. This can also be computed from

$$(0.40)(0.75) + (0.40)(0.25) = 0.40.$$

14. Let A be the event that a randomly selected child is the kth born of his or her family. Let B_j be the event that he or she is from a family with j children. Then

$$P(A) = \sum_{j=k}^{c} P(A \mid B_j)P(B_j),$$

where, clearly, $P(A \mid B_j) = 1/j$. To find $P(B_j)$, note that there are $\alpha_i N$ families with j children. Therefore, the total number of children in the world is $\sum_{i=0}^{c} i(\alpha_i N)$ of which $j(N\alpha_j)$ are from families with j children. Hence

$$P(B_j) = \frac{j(N\alpha_j)}{\sum_{i=0}^{c} i(\alpha_i N)} = \frac{j\alpha_j}{\sum_{i=0}^{c} i\alpha_i}.$$

This shows that the desired fraction is given by

$$\begin{aligned} P(A) &= \sum_{j=k}^{c} P(A \mid B_j)P(B_j) = \sum_{j=k}^{c} \frac{1}{j} \cdot \frac{j\alpha_j}{\sum_{i=0}^{c} i\alpha_i} \\ &= \sum_{j=k}^{c} \frac{\alpha_j}{\sum_{i=0}^{c} i\alpha_i} = \frac{\sum_{j=k}^{c} \alpha_j}{\sum_{i=0}^{c} i\alpha_i}. \end{aligned}$$

15. $$Q(E \mid F) = \frac{Q(EF)}{Q(F)} = \frac{P(EF \mid B)}{P(F \mid B)} = \frac{\dfrac{P(EFB)}{P(B)}}{\dfrac{P(FB)}{P(B)}} = \frac{P(EFB)}{P(FB)} = P(E \mid FB).$$

16. Let M, C, and F denote the events that the random student is married, is married to a student at the same campus, and is female, respectively. We have that

$$P(F \mid M) = P(F \mid MC)P(C \mid M) + P(F \mid MC^c)P(C^c \mid M) = (0.40)\frac{1}{3} + (0.30)\frac{2}{3} = 0.333.$$

17. Let $p(k, n)$ be the probability that exactly k of the first n seeds planted in the farm germinated. Using induction on n, we will show that $p(k, n) = 1/(n - 1)$ for all $k < n$. For $n = 2$, $p(1, 2) = 1 = 1/(2 - 1)$ is true. If $p(k, n - 1) = 1/(n - 2)$ for all $k < n - 1$, then, by the law of total probability,

$$\begin{aligned} p(k, n) &= \frac{k - 1}{n - 1} p(k - 1, n - 1) + \frac{n - k - 1}{n - 1} p(k, n - 1) \\ &= \frac{k - 1}{n - 1} \cdot \frac{1}{n - 2} + \frac{n - k - 1}{n - 1} \cdot \frac{1}{n - 2} = \frac{1}{n - 1}. \end{aligned}$$

This proves the induction hypothesis.

18. Reducing the sample space, we have that the answer is $7/10$.

19. $$\frac{\binom{8}{3}}{\binom{18}{3}} \times \frac{\binom{10}{3}}{\binom{18}{3}} + \frac{\binom{7}{3}}{\binom{18}{3}} \times \frac{\binom{10}{2}\binom{8}{1}}{\binom{18}{3}} + \frac{\binom{6}{3}}{\binom{18}{3}} \times \frac{\binom{10}{1}\binom{8}{2}}{\binom{18}{3}} + \frac{\binom{5}{3}}{\binom{18}{3}} \times \frac{\binom{8}{3}}{\binom{18}{3}} = 0.0383.$$

20. We have that

$$\begin{aligned} P(A \mid G) &= P(A \mid GO)P(O \mid G) + P(A \mid GM)P(M \mid G) + P(A \mid GY)P(Y \mid G) \\ &= 0 \times \frac{1}{3} + \frac{1}{2} \times \frac{1}{3} + \frac{3}{4} \times \frac{1}{3} = \frac{5}{12}. \end{aligned}$$

21. Let E be the event that the third number falls between the first two. Let A be the event that the first number is smaller than the second number. We have that

$$P(E \mid A) = \frac{P(EA)}{P(A)} = \frac{1/6}{1/2} = \frac{1}{3}.$$

Intuitively, the fact that $P(A) = 1/2$ and $P(EA) = 1/6$ should be clear (say, by symmetry). However, we can prove these rigorously. We show that $P(A) = 1/2$; $P(EA) = 1/6$ can be proved similarly. Let B be the event that the second number selected is smaller than the first number. Clearly $A = B^c$ and we only need to show that $P(B) = 1/2$. To do this, let B_i be the event that the first number drawn is i, $1 \le i \le n$. Since $\{B_1, B_2, \dots, B_n\}$ is a partition of the sample space,

$$P(B) = \sum_{i=1}^{n} P(B \mid B_i)P(B_i).$$

Now $P(B \mid B_1) = 0$ because if the first number selected is 1, the second number selected cannot be smaller. $P(B \mid B_i) = \dfrac{i-1}{n-1}$, $1 \le i \le n$ since if the first number is i, the second number must be one of $1, 2, 3, \dots, i-1$ if it is to be smaller. Thus

$$\begin{aligned} P(B) &= \sum_{i=1}^{n} P(B \mid B_i)P(B_i) = \sum_{i=2}^{n} \frac{i-1}{n-1} \cdot \frac{1}{n} = \frac{1}{(n-1)n} \sum_{i=2}^{n} (i-1) \\ &= \frac{1}{(n-1)n}\big[1 + 2 + 3 + \cdots + (n-1)\big] = \frac{1}{(n-1)n} \cdot \frac{(n-1)n}{2} = \frac{1}{2}. \end{aligned}$$

22. Let E_m be the event that Avril selects the best suitor given her strategy. Let B_i be the event that the best suitor is the ith of Avril's dates. By the law of total probability,

$$P(E_m) = \sum_{i=1}^{n} P(E_m \mid B_i)P(B_i) = \frac{1}{n} \sum_{i=1}^{n} P(E_m \mid B_i).$$

Clearly, $P(E_m \mid B_i) = 0$ for $1 \le i \le m$. For $i > m$, if the ith suitor is the best, then Avril chooses him if and only if among the first $i-1$ suitors Avril dates, the best is one of the first m. So

$$P(E_m \mid B_i) = \frac{m}{i-1}.$$

Therefore,

$$P(E_m) = \frac{1}{n}\sum_{i=m+1}^{n} \frac{m}{i-1} = \frac{m}{n}\sum_{i=m+1}^{n} \frac{1}{i-1}.$$

Now

$$\sum_{i=m+1}^{n} \frac{1}{i-1} \approx \int_m^n \frac{1}{x}\,dx = \ln\Big(\frac{n}{m}\Big).$$

Thus

$$P(E_m) \approx \frac{m}{n} \ln\Big(\frac{n}{m}\Big).$$

To find the maximum of $P(E_m)$, consider the differentiable function

$$h(x) = \frac{x}{n} \ln\Big(\frac{n}{x}\Big).$$

Since

$$h'(x) = \frac{1}{n} \ln\Big(\frac{n}{x}\Big) - \frac{1}{n} = 0$$

implies that $x = n/e$, the maximum of $P(E_m)$ is at $m = [n/e]$, where $[n/e]$ is the greatest integer less than or equal to n/e. Hence Avril should dump the first $[n/e]$ suitors she dates and marry the first suitor she dates afterward who is better than all those preceding him. The probability that with such a strategy she selects the best suitor of all n is approximately

$$h\Big(\frac{n}{e}\Big) = \frac{1}{e}\ln e = \frac{1}{e} \approx 0.368.$$

23. Let $\mathbf{N}$ be the set of nonnegative integers. The domain of f is

$$\big\{(g, r) \in \mathbf{N} \times \mathbf{N}:\ 0 \le g \le N,\ 0 \le r \le N,\ 0 < g + r < 2N\big\}.$$

Extending the domain of f to all points $(g, r) \in \mathbf{R} \times \mathbf{R}$, we find that $\dfrac{\partial f}{\partial g} = \dfrac{\partial f}{\partial r} = 0$ gives $g = r = N/2$ and $f(N/2, N/2) = 1/2$. However, this is not the maximum value because on the boundary of the domain of f along $r = 0$, we find that

$$f(g, 0) = \frac{1}{2}\Big(1 + \frac{N-g}{2N-g}\Big)$$

is maximum at $g = 1$ and

$$f(1, 0) = \frac{1}{2}\Big(\frac{3N-2}{2N-1}\Big) \ge \frac{1}{2}.$$

We also find that on the boundary along $r = N$,

$$f(g, N) = \frac{1}{2}\Big(\frac{g}{g+N} + 1\Big)$$

is maximum at $g = N - 1$ and

$$f(N-1, N) = \frac{1}{2}\Big(\frac{3N-2}{2N-1}\Big) \geq \frac{1}{2}.$$

The maximums of f along other sides of the boundary are all less than $\frac{1}{2}\Big(\frac{3N-2}{2N-1}\Big)$. Therefore, there are exactly two maximums and they occur at $(1, 0)$ and $(N-1, N)$. That is, the maximum of f occurs if one urn contains one green and 0 red balls and the other one contains $N-1$ green and N red balls. For large N, the probability that the prisoner is freed is $\frac{1}{2}\Big(\frac{3N-2}{2N-1}\Big) \approx \frac{3}{4}$.

3.4 BAYES' FORMULA

1. $$\frac{(3/4)(0.40)}{(3/4)(0.40) + (1/3)(0.60)} = \frac{3}{5}.$$

2. $$\frac{1(2/3)}{1(2/3) + (1/4)(1/3)} = \frac{8}{9}.$$

3. Let G and I be the events that the suspect is guilty and innocent, respectively. Let A be the event that the suspect is left-handed. Since $\{G, I\}$ is a partition of the sample space, we can use Bayes' formula to calculate $P(G \mid A)$, the probability that the suspect has committed the crime in view of the new evidence.

$$P(G \mid A) = \frac{P(A \mid G)P(G)}{P(A \mid G)P(G) + P(A \mid I)P(I)} = \frac{(0.85)(0.65)}{(0.85)(0.65) + (0.23)(0.35)} \approx 0.87.$$

4. Let G be the event that Susan is guilty. Let C be the event that Robert and Julie give conflicting testimony. By Bayes' formula,

$$P(G \mid C) = \frac{P(C \mid G)P(G)}{P(C \mid G)P(G) + P(C \mid G^c)P(G^c)} = \frac{(0.25)(0.65)}{(0.25)(0.65) + (0.30)(0.35)} = 0.607.$$

5. $$\frac{(0.02)(0.30)}{(0.02)(0.30) + (0.05)(0.70)} = 0.1463.$$

6. $$\frac{\Big[\binom{6}{3}\Big/\binom{11}{3}\Big]\Big(\frac{1}{2}\Big)}{\Big[\binom{6}{3}\Big/\binom{11}{3}\Big]\Big(\frac{1}{2}\Big) + 1\Big(\frac{1}{2}\Big)} = \frac{4}{37}.$$

7. $$\frac{(0.92)(1/5000)}{(0.92)(1/5000)+(1/500)(4999/5000)}=0.084.$$

8. Let A be the event that two of the three coins are dimes. Let B be the event that the coin selected from urn I is a dime. Then

$$P(B \mid A)=\frac{P(A \mid B)P(B)}{P(A \mid B)P(B)+P(A \mid B^c)P(B^c)}=\frac{\Big(\frac{5}{7}\cdot\frac{3}{4}+\frac{2}{7}\cdot\frac{1}{4}\Big)\frac{4}{7}}{\Big(\frac{5}{7}\cdot\frac{3}{4}+\frac{2}{7}\cdot\frac{1}{4}\Big)\frac{4}{7}+\Big(\frac{5}{7}\cdot\frac{1}{4}\Big)\frac{3}{7}}=\frac{68}{83}.$$

9. $$\frac{(0.15)(0.25)}{(0.15)(0.25)+(0.85)(0.75)}=0.056.$$

10. Let R be the event that the upper side of the card selected is red. Let BB be the event that the card with both sides black is selected. Define RR and RB similarly. By Bayes' Formula,

$$\begin{aligned}P(RB \mid R)&=\frac{P(R \mid RB)P(RB)}{P(R \mid RB)P(RB)+P(R \mid RR)P(RR)+P(R \mid BB)P(BB)}\\&=\frac{(1/2)(1/3)}{(1/2)(1/3)+1(1/3)+0(1/3)}=\frac{1}{3}.\end{aligned}$$

11. $$\frac{1\Big(\frac{1}{6}\Big)}{\displaystyle\sum_{i=0}^{5}\Big[\binom{1000-i}{100}\Big/\binom{1000}{100}\Big]\Big(\frac{1}{6}\Big)}=0.21.$$

12. Let A be the event that the wallet originally contained a \$2 bill. Let B be the event that the bill removed is a \$2 bill. The desired probability is given by

$$\begin{aligned}P(A \mid B)&=\frac{P\big(B \mid A\big)P(A)}{P\big(B \mid A\big)P(A)+P\big(B \mid A^c\big)P(A^c)}\\&=\frac{1\times\frac{1}{2}}{1\times\frac{1}{2}+\frac{1}{2}\times\frac{1}{2}}=\frac{2}{3}.\end{aligned}$$

13. By Bayes' formula, the probability that the horse that comes out is from stable I equals

$$\frac{(20/33)(1/2)}{(20/33)(1/2)+(25/33)(1/2)}=\frac{4}{9}.$$

The probability that it is from stable II is $5/9$; hence the desired probability is

$$\frac{20}{33}\cdot\frac{4}{9}+\frac{25}{33}\cdot\frac{5}{9}=\frac{205}{297}=0.69.$$

14.
$$\frac{\dfrac{2}{4}\cdot\dfrac{\binom{5}{2}\binom{3}{2}}{\binom{8}{4}}}{0\cdot\dfrac{\binom{5}{4}}{\binom{8}{4}}+\dfrac{1}{4}\cdot\dfrac{\binom{5}{3}\binom{3}{1}}{\binom{8}{4}}+\dfrac{2}{4}\cdot\dfrac{\binom{5}{2}\binom{3}{2}}{\binom{8}{4}}+\dfrac{3}{4}\cdot\dfrac{\binom{5}{1}\binom{3}{3}}{\binom{8}{4}}}=0.571.$$

15. Let I be the event that the person is ill with the disease, N be the event that the result of the test on the person is negative, and R denote the event that the person has the rash. We are interested in $P(I \mid R)$:

$$P(I \mid R) = P(IN \mid R) + P(IN^c \mid R) = 0 + P(IN^c \mid R).$$

Since $\{IN, IN^c, I^cN, I^cN^c\}$ is a partition of the sample space, by Bayes' Formula,

$$\begin{aligned}
P(I \mid R) &= P(IN^c \mid R) \\
&= \frac{P(R \mid IN^c)P(IN^c)}{P(R \mid IN)P(IN) + P(R \mid IN^c)P(IN^c) + P(R \mid I^cN)P(I^cN) + P(R \mid I^cN^c)P(I^cN^c)} \\
&= \frac{(0.2)(0.30 \times 0.90)}{0(0.30 \times 0.10) + (0.2)(0.30 \times 0.90) + 0(0.70 \times 0.75) + (0.2)(0.70 \times 0.25)} = 0.61.
\end{aligned}$$

3.5 INDEPENDENCE

1. No, because by independence, regardless of the number of heads that have previously occurred, the probability of tails remains to be $1/2$ on each flip.

2. A and B are mutually exclusive; therefore, they are dependent. If A occurs, then the probability that B occurs is 0 and vice versa.

3. Neither. Since the probability that a fighter plane returns from a mission without mishap is 49/50 *independent* of other missions, the probability that a pilot who flew 49 consecutive missions without mishap making another successful flight is still 49/50=0.98; neither higher nor lower than the probability of success in any other mission.

4. $P(AB) = 1/12 = (1/2)(1/6)$; so A and B are independent.

5. $(3/8)^3(5/8)^5 = 0.00503.$

6. $(3/4)^2 = 0.5625.$

7. **(a)** $(0.725)^2 = 0.526$; **(b)** $(1 - 0.725)^2 = 0.076$.

8. Suppose that for an event A, $P(A) = 3/4$. Then the probability that A occurs in two consecutive independent experiments is $9/16$. So the correct odds are 9 to 7, not 9 to 1. In later computations, Cardano, himself, had realized that the correct answer is 9 to 7 and not 9 to 1.

9. We have that

$$\begin{aligned}
&P(A \text{ beats } B) = P(A \text{ rolls } 4) = \frac{4}{6},\\
&P(B \text{ beats } A) = 1 - P(A \text{ beats } B) = 1 - \frac{4}{6} = \frac{2}{6},\\
&P(B \text{ beats } C) = P(C \text{ rolls } 2) = \frac{4}{6},\\
&P(C \text{ beats } B) = 1 - P(B \text{ beats } C) = 1 - \frac{4}{6} = \frac{2}{6},\\
&P(C \text{ beats } D) = P(C \text{ rolls } 6) + P(C \text{ rolls } 2 \text{ and } D \text{ rolls } 1) = \frac{2}{6} + \frac{4}{6} \times \frac{3}{6} = \frac{4}{6},\\
&P(D \text{ beats } C) = 1 - P(C \text{ beats } D) = 1 - \frac{4}{6} = \frac{2}{6},\\
&P(D \text{ beats } A) = P(D \text{ rolls } 5) + P(D \text{ rolls } 1 \text{ and } A \text{ rolls } 0) = \frac{3}{6} + \frac{3}{6} \times \frac{2}{6} = \frac{4}{6}.
\end{aligned}$$

10. For $1 \leq i \leq 4$, let A_i be the event of obtaining 6 on the ith toss. Chevalier de Méré had implicitly thought that A_i's are mutually exclusive and so

$$P\big(A_1 \cup A_2 \cup A_3 \cup A_4\big) = \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = 4 \times \frac{1}{6}.$$

Clearly A_i's are not mutually exclusive. The correct answers are $1 - (5/6)^4 = 0.5177$ and $1 - (35/36)^{24} = 0.4914$.

11. $(1 - 0.0001)^{64} = 0.9936$.

12. In the experiment of tossing a coin, let A be the event of obtaining heads and B be the event of obtaining tails.

13. **(a)** $P(A \cup B) \geq P(A) = 1$, so $P(A \cup B) = 1$. Now

$$1 = P(A \cup B) = P(A) + P(B) - P(AB) = 1 + P(B) - P(AB)$$

gives $P(B) = P(AB)$.

(b) If $P(A) = 0$, then $P(AB) = 0$; so $P(AB) = P(A)P(B)$ is valid. If $P(A) = 1$, by part (a), $P(AB) = P(B) = P(A)P(B)$.

14. $P(AA) = P(A)P(A)$ implies that $P(A) = \big[P(A)\big]^2$. This gives $P(A) = 0$ or $P(A) = 1$.

15. $P(AB) = P(A)P(B)$ implies that $P(A) = P(A)P(B)$. This gives $P(A)\big[1 - P(B)\big] = 0$; so $P(A) = 0$ or $P(B) = 1$.

16. $1 - (0.45)^6 = 0.9917$.

17. $1 - (0.3)(0.2)(0.1) = 0.994$.

18. There are

$$(100 \times 10^9) \times (300 \times 10^9) - 1 = 30 \times 10^{21} - 1$$

other stars in the universe. Provided that Aczel's estimate is correct, the probability of no life in orbit around any one given star in the known universe is

$$0.99999999999995$$

independently of other stars. Therefore, the probability of no life in orbit around any other star is

$$(0.99999999999995)^{30,000,000,000,000,000,000,000-1}.$$

Using Aczel's words, "this number is indistinguishable from 0 at any level of decimal accuracy reported by the computer." Hence the probability that there is life in orbit around at least one other star is 1 for all practical purposes. If there were only a billion galaxies each having 10 billion stars, still the probability of life would have been indistinguishable from 1.0 at any level of accuracy reported by the computer. In fact, if we divide the stars into mutually exclusive groups with each group containing billions of stars, then the argument above and Exercise 8 of Section 1.7 imply that the probability of life in orbit around many other stars is a number practically indistinguishable from 1.

19. $1 - (0.94)^{15} - 15(0.94)^{14}(0.06) = 0.226$.

20. A and B are independent if and only if $P(AB) = P(A)P(B)$, or, equivalently, if and only if

$$\frac{m}{M+W} = \frac{M}{M+W} \cdot \frac{m+w}{M+W}.$$

This implies that $m/M = w/W$. Therefore, A and B are independent if and only if the fraction of the men who smoke is equal to the fraction of the women who smoke.

21. **(a)** By Theorem 1.6,

$$\begin{aligned} P\big(A(B \cup C)\big) &= P(AB \cup AC) = P(AB) + P(AC) - P(ABC) \\ &= P(A)P(B) + P(A)P(C) - P(A)P(B)P(C) \\ &= P(A)\big[P(B) + P(C) - P(B)P(C)\big] = P(A)P(B \cup C). \end{aligned}$$

(b) $P\big((A - B)C\big) = P(AB^cC) = P(A)P(B^c)P(C) = P(AB^c)P(C) = P(A - B)P(C)$.

22. $1 - (5/6)^6 = 0.6651$.

23. **(a)** $1 - \big[(n-1)/n\big]^n$. **(b)** As $n \to \infty$, this approaches $1 - (1/e) = 0.6321$.

24. $\dfrac{1 - (0.85)^{10} - 10(0.85)^9(0.15)}{1 - (0.85)^{10}} = 0.567.$

25. No. In the experiment of choosing a random number from (0, 1), let A, B, and C denote the events that the point lies in (0, 1/2), (1/4, 3/4), and (1/2, 1), respectively.

26. Denote a family with two girls and one boy by ggb, with similar representations for other cases. The sample space is $S = \{ggg, bbb, ggb, gbb\}$. we have

$$P\big(\{ggg\}\big) = P\big(\{bbb\}\big) = 1/8, \quad P\big(\{ggb\}\big) = P\big(\{gbb\}\big) = 3/8.$$

Clearly, $P(A) = 6/8 = 3/4$, $P(B) = 4/8 = 1/2$, and $P(AB) = 3/8$. Since $P(AB) = P(A)P(B)$, the events A and B are independent. Using the same method, we can show that for families with two children and for families with four children, A and B are *not* independent.

27. If p is the probability of its occurrence in one trial, $1 - (1-p)^4 = 0.59$. This implies that $p = 0.2$.

28. **(a)** $1 - (1-p_1)(1-p_2)\cdots(1-p_n)$. **(b)** $(1-p_1)(1-p_2)\cdots(1-p_n)$.

29. Let E_i be the event that the switch located at i is closed. The desired probability is

$$P(E_1E_2E_4E_6 \cup E_1E_3E_5E_6) = P(E_1E_2E_4E_6) + P(E_1E_3E_5E_6) - P(E_1E_2E_3E_4E_5E_6) = 2p^4 - p^6.$$

30. $\dbinom{5}{3}\Big(\dfrac{2}{3}\Big)^3\Big(\dfrac{1}{3}\Big)^2 = 0.329.$

31. For $n = 3$, the probabilities of the given events, respectively, are

$$\binom{3}{2}\Big(\frac{1}{2}\Big)^2\Big(\frac{1}{2}\Big) + \Big(\frac{1}{2}\Big)^3 = \frac{1}{2},$$

and

$$\binom{3}{1}\Big(\frac{1}{2}\Big)\Big(\frac{1}{2}\Big)^2 + \binom{3}{2}\Big(\frac{1}{2}\Big)^2\Big(\frac{1}{2}\Big) = \frac{3}{4}.$$

The probability of their joint occurrence is

$$\binom{3}{2}\Big(\frac{1}{2}\Big)^2\Big(\frac{1}{2}\Big) = \frac{3}{8} = \frac{1}{2}\cdot\frac{3}{4}.$$

So the given events are independent. For $n = 4$, similar calculations show that the given events are *not* independent.

32. **(a)** $1-(1/2)^n$. **(b)** $\binom{n}{k}\left(\frac{1}{2}\right)^n$.

(c) Let A_n be the event of getting n heads in the first n flips. We have

$$A_1 \supseteq A_2 \supseteq A_3 \supseteq \cdots \supseteq A_n \supseteq A_{n+1} \supseteq \cdots .$$

The event of getting heads in all of the flips indefinitely is $\bigcap_{n=1}^{\infty} A_n$. By the continuity property of probability function (Theorem 1.8), its probability is

$$P\Big(\bigcap_{n=1}^{\infty} A_n\Big) = \lim_{n\to\infty} P(A_n) = \lim_{n\to\infty}\left(\frac{1}{2}\right)^n = 0.$$

33. Let A_i be the event that the sixth sum obtained is i, $i = 2, 3, \ldots, 12$. Let B be the event that the sixth sum obtained is not a repetition. By the law of total probability,

$$P(B) = \sum_{i=2}^{12} P(B \mid A_i)P(A_i).$$

Note that in this sum, the terms for $i = 2$ and $i = 12$ are equal. This is true also for the terms for $i = 3$ and 11, for the terms for $i = 4$ and 10, for the terms for $i = 5$ and 9, and for the terms for $i = 6$ and 8. So

$$\begin{aligned} P(B) &= 2\Big[\sum_{i=2}^{6} P(B \mid A_i)P(A_i)\Big] + P(B \mid A_7)P(A_7) \\ &= 2\Big[\Big(\frac{35}{36}\Big)^5\Big(\frac{1}{36}\Big) + \Big(\frac{34}{36}\Big)^5\Big(\frac{2}{36}\Big) + \Big(\frac{33}{36}\Big)^5\Big(\frac{3}{36}\Big) + \Big(\frac{32}{36}\Big)^5\Big(\frac{4}{36}\Big) \\ &\quad + \Big(\frac{31}{36}\Big)^5\Big(\frac{5}{36}\Big)\Big] + \Big(\frac{30}{36}\Big)^5\Big(\frac{6}{36}\Big) = 0.5614. \end{aligned}$$

34. **(a)** Let E be the event that Dr. May's suitcase does not reach his destination with him. We have

$$P(E) = (0.04) + (0.96)(0.05) + (0.96)(0.95)(0.05) + (0.96)(0.95)(0.95)(0.04) = 0.168,$$

or simply, $P(E) = 1 - (0.96)(0.95)(0.96) = 0.168$.

(b) Let D be the event that the suitcase is lost in Da Vinci airport in Rome. Then, by Bayes' formula,

$$P(D \mid E) = \frac{P(D)}{P(E)} = \frac{(0.96)(0.05)}{0.168} = 0.286.$$

35. Let E be the event of obtaining heads on the coin before an ace from the cards. Let H, T, A, and N denote the events of heads, tails, ace, and not ace in the first experiment, respectively. We use two different techniques to solve this problem.

***Technique* 1:** By the law of total probability,

$$P(E) = P(E \mid H)P(H) + P(E \mid T)P(T) = 1 \cdot \frac{1}{2} + P(E \mid T) \cdot \frac{1}{2},$$

where

$$P(E \mid T) = P(E \mid TA)P(A \mid T) + P(E \mid TN)P(N \mid T) = 0 \cdot \frac{1}{13} + P(E) \cdot \frac{12}{13}.$$

Thus

$$P(E) = \frac{1}{2} + \Big[P(E)\frac{12}{13}\Big]\frac{1}{2},$$

which gives $P(E) = 13/14$.

***Technique* 2:** We have that

$$P(E) = P(E \mid HA)P(HA) + P(E \mid TA)P(TA) + P(E \mid HN)P(HN) + P(E \mid TN)P(TN).$$

Thus

$$P(E) = 1 \times \frac{1}{2} \times \frac{1}{13} + 0 \times \frac{1}{2} \times \frac{1}{13} + 1 \times \frac{1}{2} \times \frac{12}{13} + P(E) \times \frac{1}{2} \times \frac{12}{13}.$$

This gives $P(E) = 13/14$.

36. Let $P(A) = p$ and $P(B) = q$. Let A_n be the event that none of A and B occurs in the first $n-1$ trials and the outcome of the nth experiment is A. The desired probability is

$$P\Big(\bigcup_{n=1}^{\infty} A_n\Big) = \sum_{n=1}^{\infty} P(A_n) = \sum_{n=1}^{\infty}(1-p-q)^{n-1}p = \frac{p}{1-(1-p-q)} = \frac{p}{p+q}.$$

37. The probability of sum 5 is 1/9 and the probability of sum 7 is 1/6. Therefore, by the result of Exercise 36, the desired probability is $\dfrac{1/9}{1/6 + 1/9} = 2/5$.

38. Let A be the event that one of them is red and the other one is blue. Let RB represent the event that the ball drawn from urn I is red and the ball drawn form urn II is blue, with similar representations for RR, BB, and BR. We have that

$$P(A) = P(A \mid RB)P(RB) + P(A \mid RR)P(RR) + P(A \mid BB)P(BB) + P(A \mid BR)P(BR)$$

$$= \frac{\binom{9}{1}\binom{5}{1}}{\binom{14}{2}}\Big(\frac{9}{10} \cdot \frac{5}{6}\Big) + \frac{\binom{8}{1}\binom{6}{1}}{\binom{14}{2}}\Big(\frac{9}{10} \cdot \frac{1}{6}\Big) + \frac{\binom{10}{1}\binom{4}{1}}{\binom{14}{2}}\Big(\frac{1}{10} \cdot \frac{5}{6}\Big) + \frac{\binom{9}{1}\binom{5}{1}}{\binom{14}{2}}\Big(\frac{1}{10} \cdot \frac{1}{6}\Big)$$

$$= 0.495.$$

39. For convenience, let $p_0 = 0$; the desired probability is

$$1 - \prod_{i=1}^{n}(1 - p_i) - \sum_{i=1}^{n}(1 - p_1)(1 - p_2)\cdots(1 - p_{i-1})p_i(1 - p_{i+1})\cdots(1 - p_n).$$

40. Let p be the probability that a randomly selected person was born on one of the first 365 days; then $365p + (p/4) = 1$ implies that $p = 4/1461$. Let E be the event that exactly four people of this group have the same birthday and that all the others have different birthdays. E is the union of the following three mutually exclusive events:

F: Exactly four people of this group have the same birthday, all the others have different birthdays, and none of the birthdays is on the 366th day.

G: Exactly four people of this group have the same birthday, all the others have different birthdays, and exactly one has his/her birthday on the 366th day.

H: Exactly four people of this group have their birthday on the 366th day and all the others have different birthdays.

We have that

$$\begin{aligned} P(E) &= P(F) + P(G) + P(H) \\ &= \binom{365}{1}\binom{30}{4}\Big(\frac{4}{1461}\Big)^4 \cdot \binom{364}{26}26!\Big(\frac{4}{1461}\Big)^{26} \\ &\quad + \binom{30}{1}\Big(\frac{1}{1461}\Big) \cdot \binom{365}{1}\binom{29}{4}\Big(\frac{4}{1461}\Big)^4 \cdot \binom{364}{25}25!\Big(\frac{4}{1461}\Big)^{25} \\ &\quad + \binom{30}{4}\Big(\frac{1}{1461}\Big)^4 \cdot \binom{365}{26}26!\Big(\frac{4}{1461}\Big)^{26} = 0.00020997237. \end{aligned}$$

If we were allowed to ignore the effect of the leap year, the solution would have been as follows.

$$\binom{365}{1}\binom{30}{1}\Big(\frac{1}{365}\Big)^4 \cdot \binom{364}{26}26!\Big(\frac{1}{365}\Big)^{26} = 0.00021029.$$

41. Let E_i be the event that the switch located at i is closed. We want to calculate the probability of $E_2E_4 \cup E_1E_5 \cup E_2E_3E_5 \cup E_1E_3E_4$. Using the rule to calculate the probability of the union of several events (the inclusion-exclusion principle) we get that the answer is $2p^2+2p^3-5p^4+p^5$.

42. Let E be the event that A will answer correctly to his or her first question. Let F and G be the corresponding events for B and C, respectively. Clearly,

$$\begin{aligned} P(ABC) = P(ABC \mid EFG)P(EFG) + P(ABC \mid E^cFG)P(E^cFG) \\ + P(ABC \mid E^cF^c)P(E^cF^c). \end{aligned} \tag{5}$$

Now

$$P(ABC \mid EFG) = P(ABC), \tag{6}$$

and

$$P(ABC \mid E^c F^c) = 1. \tag{7}$$

To calculate $P(ABC \mid E^c FG)$, note that since A has already lost, the game continues between B and C. Let BC be the event that B loses and C wins. Then

$$P(ABC \mid E^c FG) = P(BC). \tag{8}$$

Let F_2 be the event that B answers the second question correctly; then

$$P(BC) = P(BC \mid F_2)P(F_2) + P(BC \mid F_2^C)P(F_2^C). \tag{9}$$

To find $P(BC \mid F_2)$, note that this quantity is the probability that B loses to C given that B did not lose the first play. So, by independence, this is the probability that B loses to C given that C plays first. Now by symmetry, this quantity is the same as C losing to B if B plays first. Thus it is equal to $P(CB)$, and hence (9) gives

$$P(BC) = P(CB) \cdot p + 1 \cdot (1 - p);$$

noting that $P(CB) = 1 - P(BC)$, this gives

$$P(BC) = \frac{1}{1 + p}.$$

Therefore, by (8),

$$P(ABC \mid E^c FG) = \frac{1}{1 + p}.$$

substituting this, (8), and (7) in (5), yields

$$P(ABC) = P(ABC) \cdot p^3 + \frac{1}{1 + p}(1 - p)p^2 + (1 - p)^2.$$

Solving this for $P(ABC)$, we obtain

$$P(ABC) = \frac{1}{(1 + p)(1 + p + p^2)}.$$

Now we find $P(BCA)$ and $P(CAB)$.

$$\begin{aligned} P(BCA) &= P(BCA \mid E)P(E) + P(BCA \mid E^c)P(E^c) \\ &= P(ABC) \cdot p + 0 \cdot (1 - p) = \frac{p}{(1 + p)(1 + p + p^2)}, \end{aligned}$$

$$\begin{aligned} P(CAB) &= P(CAB \mid E)P(E) + P(CAB \mid E^c)P(E^c) \\ &= P(BCA) \cdot p + 0 \cdot (1 - p) = \frac{p^2}{(1 + p)(1 + p + p^2)}. \end{aligned}$$

43. We have that

$$P(H_1) = P(H_1 \mid H)P(H) + P(H_1 \mid H^c)P(H^c) = \frac{1}{2} \cdot \frac{1}{4} + 0 \cdot \frac{3}{4} = \frac{1}{8}.$$

Similarly, $P(H_2) = 1/8$. To calculate $P(H_1^c H_2^c)$, the probability that none of her sons is hemophiliac, we condition on H again.

$$P(H_1^c H_2^c) = P(H_1^c H_2^c \mid H)P(H) + P(H_1^c H_2^c \mid H^c)P(H^c).$$

Clearly, $P(H_1^c H_2^c \mid H^c) = 1$. To find $P(H_1^c H_2^c \mid H)$, we use the fact that H_1 and H_2 are conditionally independent given H.

$$P(H_1^c H_2^c \mid H) = P(H_1^c \mid H)P(H_2^c \mid H) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}.$$

Thus

$$P(H_1^c H_2^c) = \frac{1}{4} \cdot \frac{1}{4} + 1 \cdot \frac{3}{4} = \frac{13}{16}.$$

44. The only quantity not calculated in the hint is $P(U_i \mid R_m)$. By Bayes' Formula,

$$P(U_i \mid R_m) = \frac{P(R_m \mid U_i)P(U_i)}{\sum_{k=0}^{n} P(R_m \mid U_k)P(U_k)} = \frac{\left(\frac{i}{n}\right)^m \left(\frac{1}{n+1}\right)}{\sum_{k=0}^{n} \left(\frac{k}{n}\right)^m \left(\frac{1}{n+1}\right)} = \frac{\left(\frac{i}{n}\right)^m}{\sum_{k=0}^{n} \left(\frac{k}{n}\right)^m}.$$

3.6 APPLICATIONS OF PROBABILITY TO GENETICS

1. Clearly, Kim and Dan both have genotype OO. With a genotype other than AO for John, it is impossible for Dan to have blood type O. Therefore, the probability is 1 that John's genotype is AO.

2. The answer is $\binom{k}{2} + k = \dfrac{k(k+1)}{2}$.

3. The genotype of the parent with wrinkled shape is necessarily rr. The genotype of the other parent is either Rr or RR. But, RR will never produce wrinkled offspring. So it must be Rr. Therefore, the parents are rr and Rr.

4. Let A represent the dominant allele for free earlobes and a represent the recessive allele for attached earlobes. Let B represent the dominant allele for freckles and b represent the recessive allele for no freckles. Since Dan has attached earlobes and no freckles, Kim and John both must be $AaBb$. This implies that Kim and John's next child is AA with probability 1/4, Aa

with probability 1/2, and aa with probability 1/4. Therefore, the next child has free earlobes with probability 3/4. Similarly, the next child is BB with probability 1/4, Bb with probability 1/2, and bb with probability 1/4. Hence he or she will have no freckles with probability 1/4. By independence, the desired probability is $(3/4)(1/4) = 3/16$.

5. If the genes are not linked, 25% of the offspring are expected to be $BbVv$, 25% are expected to be $bbvv$, 25% are expected to be $Bbvv$, and 25% are expected to be $bbVv$. The observed data shows that the genes are linked.

6. Clearly, John's genotype is either Dd or dd. Let E be the event that it is dd. Then E^c is the event that John's genotype is Dd. Let F be the event that Dan is deaf. That is, his genotype is dd. We use Bayes' theorem to calculate the desired probability.

$$\begin{aligned} P(E \mid F) &= \frac{P(F \mid E)P(E)}{P(F \mid E)P(E) + P(F \mid E^c)P(E^c)} \\ &= \frac{1 \cdot (0.01)}{1 \cdot (0.01) + (1/2)(0.99)} = 0.0198. \end{aligned}$$

Therefore, the probability is 0.0198 that John is also deaf.

7. A person who has cystic fibrosis carries two mutant alleles. Applying the Hardy-Weinberg law, we have that $q^2 = 0.0529$, or $q = 0.23$. Therefore, $p = 0.77$. Since $q^2 + 2pq = 1 - p^2 = 0.4071$, the percentage of the people who carry at least one mutant allele of the disease is 40.71%.

8. Dan inherits all of his sex-linked genes from his mother. Therefore, John being normal has no effect on whether or not Dan has hemophilia or not. Let E be the event that Kim is Hh. Then E^c is the event that Kim is HH. Let F be the event that Dan has hemophilia. By the law of total probability,

$$\begin{aligned} P(F) &= P(F \mid E)P(E) + P(F \mid E^c)P(E^c) \\ &= (1/2)\big[2(0.98)(0.02)\big] + 0 \cdot (0.98)(0.98) = 0.0196. \end{aligned}$$

9. Dan has inherited all of his sex-linked genes from his mother. Let E_1 be the event that Kim is CC, E_2 be the event that she is Cc, and E_3 be the event that she is cc. Let F be the event that Dan is color-blind. By Bayes' formula, the desired probability is

$$\begin{aligned} P(E_3 \mid F) &= \frac{P(F \mid E_3)P(E_3)}{P(F \mid E_1)P(E_1) + P(F \mid E_2)P(E_2) + P(F \mid E_3)P(E_3)} \\ &= \frac{1 \cdot (0.17)(0.17)}{0 \cdot (0.83)(0.83) + (1/2)\big[2(0.83)(0.17)\big] + 1 \cdot (0.17)(0.17)} = 0.17. \end{aligned}$$

10. Since Ann is hh and John is hemophiliac, Kim is either Hh or hh. Let E be the event that she is Hh. Then E^c is the event that she is hh. Let F be the event that Ann has hemophilia. By

Bayes' formula, the desired probability is

$$P(E \mid F) = \frac{P(F \mid E)P(E)}{P(F \mid E)P(E) + P(F \mid E^c)P(E^c)}$$
$$= \frac{(1/2)\big[2(0.98)(0.02)\big]}{(1/2)\big[2(0.98)(0.02)\big] + 1 \cdot (0.02)(0.02)} = 0.98.$$

11. Clearly, both parents of Mr. J must be Cc. Since Mr. J has survived to adulthood, he is not cc. Therefore, he is either CC or Cc. We have

$$P(\text{he is } CC \mid \text{he is } CC \text{ or } Cc) = \frac{P(\text{he is } CC)}{P(\text{he is } CC \text{ or } Cc)} = \frac{1/4}{3/4} = \frac{1}{3}.$$
$$P(\text{he is } Cc \mid \text{he is } CC \text{ or } Cc) = \frac{2}{3}.$$

Mr. J's wife is either CC with probability $1 - p$ or Cc with probability p. Let E be the event that Mr. J is Cc, F be the event that his wife is Cc, and H be the event that their next child is cc. The desired probability is

$$P(H) = P(HEF) = P(H \mid EF)P(EF)$$
$$= P(H \mid EF)P(E)P(F) = \frac{1}{4} \cdot \frac{2}{3} \cdot p = \frac{p}{6}.$$

12. Let E_1 be the event that both parents are of genotype AA, let E_2 be the event that one parent is of genotype Aa and the other of genotype AA, and let E_3 be the event that both parents are of genotype Aa. Let F be the event that the man is of genotype AA. By Bayes' formula,

$$P(E_1 \mid F) = \frac{P(F \mid E_1)P(E_1)}{P(F \mid E_1)P(E_1) + P(F \mid E_2)P(E_2) + P(F \mid E_3)P(E_3)}$$
$$= \frac{1 \cdot p^4}{1 \cdot p^4 + (1/2) \cdot 4p^3q + (1/4) \cdot 4p^2q^2} = \frac{p^2}{(p+q)^2} = p^2.$$

Similarly, $P(E_2 \mid F) = 2pq$ and $P(E_3 \mid F) = q^2$. Let B be the event that the brother is AA. We have

$$P(B \mid F) = P(B \mid FE_1)P(E_1 \mid F) + P(B \mid FE_2)P(E_2 \mid F) + P(B \mid FE_3)P(E_3 \mid F)$$
$$= P(B \mid E_1)P(E_1 \mid F) + P(B \mid E_2)P(E_2 \mid F) + P(B \mid E_3)P(E_3 \mid F)$$
$$= 1 \cdot p^2 + \frac{1}{2} \cdot 2pq + \frac{1}{4} \cdot q^2 = \frac{(2p+q)^2}{4} = \frac{(1+p)^2}{4}.$$

REVIEW PROBLEMS FOR CHAPTER 3

1. $\dfrac{12}{30} \cdot \dfrac{13}{30} + \dfrac{13}{30} \cdot \dfrac{12}{30} = \dfrac{26}{75} = 0.347.$

2. $1 - (0.97)^6 = 0.167.$

3. $(0.48)(0.30) + (0.67)(0.53) + (0.89)(0.17) = 0.65.$

4. $(0.5)(0.05) + (0.7)(0.02) + (0.8)(0.035) = 0.067.$

5. **(a)** $(0.95)(0.97)(0.85) = 0.783$; **(b)** $1 - (0.05)(0.03)(0.05) = 0.999775$;
(c) $1 - (0.95)(0.97)(0.85) = 0.217$; **(d)** $(0.05)(0.03)(0.15) = 0.000225.$

6. $103/132 = 0.780.$

7. $\dfrac{(0.08)(0.20)}{(0.2)(0.3) + (0.25)(0.5) + (0.08)(0.20)} = 0.0796.$

8. $1 - \left[\binom{26}{6}\Big/\binom{39}{6}\right] = 0.929.$

9. $1/6.$

10. $\dfrac{1 - \left(\dfrac{5}{6}\right)^{10} - 10\left(\dfrac{5}{6}\right)^{9}\left(\dfrac{1}{6}\right)}{1 - \left(\dfrac{5}{6}\right)^{10}} = 0.615.$

11. $\dfrac{\dfrac{2}{7} \cdot \dfrac{4}{7}}{\dfrac{2}{7} \cdot \dfrac{4}{7} + \dfrac{5}{7} \cdot \dfrac{3}{7}} = \dfrac{8}{23} = 0.35.$

12. Let A be the event of "head on the coin." Let B be the event of "tail on the coin and 1 or 2 on the die." Then A and B are mutually exclusive, and by the result of Exercise 36 of Section 3.5, the answer is $\dfrac{1/2}{(1/2) + (1/6)} = \dfrac{3}{4}.$

13. The probability that the number of 1's minus the number of 2's will be 3 is

$$P(\text{four 1's and one 2}) + P(\text{three 1's and no 2's})$$
$$= \binom{6}{4}\left(\frac{1}{6}\right)^4\binom{2}{1}\left(\frac{1}{6}\right)\left(\frac{4}{6}\right) + \binom{6}{3}\left(\frac{1}{6}\right)^3\left(\frac{4}{6}\right)^3 = 0.03.$$

14. The probability that the first urn was selected in the first place is

$$\frac{\frac{20}{45}\cdot\frac{1}{2}}{\frac{20}{45}\cdot\frac{1}{2}+\frac{10}{25}\cdot\frac{1}{2}}=\frac{10}{19}.$$

The desired probability is

$$\frac{20}{45}\cdot\frac{10}{19}+\frac{10}{25}\cdot\frac{9}{19}\approx 0.42.$$

15. Let B be the event that the ball removed from the third urn is blue. Let BR be the event that the ball drawn from the first urn is blue and the ball drawn from the second urn is red. Define BB, RB, and RR similarly. We have that

$$\begin{aligned}P(B)&=P(B\mid BB)P(BB)+P(B\mid RB)P(RB)+P(B\mid RR)P(RR)+P(B\mid BR)P(BR)\\&=\frac{4}{14}\cdot\frac{1}{10}\frac{5}{6}+\frac{5}{14}\cdot\frac{9}{10}\frac{5}{6}+\frac{6}{14}\cdot\frac{9}{10}\frac{1}{6}+\frac{5}{14}\cdot\frac{1}{10}\frac{1}{6}=\frac{38}{105}=0.36.\end{aligned}$$

16. Let E be the event that Lorna guesses correctly. Let R be the event that a red hat is placed on Lorna's head, and B be the event that a blue hat is placed on her head. By the law of total probability,

$$\begin{aligned}P(E)&=P(E\mid R)P(R)+P(E\mid B)P(B)\\&=\alpha\cdot\frac{1}{2}+(1-\alpha)\cdot\frac{1}{2}=\frac{1}{2}\end{aligned}$$

This shows that Lorna's chances are 50% to guess correctly no matter what the value of α is. This should be intuitively clear.

17. Let F be the event that the child is found; E be the event that he is lost in the east wing, and W be the event that he is lost in the west wing. We have

$$\begin{aligned}P(F)&=P(F\mid E)P(E)+P(F\mid W)P(W)\\&=\big[1-(0.6)^3\big](0.75)+\big[1-(0.6)^2\big](0.25)=0.748.\end{aligned}$$

18. The answer is that it is the same either way. Let W be the event that they win one of the nights to themselves. Let F be the event that they win Friday night to themselves. Then

$$P(W)=P(W\mid F)P(F)+P(W\mid F^c)P(F^c)=1\cdot\frac{1}{3}+\frac{1}{2}\cdot\frac{2}{3}=\frac{2}{3}.$$

19. Let A be the event that Kevin is prepared. We have that

$$\begin{aligned}P(R\mid B^cS^c)&=\frac{P(RB^cS^c)}{P(B^cS^c)}=\frac{P(RB^cS^c\mid A)P(A)+P(RB^cS^c\mid A^c)P(A^c)}{P(B^cS^c\mid A)P(A)+P(B^cS^c\mid A^c)P(A^c)}\\&=\frac{(0.85)(0.15)^2(0.85)+(0.20)(0.80)^2(0.15)}{(0.15)^2(0.85)+(0.80)^2(0.15)}=0.308.\end{aligned}$$

Note that

$$P(R) = P(R \mid A)P(A) + P(R \mid A^c)P(A^c) = (0.85)(0.85) + (0.20)(0.15) = 0.7525.$$

Since $P(R \mid B^cS^c) \neq P(R)$, the events R, B, and S are not independent. However, it must be clear that R, B, and S are conditionally independent given that Kevin is prepared and they are conditionally independent given that Kevin is unprepared. To explain this, suppose that we are given that, for example, Smith and Brown both failed a student. This information will increase the probability that the student was unprepared. Therefore, it increases the probability that Rose will also fails the student. However, if we know that the student was unprepared, the knowledge that Smith and Brown failed the student does not affect the probability that Rose will also fail the student.

20. **(a)** Let A be the event that Adam has at least one king; B be the event that he has at least two kings. We have

$$P(B \mid A) = \frac{P(AB)}{P(A)} = \frac{P(\text{Adam has at least two kings})}{P(\text{Adam has at least one king})}$$

$$= \frac{1 - \dfrac{\binom{48}{13}}{\binom{52}{13}} - \dfrac{\binom{48}{12}\binom{4}{1}}{\binom{52}{13}}}{1 - \dfrac{\binom{48}{13}}{\binom{52}{13}}} = 0.3696.$$

(b) Let A be the event that Adam has the king of diamonds. Let B be the event that he has the king of diamonds and at least one other king. Then

$$P(B \mid A) = \frac{P(BA)}{P(A)} = \frac{\dfrac{\binom{48}{11}\binom{3}{1} + \binom{48}{10}\binom{3}{2} + \binom{48}{9}\binom{3}{3}}{\binom{52}{13}}}{\dfrac{\binom{51}{12}}{\binom{52}{13}}} = 0.5612.$$

Knowing that Adam has the king of diamonds reduces the sample space to a size considerably smaller than the case in which we are given that he has a king. This is why the answer to part (b) is larger than the answer to part (a). If one is not convinced of this, he or she should solve the problem in a simpler case. For example, a case in which there are four cards, say, king of diamonds, king of hearts, jack of clubs, and eight of spade. If two cards are drawn, the reduced sample space in the case Adam announces that he has a king is

$$\{K_d K_h,\ K_d J_c,\ K_d 8_s,\ K_h J_c,\ K_h 8_s\},$$

while the reduced sample space in the case Adam announces that he has the king of diamonds is

$$\{K_d K_h,\ K_d J_c,\ K_d 8_s\}.$$

In the first case, the probability of more kings is $1/5$; in the second case the probability of more kings is $1/3$.

Chapter 4

Distribution Functions and Discrete Random Variables

4.2 DISTRIBUTION FUNCTIONS

1. The set of possible values of X is $\{0, 1, 2, 3, 4, 5\}$. The probabilities associated with these values are

x	0	1	2	3	4	5
$P(X = x)$	6/36	10/36	8/36	6/36	4/36	2/36

2. The set of possible values of X is $\{-6, -2, -1, 2, 3, 4\}$. The probabilities associated with these values are

$$P(X = -6) = P(X = 2) = P(X = 4) = \frac{\binom{5}{2}}{\binom{15}{2}} = 0.095,$$

$$P(X = -2) = P(X = -1) = P(X = 3) = \frac{\binom{5}{1}\binom{5}{1}}{\binom{15}{2}} = 0.238.$$

3. The set of possible values of X is $\{0, 1, 2 \dots, N\}$. Assuming that people have the disease independent of each other,

$$P(X = i) = \begin{cases} (1 - p)^{i-1} p & 1 \le i \le N \\ (1 - p)^N & i = 0. \end{cases}$$

4. Let X be the length of the side of a randomly chosen plastic die manufactured by the factory, then

$$P(X^3 > 1.424) = P(X > 1.125) = \frac{1.25 - 1.125}{1.25 - 1} = \frac{1}{2}.$$

5. $P(X < 1) = F(1-) = 1/2.$

$P(X = 1) = F(1) - F(1-) = 1/6.$

$P(1 \leq X < 2) = F(2-) - F(1-) = 1/4.$

$P(X > 1/2) = 1 - F(1/2) = 1 - 1/2 = 1/2.$

$P(X = 3/2) = 0.$

$P(1 < X \leq 6) = F(6) - F(1) = 1 - 2/3 = 1/3.$

6. Let F be the distribution function of X. Then

$$F(t) = \begin{cases} 0 & t < 0 \\ 1/8 & 0 \leq t < 1 \\ 1/2 & 1 \leq t < 2 \\ 7/8 & 2 \leq t < 3 \\ 1 & t \geq 3. \end{cases}$$

7. Note that X is neither continuous nor discrete. The answers are

(a) $F(6-) = 1$ implies that $k(-36 + 72 - 3) = 1$; so $k = 1/33$.

(b) $F(4) - F(2) = 29/33 - 4/33 = 25/33.$

(c) $1 - F(3) = 1 - (24/33) = 9/33.$

(d) $$P(X \leq 4 \mid X \geq 3) = \frac{F(4) - F(3-)}{1 - F(3-)} = \frac{\dfrac{29}{33} - \dfrac{9}{33}}{1 - \dfrac{9}{33}} = \frac{5}{6}.$$

8. $F(Q_{0.5}) = 1/2$ implies that $1 + e^{-x} = 2$. The only solution of this question is $x = 0$. So $x = 0$ is the median of F. Similarly, $F(Q_{0.25}) = 1/4$ implies that $1 + e^{-x} = 4$, the solution of which is $x = -\ln 3$. $F(Q_{0.75}) = 3/4$ implies that $1 + e^{-x} = 4/3$, the solution of which is $x = \ln 3$. So $-\ln 3$ and $\ln 3$ are the first and the third quartiles of F, respectively. Therefore, 50% of the years the rate at which the price of oil per gallon changes is negative or zero, 25% of the years the rate is $-\ln 3 \approx -1.0986$ or less, and 75% of the years the rate is $\ln 3 \approx 1.0986$ or less.

9. **(a)**

$$\begin{aligned} P(|X| \leq t) &= P(-t \leq X \leq t) = P(X \leq t) - P(X < -t) \\ &= F(t) - \big[1 - P(X \geq -t)\big] = F(t) - \big[1 - P(x \leq t)\big] = 2F(t) - 1. \end{aligned}$$

(b) Using part (a), we have

$$P(|X| > t) = 1 - P(|X| \leq t) = 1 - \big[2F(t) - 1\big] = 2\big[1 - F(t)\big].$$

(c)

$$\begin{aligned}P(X=t) &= 1+P(X=t)-1 = P(X \le t)+P(X>t)+P(X=t)-1\\ &= P(X\le t)+P(X\ge t)-1 = P(X\le t)+P(X\le -t)-1\\ &= F(t)+F(-t)-1.\end{aligned}$$

10. F is a distribution function because $F(-\infty)=0$, $F(\infty)=1$, F is right continuous, and $F'(t)=\dfrac{1}{\pi}e^{-t}>0$ implies that F is nondecreasing.

11. F is a distribution function because $F(-\infty)=0$, $F(\infty)=1$, F is right continuous, and $F'(t)=\dfrac{1}{(1+t)^2}>0$ implies that it is nondecreasing.

12. Clearly, F is right continuous. On $t<0$ and on $t\ge 0$, it is increasing, $\lim_{t\to\infty}F(t)=1$, and $\lim_{t\to-\infty}F(t)=0$. It looks like F satisfies all of the conditions necessary to make it a distribution function. However, $F(0-)=1/2>F(0+)=1/4$ shows that F is not nondecreasing. Therefore, F is not a probability distribution function.

13. Let the departure time of the last flight before the passenger arrives be 0. Then Y, the arrival time of the passenger is a random number from $(0, 45)$. The waiting time is $X=45-Y$. We have that for $0\le t\le 45$,

$$P(X\le t)=P(45-Y\le t)=P(Y\ge 45-t)=\frac{45-(45-t)}{45}=\frac{t}{45}.$$

So F, the distribution function of X is

$$F(t)=\begin{cases}0 & t<0\\ t/45 & 0\le t<45\\ 1 & t\ge 45.\end{cases}$$

14. Let X be the first two-digit number selected from the set $\{00, 01, 02, \dots, 99\}$ which is between 4 and 18. Since for $i=4, 5, \dots, 18$,

$$P(X=i\mid 4\le X\le 18)=\frac{P(X=i)}{P(4\le X\le 18)}=\frac{1/100}{15/100}=\frac{1}{15},$$

we have that X is chosen randomly from the set $\{4, 5, \dots, 18\}$.

15. Let X be the minimum of the three numbers,

$$P(X<5)=1-P(X\ge 5)=1-\frac{\binom{36}{3}}{\binom{40}{3}}=0.277.$$

16.

$$P(X^2 - 5X + 6 > 0) = P\big((X-2)(X-3) > 0\big) = P(X < 2) + P(X > 3) = \frac{2-0}{3-0} + 0 = \frac{2}{3}.$$

17.

$$F(t) = \begin{cases} 0 & t < 0 \\ \dfrac{t}{1-t} & 0 \le t < 1/2 \\ 1 & t \ge 1/2. \end{cases}$$

18. The distribution function of X is $F(t) = 0$ if $t < 1$; $F(t) = 1 - (89/90)^n$ if $n \le t < n+1$, $n \ge 1$. Since

$$F(26-) = 1 - \Big(\frac{89}{90}\Big)^{25} = 0.244 < 0.25 < 1 - \Big(\frac{89}{90}\Big)^{26} = 0.252 = F(26),$$

26 is the first quartile. Since

$$F(63-) = 1 - \Big(\frac{89}{90}\Big)^{62} = 0.4998 < 0.5 < 1 - \Big(\frac{89}{90}\Big)^{63} = 0.505 = F(63),$$

63 is the median of X. Similarly,

$$F(125-) = 1 - \Big(\frac{89}{90}\Big)^{124} = 0.7498 < 0.75 < 1 - \Big(\frac{89}{90}\Big)^{125} = 0.753 = F(125),$$

implies that 125 is the third quartile of X.

19.

$$G(t) = \begin{cases} F(t) & t < 5 \\ 1 & t \ge 5. \end{cases}$$

4.3 DISCRETE RANDOM VARIABLES

1. F, the distribution functions of X is given by

$$F(x) = \begin{cases} 0 & \text{if } x < 1 \\ 1/15 & \text{if } 1 \le x < 2 \\ 3/15 & \text{if } 2 \le x < 3 \\ 6/15 & \text{if } 3 \le x < 4 \\ 10/15 & \text{if } 4 \le x < 5 \\ 1 & \text{if } x \ge 5. \end{cases}$$

2. p, the probability mass function of X, is given by

x	1	2	3	4	5	6
$p(x)$	11/36	9/36	7/36	5/36	3/36	1/36

F, the probability distribution function of X, is given by

$$F(x) = \begin{cases} 0 & \text{if } x < 1 \\ 11/36 & \text{if } 1 \le x < 2 \\ 20/36 & \text{if } 2 \le x < 3 \\ 27/36 & \text{if } 3 \le x < 4 \\ 32/36 & \text{if } 4 \le x < 5 \\ 35/36 & \text{if } 5 \le x < 6 \\ 1 & \text{if } x \ge 6. \end{cases}$$

3. The possible values of X are $2, 3, \dots, 12$. The sample space of this experiment consists of 36 equally likely outcomes. Hence the probability of any of them is 1/36. Thus

$$p(2) = P(X = 2) = P\big(\{(1,1)\}\big) = 1/36,$$
$$p(3) = P(X = 3) = P\big(\{(1,2), (2,1)\}\big) = 2/36,$$
$$p(4) = P(X = 4) = P\big(\{(1,3), (2,2), (3,1)\}\big) = 3/36.$$

Similarly,

i	5	6	7	8	9	10	11	12
$p(i)$	4/36	5/36	6/36	5/36	4/36	3/36	2/36	1/36

4. Let p be the probability mass function of X. We have

x	−2	2	4	6
$p(x)$	1/2	1/10	13/45	1/9

5. Let p be the probability mass function of X and q be the probability mass function of Y. We have

$$p(i) = \Big(\frac{9}{10}\Big)^{i-1}\Big(\frac{1}{10}\Big), \quad i = 1, 2, \dots .$$

$$q(j) = P(Y = j) = P\Big(X = \frac{j-1}{2}\Big) = \Big(\frac{9}{10}\Big)^{(j-3)/2}\Big(\frac{1}{10}\Big), \quad j = 3, 5, 7, \dots .$$

6. Mode of $p = 1$; mode of $q = 1$.

7. **(a)** $\sum_{k=1}^{5} kx = 1 \Rightarrow k = 1/15.$

(b) $k(-1)^2 + k + 4k + 9k = 1 \Rightarrow k = 1/15.$

(c) $\displaystyle\sum_{x=1}^{\infty} k\Big(\frac{1}{9}\Big)^x = 1 \Rightarrow k = \frac{1}{\sum_{x=1}^{\infty}(1/9)^x} = 1\Big/\Big[\frac{1/9}{1-(1/9)}\Big] = 8.$

(d) $\displaystyle k(1+2+\cdots+n) = 1 \Rightarrow k = \frac{1}{\big[n(n+1)\big]/2} = \frac{2}{n(n+1)}.$

(e) $\displaystyle k(1^2+2^2+\cdots+n^2) = 1 \Rightarrow k = \frac{6}{n(n+1)(2n+1)}.$

8. Let p be the probability mass function of X; then

$$p(i) = P(X=i) = \frac{\binom{18}{i}\binom{28}{12-i}}{\binom{46}{12}} \qquad i = 0, 1, 2, \dots, 12.$$

9. For $x < 0$, $F(x) = 0$. If $x \geq 0$, for some nonnegative integer n, $n \leq x < n+1$, and we have that

$$F(x) = \sum_{i=0}^{n} \frac{3}{4}\Big(\frac{1}{4}\Big)^i = \frac{3}{4}\Big[1 + \Big(\frac{1}{4}\Big) + \Big(\frac{1}{4}\Big)^2 + \cdots + \Big(\frac{1}{4}\Big)^n\Big]$$
$$= \frac{3}{4}\cdot\frac{1-(1/4)^{n+1}}{1-(1/4)} = 1 - \Big(\frac{1}{4}\Big)^{n+1}.$$

Thus

$$F(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1-(1/4)^{n+1} & \text{if } n \leq x < n+1,\ n = 0, 1, 2, \dots. \end{cases}$$

10. Let p be the probability mass function of X and F be its distribution function. We have

$$p(i) = \Big(\frac{5}{6}\Big)^{i-1}\Big(\frac{1}{6}\Big), \quad i = 1, 2, 3, \dots.$$

$F(x) = 0$ for $x < 1$. If $x \geq 1$, for some positive integer n, $n \leq x < n+1$, and we have that

$$F(x) = \sum_{i=1}^{n}\Big(\frac{5}{6}\Big)^{i-1}\Big(\frac{1}{6}\Big) = \frac{1}{6}\Big[1 + \Big(\frac{5}{6}\Big) + \Big(\frac{5}{6}\Big)^2 + \cdots + \Big(\frac{5}{6}\Big)^{n-1}\Big]$$
$$= \frac{1}{6}\cdot\frac{1-(5/6)^n}{1-(5/6)} = 1 - \Big(\frac{5}{6}\Big)^n.$$

Hence

$$F(x) = \begin{cases} 0 & \text{if } x < 1 \\ 1 - \left(\frac{5}{6}\right)^n & \text{if } n \le x < n+1, \quad n = 1, 2, 3, \dots. \end{cases}$$

11. The set of possible values of X is $\{2, 3, 4, \dots\}$. For $n \ge 2$, $X = n$ if and only if either all of the first $n-1$ bits generated are 0 and the nth bit generated is 1, or all of the first $n-1$ bits generated are 1 and the nth bit generated is 0. Therefore, by independence,

$$P(X = n) = \left(\frac{1}{2}\right)^{n-1} \cdot \frac{1}{2} + \left(\frac{1}{2}\right)^{n-1} \cdot \frac{1}{2} = \left(\frac{1}{2}\right)^{n-1}, \quad n \ge 2.$$

12. The event $Z > i$ occurs if and only if Liz has not played with Bob since i Sundays ago, and the earliest she will play with him is next Sunday. Now the probability is i/k that Liz will play with Bob if last time they played was i Sundays ago; hence

$$P(Z > i) = 1 - \frac{i}{k}, \quad i = 1, 2, \dots, k-1.$$

Let p be the probability mass function of Z. Then, using this fact for $1 \le i \le k$, we obtain

$$p(i) = P(Z = i) = P(Z > i-1) - P(Z > i) = \left(1 - \frac{i-1}{k}\right) - \left(1 - \frac{i}{k}\right) = \frac{1}{k}.$$

13. The possible values of X are 0, 1, 2, 3, 4, and 5. For i, $0 \le i \le 5$,

$$P(X = i) = \frac{\binom{5}{i} {}_6P_i \cdot {}_9P_{5-i} \cdot 10!}{15!}.$$

The numerical values of these probabilities are as follows.

i	0	1	2	3	4	5
$P(X = i)$	42/1001	252/1001	420/1001	240/1001	45/1001	2/1001

14. For $i = 0,\ 1, 2$, and 3, we have

$$P(X = i) = \frac{\binom{10}{i}\binom{10-i}{6-2i} 2^{6-2i}}{\binom{20}{6}}.$$

The numerical values of these probabilities are as follows.

i	0	1	2	3
$p(i)$	112/323	168/323	42/323	1/323

15. Clearly,

$$P(X > n) = P\Big(\bigcup_{i=1}^{6} E_i\Big).$$

To calculate $P\big(E_1 \cup E_2 \cup \cdots \cup E_6\big)$, we use the inclusion-exclusion principle. To do so, we must calculate the probabilities of all possible intersections of the events from $E_1, \ldots, E_6$, add the probabilities that are obtained by intersecting an odd number of events, and subtract all the probabilities that are obtained by intersecting an even number of events. Clearly, there are $\binom{6}{1}$ terms of the form $P(E_i)$, $\binom{6}{2}$ terms of the form $P(E_iE_j)$, $\binom{6}{3}$ terms of the form $P(E_iE_jE_k)$, and so on. Now for all i, $P(E_i) = (5/6)^n$; for all i and j, $P(E_iE_j) = (4/6)^n$; for all i, j, and k, $P(E_iE_jE_k) = (3/6)^n$; and so on. Thus

$$\begin{aligned}
P(X > n) &= P(E_1 \cup E_2 \cup \cdots \cup E_6)\\
&= \binom{6}{1}\Big(\frac{5}{6}\Big)^n - \binom{6}{2}\Big(\frac{4}{6}\Big)^n + \binom{6}{3}\Big(\frac{3}{6}\Big)^n - \binom{6}{4}\Big(\frac{2}{6}\Big)^n + \binom{6}{5}\Big(\frac{1}{6}\Big)^n\\
&= 6\Big(\frac{5}{6}\Big)^n - 15\Big(\frac{4}{6}\Big)^n + 20\Big(\frac{3}{6}\Big)^n - 15\Big(\frac{2}{6}\Big)^n + 6\Big(\frac{1}{6}\Big)^n.
\end{aligned}$$

Let p be the probability mass function of X. The set of all possible values of X is $\{6, 7, 8, \ldots\}$, and

$$\begin{aligned}
p(n) = P(X = n) &= P(X > n-1) - P(X > n)\\
&= \Big(\frac{5}{6}\Big)^{n-1} - 5\Big(\frac{4}{6}\Big)^{n-1} + 10\Big(\frac{3}{6}\Big)^{n-1} - 10\Big(\frac{2}{6}\Big)^{n-1} + 5\Big(\frac{1}{6}\Big)^{n-1}, \quad n \geq 6.
\end{aligned}$$

16. Put the students in some random order. Suppose that the first two students form the first team, the third and fourth students form the second team, the fifth and sixth students form the third team, and so on. Let F stand for "female" and M stand for "male." Since our only concern is gender of the students, the total number of ways we can form 13 teams, each consisting of two students, is equal to the number of distinguishable permutations of a sequence of 23 M's and three F's. By Theorem 2.4, this number is $\dfrac{26!}{23!\,3!} = \dbinom{26}{3}$. The set of possible values of the random variable X is $\{2, 4, \ldots, 26\}$. To calculate the probabilities associated with these values, note that for $k = 1, 2, \ldots, 13$, $X = 2k$ if and only if one of the following events occurs:

A: One of the first $k-1$ teams is a female-female team, the kth team is either a male-female or a female-male team, and the remaining teams are all male-male teams.

B: The first $k-1$ teams are all male-male teams, and the kth team is either a male-female team or a female-male team.

To find $P(A)$, note that for A to occur, there are $k-1$ possibilities for one of the first $k-1$ teams to be a female-female team, two possibilities for the kth team (male-female and female-male), and one possibility for the remaining teams to be all male-male teams. Therefore,

$$P(A) = \frac{2(k-1)}{\binom{26}{3}}.$$

To find $P(B)$, note that for B to occur, there is one possibility for the first $k-1$ teams to be all male-male, and two possibilities for the kth team: male-female and female-male. The number of possibilities for the remaining $13-k$ teams is equal to the number of distinguishable permutations of two F's and $(26-2k)-2$ M's, which, by Theorem 2.4, is $\dfrac{26-2k)!}{2!\,(26-2k-2)!} = \dbinom{26-2k}{2}$. Therefore,

$$P(B) = \frac{2\binom{26-2k}{2}}{\binom{26}{3}}.$$

Hence, for $1 \leq k \leq 13$,

$$P(X = 2k) = P(A) + P(B) = \frac{2(k-1) + 2\binom{26-2k}{2}}{\binom{26}{3}} = \frac{1}{650}k^2 - \frac{1}{26}k + \frac{1}{4}.$$

4.4 EXPECTATIONS OF DISCRETE RANDOM VARIABLES

1. Yes, of course there is a fallacy in Dickens' argument. If, in England, at that time there were exactly two train accidents each month, then Dickens would have been right. Usually, for all $n > 0$ and for any two given days, the probability of n train accidents in day 1 is equal to the probability of n accidents in day 2. Therefore, in all likelihood the risk of train accidents on the final day in March and the risk of such accidents on the first day in April would have been about the same. The fact that train accidents occurred at random days, two per month on the average, imply that in some months more than two and in other months two or less accidents were occurring.

2. Let X be the fine that the citizen pays on a random day. Then

$$E(X) = 25(0.60) + 0(0.40) = 15.$$

Therefore, it is much better to park legally.

3. The expected value of the winning amount is

$$30\Big(\frac{4000}{2,000,000}\Big) + 800\Big(\frac{500}{2,000,000}\Big) + 1,200,000\Big(\frac{1}{2,000,000}\Big) = 0.86.$$

Considering the cost of the ticket, the expected value of the player's *gain* in one game is $-1 + 0.86 = -0.14$.

4. Let X be the amount that the player gains in one game, then

$$P(X = 4) = \frac{\binom{4}{3}\binom{6}{1}}{\binom{10}{4}} = 0.114, \qquad P(X = 9) = \frac{1}{\binom{10}{4}} = 0.005,$$

and $P(X = -1) = 1 - 0.114 - 0.005 = 0.881$. Thus

$$E(X) = -1(0.881) + 4(0.114) + 9(0.005) = -0.38.$$

Therefore, on the average, the player loses 38 cents per game.

5. Let X be the net gain in one play of the game. The set of possible values of X is $\{-8, -4, 0, 6, 10\}$. The probabilities associated with these values are

$$p(-8) = p(0) = \frac{1}{\binom{5}{2}} = \frac{1}{10}, \qquad p(-4) = \frac{\binom{2}{1}\binom{2}{1}}{\binom{5}{2}} = \frac{4}{10},$$

and $p(6) = p(10) = \dfrac{\binom{2}{1}}{\binom{5}{2}} = \dfrac{2}{10}$. Hence

$$E(X) = -8 \cdot \frac{1}{10} - 4 \cdot \frac{4}{10} + 0 \cdot \frac{1}{10} + 6 \cdot \frac{2}{10} + 10 \cdot \frac{2}{10} = \frac{4}{5}.$$

Since $E(X) > 0$, the game is not fair.

6. The expected number of defective items is

$$\sum_{i=0}^{3} i \cdot \frac{\binom{5}{i}\binom{15}{5-i}}{\binom{20}{3}} = 0.75.$$

7. For $i = 4, 5, 6, 7$, let X_i be the profit if i magazines are ordered. Then

$$E(X_4) = \frac{4a}{3},$$

$$E(X_5) = \frac{2a}{3} \cdot \frac{6}{18} + \frac{5a}{3} \cdot \frac{12}{18} = \frac{4a}{3},$$

$$E(X_6) = 0 \cdot \frac{6}{18} + a \cdot \frac{5}{18} + \frac{6a}{3} \cdot \frac{7}{18} = \frac{19a}{18},$$

$$E(X_7) = -\frac{2a}{3} \cdot \frac{6}{18} + \frac{a}{3} \cdot \frac{5}{18} + \frac{4a}{3} \cdot \frac{4}{18} + \frac{7a}{3} \cdot \frac{3}{18} = \frac{10a}{18}.$$

Since $4a/3 > 19a/18$ and $4a/3 > 10a/18$, either 4, or 5 magazines should be ordered to maximize the profit in the long run.

8. **(a)** $$\sum_{x=1}^{\infty} \frac{6}{\pi^2 x^2} = \frac{6}{\pi^2} \sum_{x=1}^{\infty} \frac{1}{x^2} = \frac{6}{\pi^2} \cdot \frac{\pi^2}{6} = 1.$$

(b) $$E(X) = \sum_{x=1}^{\infty} x \frac{6}{\pi^2 x^2} = \frac{6}{\pi^2} \sum_{x=1}^{\infty} \frac{1}{x} = \infty.$$

9. **(a)** $$\sum_{i=-2}^{2} p(x) = \frac{9}{27} + \frac{4}{27} + \frac{1}{27} + \frac{4}{27} + \frac{9}{27} = 1.$$

(b) $E(X) = \sum_{x=-2}^{2} x p(x) = 0$, $E(|X|) = \sum_{x=-2}^{2} |x| p(x) = 44/27$,

$E(X^2) = \sum_{x=-2}^{2} x^2 p(x) = 80/27$. Hence

$$E(2X^2 - 5X + 7) = 2(80/27) - 5(0) + 7 = 349/27.$$

10. Let R be the radius of the randomly selected disk; then $E(2\pi R) = 2\pi \sum_{i=1}^{10} i \frac{1}{10} = 11\pi$.

11. $p(x)$ the probability mass function of X is given by

x	-3	0	3	4
$p(x)$	3/8	1/8	1/4	1/4

Hence

$$E(X) = -3 \cdot \frac{3}{8} + 0 \cdot \frac{1}{8} + 3 \cdot \frac{1}{4} + 4 \cdot \frac{1}{4} = \frac{5}{8},$$

$$E(X^2) = 9 \cdot \frac{3}{8} + 0 \cdot \frac{1}{8} + 9 \cdot \frac{1}{4} + 16 \cdot \frac{1}{4} = \frac{77}{8},$$

$$E(|X|) = 3 \cdot \frac{3}{8} + 0 \cdot \frac{1}{8} + 3 \cdot \frac{1}{4} + 4 \cdot \frac{1}{4} = \frac{23}{8},$$

$$E(X^2 - 2|X|) = \frac{77}{8} - 2\Big(\frac{23}{8}\Big) = \frac{31}{8},$$

$$E(X|X|) = -9 \cdot \frac{3}{8} + 0 \cdot \frac{1}{8} + 9 \cdot \frac{1}{4} + 16 \cdot \frac{1}{4} = \frac{23}{8}.$$

12. $E(X) = \sum_{i=1}^{10} i \cdot \frac{1}{10} = \frac{11}{2}$ and $E(X^2) = \sum_{i=1}^{10} i^2 \cdot \frac{1}{10} = \frac{77}{2}$. So

$$E\big[X(11 - X)\big] = E(11X - X^2) = 11 \cdot \frac{11}{2} - \frac{77}{2} = 22.$$

13. Let X be the number of different birthdays; we have

$$P(X = 4) = \frac{365 \times 364 \times 363 \times 362}{365^4} = 0.9836,$$

$$P(X = 3) = \frac{\binom{4}{2} 365 \times 364 \times 363}{365^4} = 0.0163,$$

$$P(X = 2) = \frac{\binom{4}{2} 365 \times 364 + \binom{4}{3} 365 \times 364}{365^4} = 0.00007,$$

$$P(X = 1) = \frac{365}{365^4} = 0.000000021.$$

Thus

$$E(X) = 4(0.9836) + 3(0.0163) + 2(0.00007) + 1(0.000, 000, 021) = 3.98.$$

14. Let X be the number of children they should continue to have until they have one of each sex. For $i \geq 2$, clearly, $X = i$ if and only if either all of their first $i - 1$ children are boys and the ith child is a girl, or all of their first $i - 1$ children are girls and the ith child is a boy. Therefore, by independence,

$$P(X = i) = \Big(\frac{1}{2}\Big)^{i-1} \cdot \frac{1}{2} + \Big(\frac{1}{2}\Big)^{i-1} \cdot \frac{1}{2} = \Big(\frac{1}{2}\Big)^{i-1}, \quad i \geq 2.$$

So

$$E(X) = \sum_{i=2}^{\infty} i\Big(\frac{1}{2}\Big)^{i-1} = -1 + \sum_{i=1}^{\infty} i\Big(\frac{1}{2}\Big)^{i-1} = -1 + \frac{1}{(1 - 1/2)^2} = 3.$$

Note that for $|r| < 1$, $\sum_{i=1}^{\infty} i r^{i-1} = 1/[(1 - r)^2]$.

15. Let A_j be the event that the person belongs to a family with j children. Then

$$P(K = k) = \sum_{j=0}^{c} P(K = k|A_j)P(A_j) = \sum_{j=k}^{c} \frac{1}{j}\alpha_j.$$

Therefore,

$$E(K) = \sum_{k=1}^{c} kP(K = k) = \sum_{k=1}^{c} k \sum_{j=k}^{c} \frac{\alpha_j}{j} = \sum_{k=1}^{c} \sum_{j=k}^{c} \frac{k\alpha_j}{j}.$$

16. Let X be the number of cards to be turned face up until an ace appears. Let A be the event that no ace appears among the first $i - 1$ cards that are turned face up. Let B be the event that the ith card turned face up is an ace. We have

$$P(X = i) = P(AB) = P(B|A)P(A) = \frac{4}{52 - (i - 1)} \cdot \frac{\dbinom{48}{i-1}}{\dbinom{52}{i-1}}.$$

Therefore,

$$E(X) = \sum_{i=1}^{49} \frac{i\dbinom{48}{i-1}4}{\dbinom{52}{i-1}(53 - i)} = 10.6.$$

To some, this answer might be counterintuitive.

17. Let X be the largest number selected. Clearly,

$$P(X = i) = P(X \le i) - P(X \le i - 1) = \Big(\frac{i}{N}\Big)^n - \Big(\frac{i-1}{N}\Big)^n, \quad i = 1, 2, \dots, N.$$

Hence

$$E(X) = \sum_{i=1}^{N} \Big[\frac{i^{n+1}}{N^n} - \frac{i(i-1)^n}{N^n}\Big] = \frac{1}{N^n} \sum_{i=1}^{N} \big[i^{n+1} - i(i - 1)^n\big]$$

$$= \frac{1}{N^n} \sum_{i=1}^{N} \big[i^{n+1} - (i - 1)^{n+1} - (i - 1)^n\big] = \frac{N^{n+1} - \sum_{i=1}^{N}(i - 1)^n}{N^n}.$$

For large N,

$$\sum_{i=1}^{N}(i - 1)^n \approx \int_0^N x^n\,dx = \frac{N^{n+1}}{n + 1}.$$

Therefore,

$$E(X) \approx \frac{N^{n+1} - \dfrac{N^{n+1}}{n+1}}{N^n} = \frac{nN}{n+1}.$$

18. **(a)** Note that

$$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}.$$

So

$$\sum_{n=1}^{k} \frac{1}{n(n+1)} = \sum_{n=1}^{k} \Big(\frac{1}{n} - \frac{1}{n+1}\Big) = 1 - \frac{1}{k+1}.$$

This implies that

$$\sum_{n=1}^{\infty} p(n) = \lim_{k\to\infty} \sum_{n=1}^{k} \frac{1}{n(n+1)} = 1 - \lim_{k\to\infty} \frac{1}{k+1} = 1.$$

Therefore, p is a probability mass function.

(b) $E(X) = \sum_{n=1}^{\infty} np(n) = \sum_{n=1}^{\infty} \frac{1}{n+1} = \infty,$

where the last equality follows since we know from calculus that the harmonic series, $1 + 1/2 + 1/3 + \cdots$, is divergent. Hence $E(X)$ does not exist.

19. By the solution to Exercise 16, Section 4.3, it should be clear that for $1 \le k \le n$,

$$P(X = 2k) = \frac{2(k-1) + 2\dbinom{2n-2k}{2}}{\dbinom{2n}{3}}.$$

Hence

$$E(X) = \sum_{k=1}^{n} 2kP(X = 2k) = \sum_{k=1}^{n} = \frac{4k(k-1) + 4k\dbinom{2n-2k}{2}}{\dbinom{2n}{3}}$$

$$= \frac{4}{\dbinom{2n}{3}}\Big[2\sum_{k=1}^{n} k^3 - (4n-2)\sum_{k=1}^{n} k^2 + (2n^2 - n - 1)\sum_{n=1}^{n} k\Big]$$

$$= \frac{4}{\dbinom{2n}{3}}\Big[2 \cdot \frac{n^2\,(n+1)^2}{4} - (4n-2)\cdot\frac{n(n+1)(2n+1)}{6} + (2n^2 - n - 1)\frac{n(n+1)}{2}\Big]$$

$$= \frac{(n+1)^2}{2n-1}.$$

4.5 VARIANCES AND MOMENTS OF DISCRETE RANDOM VARIABLES

1. On average, in the long run, the two businesses have the same profit. The one that has a profit with lower standard deviation should be chosen by Mr. Jones because he's interested in steady income. Therefore, he should choose the first business.

2. The one with lower standard deviation, namely, the second device.

3. $E(X) = \sum_{x=-3}^{3} xp(x) = -1$, $E(X^2) = \sum_{x=-3}^{3} x^2 p(x) = 4$. Therefore, $\text{Var}(X) = 4 - 1 = 3$.

4. p, the probability mass function of X is given by

x	-3	0	6
$p(x)$	3/8	3/8	2/8

Thus

$$E(X) = -\frac{9}{8} + \frac{12}{8} = \frac{3}{8}, \qquad E(X^2) = \frac{27}{8} + \frac{72}{8} = \frac{99}{8},$$

$$\text{Var}(X) = \frac{99}{8} - \frac{9}{64} = \frac{783}{64} = 12.234, \qquad \sigma_X = \sqrt{12.234} = 3.498.$$

5. By straightforward calculations,

$$E(X) = \sum_{i=1}^{N} i \cdot \frac{1}{N} = \frac{1}{N} \cdot \frac{N(N+1)}{2} = \frac{N+1}{2},$$

$$E(X^2) = \sum_{i=1}^{N} i^2 \cdot \frac{1}{N} = \frac{1}{N} \cdot \frac{N(N+1)(2N+1)}{6} = \frac{(N+1)(2N+1)}{6},$$

$$\text{Var}(X) = \frac{(N+1)(2N+1)}{6} - \frac{(N+1)^2}{4} = \frac{N^2-1}{12},$$

$$\sigma_X = \sqrt{\frac{N^2-1}{12}}.$$

6. Clearly,

$$E(X) = \sum_{i=0}^{5} i \cdot \frac{\binom{13}{i}\binom{39}{5-i}}{\binom{52}{5}} = 1.25,$$

$$E(X^2) = \sum_{i=0}^{5} i^2 \cdot \frac{\binom{13}{i}\binom{39}{5-i}}{\binom{52}{5}} = 2.426.$$

Therefore, $\text{Var}(X) = 2.426 - (1.25)^2 = 0.864$, and hence $\sigma_X = \sqrt{0.864} = 0.9295$.

7. By the Corollary of Theorem 4.2, $E(X^2 - 2X) = 3$ implies that $E(X^2) - 2E(X) = 3$. Substituting $E(X) = 1$ in this relation gives $E(X^2) = 5$. Hence, by Theorem 4.3,

$$\text{Var}(X) = E(X^2) - \big[E(X)\big]^2 = 5 - 1 = 4.$$

By Theorem 4.5,

$$\text{Var}(-3X + 5) = 9\text{Var}(X) = 9 \times 4 = 36.$$

8. Let X be Harry's net gain. Then

$$X = \begin{cases} -2 & \text{with probability } 1/8 \\ 0.25 & \text{with probability } 3/8 \\ 0.50 & \text{with probability } 3/8 \\ 0.75 & \text{with probability } 1/8. \end{cases}$$

Thus

$$E(X) = -2 \cdot \frac{1}{8} + 0.25 \cdot \frac{3}{8} + 0.50 \cdot \frac{3}{8} + 0.75 \cdot \frac{1}{8} = 0.125$$

$$E(X^2) = (-2)^2 \cdot \frac{1}{8} + 0.25^2 \cdot \frac{3}{8} + 0.50^2 \cdot \frac{3}{8} + 0.75^2 \cdot \frac{1}{8} = 0.6875.$$

These show that the expected value of Harry's net gain is 12.5 cents. Its variance is

$$\text{Var}(X) = 0.6875 - 0.125^2 = 0.671875.$$

9. Note that $E(X) = E(Y) = 0$. Clearly,

$$P\big(|X - 0| \leq t\big) = \begin{cases} 0 & \text{if } t < 1 \\ 1 & \text{if } t \geq 1, \end{cases}$$

$$P\big(|Y - 0| \leq t\big) = \begin{cases} 0 & \text{if } t < 10 \\ 1 & \text{if } t \geq 10. \end{cases}$$

These relations, clearly, show that for all $t > 0$,

$$P\big(|Y - 0| \leq t\big) \leq P\big(|X - 0| \leq t\big).$$

Therefore, X is more concentrated about 0 than Y is.

10. **(a)** Let X be the number of trials required to open the door. Clearly,

$$P(X = x) = \Big(1 - \frac{1}{n}\Big)^{x-1} \frac{1}{n}, \quad x = 1, 2, 3, \dots .$$

Thus

$$E(X) = \sum_{x=1}^{\infty} x\Big(1 - \frac{1}{n}\Big)^{x-1}\frac{1}{n} = \frac{1}{n}\sum_{x=1}^{\infty} x\Big(1 - \frac{1}{n}\Big)^{x-1}. \tag{10}$$

We know from calculus that $\forall r$, $|r| < 1$,

$$\sum_{x=1}^{\infty} x r^{x-1} = \frac{1}{(1-r)^2}. \tag{11}$$

Thus

$$\sum_{x=1}^{\infty} x\Big(1 - \frac{1}{n}\Big)^{x-1} = \frac{1}{\Big[1 - \Big(1 - \frac{1}{n}\Big)\Big]^2} = n^2. \tag{12}$$

Substituting (12) in (10), we obtain $E(X) = n$. To calculate $\text{Var}(X)$, first we find $E(X^2)$. We have

$$E(X^2) = \sum_{x=1}^{\infty} x^2\Big(1 - \frac{1}{n}\Big)^{x-1}\Big(\frac{1}{n}\Big) = \frac{1}{n}\sum_{x=1}^{\infty} x^2\Big(1 - \frac{1}{n}\Big)^{x-1}. \tag{13}$$

Now to calculate this sum, we multiply both sides of (11) by r and then differentiate it with respect to r; we get

$$\sum_{x=1}^{\infty} x^2 r^{x-1} = \frac{1+r}{(1-r)^3}.$$

Using this relation in (13), we obtain

$$E(X^2) = \frac{1}{n} \cdot \frac{1 + 1 - \frac{1}{n}}{\Big[1 - \Big(1 - \frac{1}{n}\Big)\Big]^3} = 2n^2 - n.$$

Therefore,

$$\text{Var}(X) = (2n^2 - n) - n^2 = n(n-1).$$

(b) Let A_i be the event that on the ith trial the door opens. Let X be the number of trials required to open the door. Then

$$P(X = 1) = \frac{1}{n},$$

$$P(X=2) = P(A_1^c A_2) = P(A_2 \mid A_1^c)P(A_1^c)$$
$$= \frac{1}{n-1} \cdot \frac{n-1}{n} = \frac{1}{n},$$
$$P(X=3) = P(A_1^c A_2^c A_3) = P(A_3 \mid A_2^c A_1^c)P(A_2^c A_1^c)$$
$$= P(A_3 \mid A_2^c A_1^c)P(A_2^c \mid A_1^c)P(A_1^c)$$
$$= \frac{1}{n-2} \cdot \frac{n-2}{n-1} \cdot \frac{n-1}{n} = \frac{1}{n}.$$

Similarly, $P(X=i) = 1/n$ for $1 \le i \le n$. Therefore, X is a random number selected from $\{1, 2, 3, \ldots, n\}$. By Exercise 5, $E(X) = (n+1)/2$ and $\text{Var}(X) = (n^2-1)/12$.

11. For $E(X^3)$ to exist, we must have $E\big(|X^3|\big) < \infty$. Now

$$\sum_{n=1}^{\infty} x_n^3\, p(x_n) = \frac{6}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n n\sqrt{n}}{n^2} = \frac{6}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}} < \infty,$$

whereas

$$E\big(|X^3|\big) = \sum_{n=1}^{\infty} |x_n^3| p(x_n) = \frac{6}{\pi^2} \sum_{n=1}^{\infty} \frac{n\sqrt{n}}{n^2} = \frac{6}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = \infty.$$

12. For $0 < s < r$, clearly,

$$|x|^s \le \max\big(1, |x|^r\big) \le 1 + |x|^r, \quad \forall x \in \mathbf{R}.$$

Let A be the set of possible values of X and p be its probability mass function. Since the rth absolute moment of X exists, $\sum_{x \in A} |x|^r p(x) < \infty$. Now

$$\sum_{x \in A} |x|^s p(x) \le \sum_{x \in A} \big(1 + |x|^r\big) p(x)$$
$$= \sum_{x \in A} p(x) + \sum_{x \in A} |x|^r p(x) = 1 + \sum_{x \in A} |x|^r p(x) < \infty,$$

implies that the absolute moment of order s of X also exists.

13. $\text{Var}(X)$=$\text{Var}(Y)$ implies that

$$E(X^2) - \big[E(X)\big]^2 = E(Y^2) - \big[E(Y)\big]^2.$$

Since $E(X) = E(Y)$, this implies that $E(X^2) = E\big(Y^2\big)$. Let

$$P(X=a) = p_1, \qquad P(X=b) = p_2, \qquad P(X=c) = p_3;$$
$$P(Y=a) = q_1, \qquad P(Y=b) = q_2, \qquad P(Y=c) = q_3.$$

Clearly,

$$p_1 + p_2 + p_3 = q_1 + q_2 + q_3 = 1.$$

This implies

$$(p_1 - q_1) + (p_2 - q_2) + (p_3 - q_3) = 0. \tag{14}$$

The relations $E(X) = E(Y)$ and $E(X^2) = E(Y^2)$ imply that

$$\begin{aligned} ap_1 + bp_2 + cp_3 &= aq_1 + bq_2 + cq_3 \\ a^2p_1 + b^2p_2 + c^2p_3 &= a^2q_1 + b^2q_2 + c^2q_3. \end{aligned}$$

These and equation (14) give us the following system of 3 equations in the 3 unknowns $p_1 - q_1$, $p_2 - q_2$, and $p_3 - q_3$.

$$\begin{cases} (p_1 - q_1) + (p_2 - q_2) + (p_3 - q_3) = 0 \\ a(p_1 - q_1) + b(p_2 - q_2) + c(p_3 - q_3) = 0 \\ a^2(p_1 - q_1) + b^2(p_2 - q_2) + c^2(p_3 - q_3) = 0. \end{cases}$$

In matrix form, this is equivalent to

$$\begin{pmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{pmatrix} \begin{pmatrix} p_1 - q_1 \\ p_2 - q_2 \\ p_3 - q_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \tag{15}$$

Now

$$\begin{aligned} \det \begin{pmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{pmatrix} &= bc^2 + ca^2 + ab^2 - ba^2 - cb^2 - ac^2 \\ &= (c-a)(c-b)(b-a) \neq 0, \end{aligned}$$

since a, b, and c are three *different* real numbers. This implies that the matrix

$$\begin{pmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{pmatrix}$$

is invertible. Hence the solution to (15) is

$$p_1 - q_1 = p_2 - q_2 = p_3 - q_3 = 0.$$

Therefore, $p_1 = q_1$, $p_2 = q_2$, $p_3 = q_3$ implying that X and Y are identically distributed.

14. Let

$$P(X = a_1) = p_1, \qquad P(X = a_2) = p_2, \qquad \dots, \qquad P(X = a_n) = p_n;$$
$$P(Y = a_1) = q_1, \qquad P(Y = a_2) = q_2, \qquad \dots, \qquad P(Y = a_n) = q_n.$$

Clearly,

$$p_1 + p_2 + \cdots + p_n = q_1 + q_2 + \cdots + q_n = 1.$$

This implies that

$$(p_1 - q_1) + (p_2 - q_2) + \cdots + (p_n - q_n) = 0.$$

The relations $E(X^r) = E(Y^r)$, for $r = 1, 2, \dots, n-1$ imply that

$$\begin{aligned} a_1 p_1 + a_2 p_2 + \cdots + a_n p_n &= a_1 q_1 + a_2 q_2 + \cdots + a_n q_n, \\ a_1^2 p_1 + a_2^2 p_2 + \cdots + a_n^2 p_n &= a_1^2 q_1 + a_2^2 q_2 + \cdots + a_n^2 q_n, \\ &\vdots \\ a_1^{n-1} p_1 + a_2^{n-1} p_2 + \cdots + a_n^{n-1} p_n &= a_1^{n-1} q_1 + a_2^{n-1} q_2 + \cdots + a_n^{n-1} q_n. \end{aligned}$$

These and the previous relation give us the following n equations in the n unknowns $p_1 - q_1$, $p_2 - q_2, \dots, p_n - q_n$.

$$\begin{cases} (p_1 - q_1) + (p_2 - q_2) + \cdots + (p_n - q_n) = 0 \\ a_1(p_1 - q_1) + a_2(p_2 - q_2) + \cdots + a_n(p_n - q_n) = 0 \\ a_1^2(p_1 - q_1) + a_2^2(p_2 - q_2) + \cdots + a_n^2(p_n - q_n) = 0 \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ a_1^{n-1}(p_1 - q_1) + a_2^{n-1}(p_2 - q_2) + \cdots + a_n^{n-1}(p_n - q_n) = 0 \end{cases}$$

In matrix form, this is equivalent to

$$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ a_1 & a_2 & \cdots & a_n \\ a_1^2 & a_2^2 & \cdots & a_n^2 \\ \vdots & \vdots & & \vdots \\ a_1^{n-1} & a_2^{n-1} & \cdots & a_n^{n-1} \end{pmatrix} \begin{pmatrix} p_1 - q_1 \\ p_2 - q_2 \\ p_3 - q_3 \\ \vdots \\ p_n - q_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \tag{16}$$

Now

$$\det \begin{pmatrix} 1 & 1 & \cdots & 1 \\ a_1 & a_2 & \cdots & a_n \\ a_1^2 & a_2^2 & \cdots & a_n^2 \\ \vdots & \vdots & & \vdots \\ a_1^{n-1} & a_2^{n-1} & \cdots & a_n^{n-1} \end{pmatrix} = \prod_{\substack{j=n,n-1,\dots,2 \\ i<j}} (a_j - a_i) \neq 0,$$

since a_i's are all *different* real numbers. The formula for the determinant of this type of matrices is well known. These are referred to as Vandermonde determinants, after the famous French mathematician A. T. Vandermonde (1735–1796). The above determinant being nonzero implies that the matrix

$$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ a_1 & a_2 & \cdots & a_n \\ a_1^2 & a_2^2 & \cdots & a_n^2 \\ \vdots & \vdots & & \vdots \\ a_1^{n-1} & a_2^{n-1} & \cdots & a_n^{n-1} \end{pmatrix}$$

is invertible. Hence the solution to (16) is

$$p_1 - q_1 = p_2 - q_2 = \cdots = p_n - q_n = 0.$$

Therefore, $p_1 = q_1$, $p_2 = q_2$, ..., $p_n = q_n$, implying that X and Y are identically distributed.

4.6 STANDARDIZED RANDOM VARIABLES

1. Let X_1 be the number of TV sets the salesperson in store 1 sells and X_2 be the number of TV sets the salesperson in store 2 sells. We have that $X_1^* = (10 - 13)/5 = -0.6$ and $X_2^* = (6 - 7)/4 = -0.25$. Therefore, the number of TV sets the salesperson in store 2 sells is 0.6 standard deviations below the mean, whereas the number of TV sets the salesperson in store 2 sells is 0.25 standard deviations below the mean. So Mr. Norton should hire the salesperson who worked in store 2.

2. Let X be the final grade comparable to Velma's 82 in the midterm. We must have

$$\frac{82 - 72}{12} = \frac{X - 68}{15}.$$

This gives $X = 80.5$.

REVIEW PROBLEMS FOR CHAPTER 4

1. Note that $\binom{10}{2} = 45$. We have

i	1, 2, 16, 17	3, 4, 14, 15	5, 6, 12, 13	7, 8, 10, 11	9
$p(i)$	1/45	2/45	3/45	4/45	5/45

2. The answer is

$$1 \cdot \frac{2}{34} + 2 \cdot \frac{5}{34} + 3 \cdot \frac{9}{34} + 4 \cdot \frac{9}{34} + 5 \cdot \frac{4}{34} + 6 \cdot \frac{5}{34} = 3.676.$$

3. Let N be the number of secretaries to be interviewed to find one who knows TEX. We must find the least n for which $P(N \le n) \ge 0.50$ or $1 - P(N > n) \ge 0.50$ or $1 - (0.98)^n \ge 0.50$. This gives $(0.98)^n \le 0.50$ or $n \ge \ln 0.50 / \ln 0.98 = 34.31$. Therefore, $n = 35$.

4. Let F be the distribution function of X, then

$$F(t) = 1 - \Big(1 + \frac{t}{200}\Big)e^{-t/200}, \quad t \ge 0.$$

Using this, we obtain

$$\begin{aligned} P(200 \le X \le 300) &= P(X \le 300) - P(X < 200) = F(300) - F(200-) \\ &= F(300) - F(200) = 0.442 - 0.264 = 0.178. \end{aligned}$$

5. Let X be the number of sections that will get a hard test. We want to calculate $E(X)$. The random variable X can only assume the values 0, 1, 2, 3, and 4; its probability mass function is given by

$$p(i) = P(X = i) = \frac{\dbinom{8}{i}\dbinom{22}{4-i}}{\dbinom{30}{4}}, \quad i = 0, 1, 2, 3, 4,$$

where the numerical values of $p(i)$'s are as follows.

i	0	1	2	3	4
$p(i)$	0.2669	0.4496	0.2360	0.0450	0.0026

Thus

$$E(X) = 0(0.2669) + 1(0.4496) + 2(0.2360) + 3(0.0450) + 4(0.00026) = 1.067.$$

6. **(a)** $1 - F(6) = 5/36$. **(b)** $F(9) = 76/81$. **(c)** $F(7) - F(2) = 44/49$.

7. We have that

$$E(X) = (15.85)(0.15) + (15.9)(0.21) + (16)(0.35) + (16.1)(0.15) + (16.2)(0.14) = 16,$$

$$\begin{aligned} \text{Var}(X) &= (15.85 - 16)^2(0.15) + (15.9 - 16)^2(0.21) + (16 - 16)^2(0.35) \\ &\quad + (16.1 - 16)^2(0.15) + (16.2 - 16)^2(0.14) = 0.013. \end{aligned}$$

$$E(Y) = (15.85)(0.14) + (15.9)(0.05) + (16)(0.64) + (16.1)(0.08) + (16.2)(0.09) = 16,$$

$$\begin{aligned} \text{Var}(Y) &= (15.85 - 16)^2(0.14) + (15.9 - 16)^2(0.05) + (16 - 16)^2(0.64) \\ &\quad + (16.1 - 16)^2(0.08) + (16.2 - 16)^2(0.09) = 0.008. \end{aligned}$$

These show that, on the average, companies A and B fill their bottles with 16 fluid ounces of soft drink. However, the amount of soda in bottles from company A vary more than in bottles from company B.

8. Let F be the distribution function of X, Then

$$F(t) = \begin{cases} 0 & t < 58 \\ 7/30 & 58 \leq t < 62 \\ 13/30 & 62 \leq t < 64 \\ 18/30 & 64 \leq t < 76 \\ 23/30 & 76 \leq t < 80 \\ 1 & t \geq 80. \end{cases}$$

9. **(a)** To determine the value of k, note that $\sum_{i=0}^{\infty} k\dfrac{(2t)^i}{i!} = 1$. Therefore, $k\sum_{i=0}^{\infty}\dfrac{(2t)^i}{i!} = 1$. This implies that $ke^{2t} = 1$ or $k = e^{-2t}$. Thus $p(i) = e^{-2t}\dfrac{(2t)^i}{i!}$.

(b)

$$P(X < 4) = \sum_{i=0}^{3} P(X = i) = e^{-2t}\big[1 + 2t + 2t^2 + (4t^3/3)\big],$$

$$P(X > 1) = 1 - P(X = 0) - P(X = 1) = 1 - e^{-2t} - 2te^{-2t}.$$

10. Let p be the probability mass function, and F be the distribution function of X. We have $p(0) = p(3) = \dfrac{1}{8}$, $p(1) = p(2) = \dfrac{3}{8}$, and

$$F(t) = \begin{cases} 0 & t < 0 \\ 1/8 & 0 \leq t < 1 \\ 4/8 & 1 \leq t < 2 \\ 7/8 & 2 \leq t < 3 \\ 1 & t \geq 3. \end{cases}$$

11. **(a)** The sample space has 52! elements because when the cards are dealt face down, any ordering of the cards is a possibility. To find $p(j)$, the probability that the 4th king will appear on the jth card, we claim that in $\binom{4}{1} \cdot {}_{(j-1)}P_3 \cdot 48!$ ways the 4th king will appear on the jth card, and the remaining 3 kings earlier. To see this, note that we have $\binom{4}{1}$ combinations for the king that appears on the jth card, and ${}_{(j-1)}P_3$ different permutations for the remaining 3 kings that appear earlier. The last term 48!, is for the remaining 48 cards that can appear in any order in the remaining 48 positions. Therefore,

$$p(j) = \frac{\binom{4}{1} \cdot {}_{(j-1)}P_3 \cdot 48!}{52!} = \frac{\binom{j-1}{3}}{\frac{52!}{4!\,48!}} = \frac{\binom{j-1}{3}}{\binom{52}{4}}.$$

(b) The probability that the player wins is $p(52) = \binom{51}{3} \Big/ \binom{52}{4} = 1/13$.

(c) To find

$$E = \sum_{j=4}^{52} j p(j) = \frac{1}{\binom{52}{4}} \sum_{j=4}^{52} j \binom{j-1}{3},$$

the expected length of the game, we use a technique introduced by Jenkyns and Muller in *Mathematics Magazine*, 54, (1981), page 203. We have the following relation which can be readily checked.

$$j\binom{j-1}{3} = \frac{4}{5}\left[(j+1)\binom{j}{4} - j\binom{j-1}{4}\right], \quad j \geq 5.$$

This gives

$$\begin{aligned}\sum_{j=5}^{52} j\binom{j-1}{3} &= \frac{4}{5}\left[\sum_{j=5}^{52}(j+1)\binom{j}{4} - \sum_{j=5}^{52} j\binom{j-1}{4}\right]\\ &= \frac{4}{5}\left[53\binom{52}{4} - 5\binom{4}{4}\right] = 11,478,736,\end{aligned}$$

where the next-to-the-last equality follows because terms cancel out in pairs. Thus

$$\begin{aligned}E &= \frac{1}{\binom{52}{4}} \sum_{j=4}^{52} j\binom{j-1}{3} = \frac{1}{\binom{52}{4}}\left[4 + \sum_{j=5}^{52} j\binom{j-1}{3}\right]\\ &= \frac{1}{\binom{52}{4}}(4 + 11,478,736) = 42.4.\end{aligned}$$

As Jenkyns and Muller have noted, “This relatively high expectation value is what makes the game interesting. However, the low probability of winning makes it frustrating!”

Chapter 5

Special Discrete Distributions

5.1 BERNOULLI AND BINOMIAL RANDOM VARIABLES

1. $\binom{8}{4}\left(\frac{1}{4}\right)^4\left(\frac{3}{4}\right)^4 = 0.087.$

2. **(a)** $64 \times \frac{1}{2} = 32.$

(b) $6 \times \frac{1}{2} + 1 = 4$ (note that we should count the mother of the family as well).

3. $\binom{6}{3}\left(\frac{1}{6}\right)^3\left(\frac{5}{6}\right)^3 = 0.054.$

4. $\binom{6}{2}\left(\frac{1}{10}\right)^2\left(\frac{9}{10}\right)^4 = 0.098.$

5. $\binom{5}{2}\left(\frac{10}{30}\right)^2\left(\frac{20}{30}\right)^3 = 0.33.$

6. Let X be the number of defective nails. If the manufacturer's claim is true, we have

$$\begin{aligned} P(X \geq 2) &= 1 - P(X = 0) - P(X = 1) \\ &= 1 - \binom{24}{0}(0.03)^0(0.97)^{24} - \binom{24}{1}(0.03)(0.97)^{23} = 0.162. \end{aligned}$$

This shows that there is 16.2% chance that two or more defective nails is found. Therefore, it is not fair to reject company's claim.

7. Let p and q be the probability mass functions of X and Y, respectively. Then

$$p(x) = \binom{4}{x}(0.60)^x(0.40)^{4-x}, \quad x = 0, 1, 2, 3, 4;$$

$$\begin{aligned} q(y) &= P(Y = y) = P\Big(X = \frac{y-1}{2}\Big) \\ &= \binom{4}{\frac{y-1}{2}}(0.60)^{(y-1)/2}(0.40)^{4-[(y-1)/2]}, \quad y = 1, 3, 5, 7, 9. \end{aligned}$$

8. $\displaystyle\sum_{i=0}^{8}\binom{15}{i}(0.8)^i(0.2)^{15-i} = 0.142.$

9. $\displaystyle\binom{10}{5}\Big(\frac{11}{36}\Big)^5\Big(\frac{25}{36}\Big)^5 = 0.108.$

10. **(a)** $\displaystyle 1 - \binom{5}{0}\Big(\frac{1}{3}\Big)^0\Big(\frac{2}{3}\Big)^5 - \binom{5}{1}\Big(\frac{1}{3}\Big)^1\Big(\frac{2}{3}\Big)^4 = 0.539.$ **(b)** $\displaystyle\binom{5}{2}\Big(\frac{1}{10}\Big)^2\Big(\frac{9}{10}\Big)^3 = 0.073.$

11. We know that $p(x)$ is maximum at $[(n+1)p]$. If $(n+1)p$ is an integer, $p(x)$ is maximum at $[(n+1)p] = np + p$. But in such a case, some straightforward algebra shows that

$$\binom{n}{np+p}p^{np+p}(1-p)^{n-np-p} = \binom{n}{np+p-1}p^{np+p-1}(1-p)^{n-np-p+1},$$

implying that $p(x)$ is also maximum at $np + p - 1$.

12. The probability of royal or straight flush is $40\Big/\binom{52}{5}$. If Ernie plays n games, he will get, on the average, $n\Big[40\Big/\binom{52}{5}\Big]$ royal or straight flushes. We want to have $40n\Big/\binom{52}{5} = 1$; this gives $n = \binom{52}{5}\Big/40 = 64,974$.

13. $\displaystyle\binom{6}{3}\Big(\frac{1}{3}\Big)^3\Big(\frac{2}{3}\Big)^3 = 0.219.$

14. $1 - (999/1000)^{100} = 0.095.$

15. The maximum occurs at $k = [11(0.45)] = 4$. The maximum probability is

$$\binom{10}{4}(0.45)^4(0.55)^6 = 0.238.$$

16. Call the event of obtaining a full house success. X, the number of full houses is n independent poker hands is a binomial random variable with parameters (n, p), where p is the probability that a random poker hand is a full house. To calculate p, note that there are $\binom{52}{5}$ possible poker hands and $\displaystyle\binom{4}{3}\binom{4}{2}\frac{13!}{11!} = 3744$ full houses. Thus $p = 3744\Big/\binom{52}{5} \approx 0.0014$. Hence

$E(X) = np \approx 0.0014n$ and $\text{Var}(X) = np(1-p) \approx 0.00144n$. Note that if n is approximately 715, then $E(X) = 1$. Thus we should expect to find, on the average, one full house in every 715 random poker hands.

17. $1 - \binom{6}{6}\Big(\frac{1}{4}\Big)^6\Big(\frac{3}{4}\Big)^0 - \binom{6}{5}\Big(\frac{1}{4}\Big)^5\Big(\frac{3}{4}\Big) \approx 0.995.$

18. $1 - \binom{3000}{0}(0.0005)^0(0.9995)^{3000} - \binom{3000}{1}(0.0005)(0.9995)^{2999} \approx 0.442.$

19. The expected value of the expenses if sent in one parcel is

$$45.20 \times 0.07 + 5.20 \times 0.93 = 8.$$

The expected value of the expenses if sent in two parcels is

$$(23.30 \times 2)(0.07)^2 + (23.30 + 3.30)\binom{2}{1}(0.07)(0.93) + (6.60)(0.93)^2 = 9.4.$$

Therefore, it is preferable to send in a single parcel.

20. Let n be the minimum number of children they should plan to have. Since the probability of all girls is $(1/2)^n$ and the probability of all boys is $(1/2)^n$, we must have $1-(1/2)^n-(1/2)^n \geq 0.95$. This gives $(1/2)^{n-1} \leq 0.05$ or $n - 1 \geq \dfrac{\ln 0.05}{\ln(0.5)} = 4.32$ or $n \geq 5.32$. Therefore, $n = 6$.

21. **(a)** For this to happen, exactly one of the N stations has to attempt transmitting a message. The probability of this is $\binom{N}{1}p(1-p)^{N-1} = Np(1-p)^{N-1}$.

(b) Let $f(p) = Np(1-p)^{N-1}$. The value of p which maximizes the probability of a message going through with no collision is the root of the equation $f'(p) = 0$. Now

$$f'(p) = N(1-p)^{N-1} - Np(N-1)(1-p)^{N-2} = 0.$$

Noting that $p \neq 1$, this equation gives $p = 1/N$. This answer makes a lot of sense because at every "suitable instance," on average, $Np = 1$ station will transmit a message.

(c) By part (b), the maximum probability is

$$f\Big(\frac{1}{N}\Big) = N\Big(\frac{1}{N}\Big)\Big(1 - \frac{1}{N}\Big)^{N-1} = \Big(1 - \frac{1}{N}\Big)^{N-1}.$$

As $N \to \infty$, this probability approaches $1/e$, showing that for large numbers of stations (in reality 20 or more), the probability of a successful transmission is approximately $1/e$ independently of the number of stations if $p = 1/N$.

22. The k students whose names have been called are not standing. Let $A_1, A_2, \ldots, A_{n-k}$ be the students whose names have not been called. For i, $1 \le i \le n-k$, call A_i a "success," if he or she is standing; failure, otherwise. Therefore, whether A_i is standing or sitting is a Bernoulli trial, and hence the random variable X is the number of successes in $n-k$ Bernoulli trials. For X to be binomial, for $i \neq j$, the event that A_i is a success must be independent of the event that A_j is a success. Furthermore, the probability that A_i is a success must be the same for all i, $1 \le i \le n-k$. The latter condition is satisfied since A_i is standing if and only if his original seat was among the first k. This happens with probability $p = k/n$ regardless of i . However, the former condition is not valid. The relation

$$P\big(A_j \text{ is standing} \mid A_i \text{ is standing}\big) = \frac{k-1}{n},$$

shows that given A_i is a success changes the probability that A_j is success. That is, A_i being a success is not independent of A_j being a success. This shows that X is not a binomial random variable.

23. Let X be the number of undecided voters who will vote for abortion. The desired probability is

$$P\big(b + (n - X) > a + X\big) = P\Big(X < \frac{n + (b-a)}{2}\Big) = \sum_{i=0}^{\left[\frac{n+(b-a)}{2}\right]} \binom{n}{i}\Big(\frac{1}{2}\Big)^i\Big(\frac{1}{2}\Big)^{n-i}$$

$$= \Big(\frac{1}{2}\Big)^n \sum_{i=0}^{\left[\frac{n+(b-a)}{2}\right]} \binom{n}{i}.$$

24. Let X be the net gain of the player per unit of stake. X is a discrete random variable with possible values -1, 1, 2, and 3. We have

$$P(X = -1) = \binom{3}{0}\Big(\frac{1}{6}\Big)^0\Big(\frac{5}{6}\Big)^3 = \frac{125}{216},$$

$$P(X = 1) = \binom{3}{1}\Big(\frac{1}{6}\Big)\Big(\frac{5}{6}\Big)^2 = \frac{75}{216},$$

$$P(X = 2) = \binom{3}{2}\Big(\frac{1}{6}\Big)^2\Big(\frac{5}{6}\Big) = \frac{15}{216},$$

$$P(X = 3) = \binom{3}{3}\Big(\frac{1}{6}\Big)^3\Big(\frac{5}{6}\Big)^0 = \frac{1}{216}.$$

Hence

$$E(X) = -1 \cdot \frac{125}{216} + 1 \cdot \frac{75}{216} + 2 \cdot \frac{15}{216} + 3 \cdot \frac{1}{216} \approx -0.08.$$

Therefore, the player loses 0.08 per unit stake.

25.

$$
\begin{aligned}
E(X^2) &= \sum_{x=1}^{n} x^2 \binom{n}{x} p^x (1-p)^{n-x} = \sum_{x=1}^{n} (x^2 - x + x) \binom{n}{x} p^x (1-p)^{n-x} \\
&= \sum_{x=1}^{n} x(x-1) \binom{n}{x} p^x (1-p)^{n-x} + \sum_{x=1}^{n} x \binom{n}{x} p^x (1-p)^{n-x} \\
&= \sum_{x=2}^{n} \frac{n!}{(x-2)!\,(n-x)!}\, p^x (1-p)^{n-x} + E(X) \\
&= n(n-1)p^2 \sum_{x=2}^{n} \binom{n-2}{x-2} p^{x-2} (1-p)^{n-x} + np \\
&= n(n-1)p^2 \big[p + (1-p)\big]^{n-2} + np = n^2p^2 - np^2 + np.
\end{aligned}
$$

26. **(a)** A four-engine plane is preferable to a two-engine plane if and only if

$$1 - \binom{4}{0} p^0 (1-p)^4 - \binom{4}{1} p(1-p)^3 > 1 - \binom{2}{0} p^0 (1-p)^2.$$

This inequality gives $p > 2/3$. Hence a four-engine plane is preferable if and only if $p > 2/3$. If $p = 2/3$, it makes no difference.

(b) A five-engine plane is preferable to a three-engine plane if and only if

$$\binom{5}{5} p^5 (1-p)^0 + \binom{5}{4} p^4 (1-p) + \binom{5}{3} p^3 (1-p)^2 > \binom{3}{2} p^2 (1-p) + p^3.$$

Simplifying this inequality, we get $3(p-1)^2(2p-1) \geq 0$ which implies that a five-engine plane is preferable if and only if $2p - 1 \geq 0$. That is, for $p > 1/2$, a five-engine plane is preferable; for $p < 1/2$, a three-engine plane is preferable; for $p = 1/2$ it makes no difference.

27. Clearly, 8 bits are transmitted. A parity check will not detect an error in the 7–bit character received erroneously *if and only if* the number of bits received incorrectly is even. Therefore, the desired probability is

$$\sum_{n=1}^{4} \binom{8}{2n} (1 - 0.999)^{2n} (0.999)^{8-2n} = 0.000028.$$

28. The message is erroneously received but the errors are not detected by the parity-check if for $1 \leq j \leq 6$, j of the characters are erroneously received but not detected by the parity–check, and the remaining $6 - j$ characters are all transmitted correctly. By the solution of the previous exercise, the probability of this event is

$$\sum_{j=1}^{6} (0.000028)^j (0.999)^{8(6-j)} = 0.000161.$$

29. The probability of a straight flush is $40\Big/\binom{52}{5} \approx 0.000015391$. Hence we must have

$$1-\binom{n}{0}(0.000015391)^0(1-0.000015391)^n \geq \frac{3}{4}.$$

This gives

$$(1-0.000015391)^n \leq \frac{1}{4}.$$

So

$$n \geq \frac{\log(1/4)}{\log(1-0.000015391)} \approx 90071.06.$$

Therefore, $n \approx 90,072$.

30. Let p, q, and r be the probabilities that a randomly selected offspring is AA, Aa, and aa, respectively. Note that both parents of the offspring are AA with probability $(\alpha/n)^2$, they are both Aa with probability $\big[1-(\alpha/n)\big]^2$, and the probability is $2(\alpha/n)\big[1-(\alpha/n)\big]$ that one parent is AA and the other is Aa. Therefore, by the law of total probability,

$$\begin{aligned}
p &= 1\cdot\Big(\frac{\alpha}{n}\Big)^2+\frac{1}{4}\cdot\Big(1-\frac{\alpha}{n}\Big)^2+\frac{1}{2}\cdot 2\Big(\frac{\alpha}{n}\Big)\Big(1-\frac{\alpha}{n}\Big)=\frac{1}{4}\Big(\frac{\alpha}{n}\Big)^2+\frac{1}{2}\Big(\frac{\alpha}{n}\Big)+\frac{1}{4},\\
q &= 0\cdot\Big(\frac{\alpha}{n}\Big)^2+\frac{1}{2}\Big(1-\frac{\alpha}{n}\Big)^2+\frac{1}{2}\cdot 2\Big(\frac{\alpha}{n}\Big)\Big(1-\frac{\alpha}{n}\Big)=\frac{1}{2}-\frac{1}{2}\Big(\frac{\alpha}{n}\Big)^2,\\
r &= 0\cdot\Big(\frac{\alpha}{n}\Big)^2+\frac{1}{4}\Big(1-\frac{\alpha}{n}\Big)^2+0\cdot 2\Big(\frac{\alpha}{n}\Big)\Big(1-\frac{\alpha}{n}\Big)=\frac{1}{4}\Big(1-\frac{\alpha}{n}\Big)^2.
\end{aligned}$$

The probability that at most two of the offspring are aa is

$$\sum_{i=0}^{2}\binom{m}{i}r^i(1-r)^{m-i}.$$

The probability that exactly i of the offspring are AA and the remaining are all Aa is

$$\binom{m}{i}p^i q^{m-i}.$$

31. The desired probability is the sum of three probabilities: probability of no customer served and two new arrivals, probability of one customer served and three new arrivals, and probability of two customers served and four new arrivals. These quantities, respectively, are $(0.4)^4\cdot\binom{4}{2}(0.45)^2(0.55)^2$, $\binom{4}{1}(0.6)(0.4)^3\cdot\binom{4}{3}(0.45)^3(0.55)$, and $\binom{4}{2}(0.6)^2(0.4)^2\cdot(0.45)^4$. The sum of these quantities, which is the answer, is 0.054.

32. **(a)** Let S be the event that the first trial is a success and E be the event that in n trials, the number of successes is even. Then

$$P(E) = P(E|S)P(S) + P(E|S^c)P(S^c).$$

Thus

$$r_n = (1 - r_{n-1})p + r_{n-1}(1 - p).$$

Using this relation, induction, and $r_0 = 1$, we find that

$$r_n = \frac{1}{2}\big[1 + (1 - 2p)^n\big].$$

(b) The left sum is the probability of 0, 2, 4, ..., or $[n/2]$ successes. Thus it is the probability of an even number of successes in n Bernoulli trials and hence it is equal to r_n.

33. For $0 \le i \le n$, let B_i be the event that i of the balls are red. Let A be the event that in drawing k balls from the urn, successively, and with replacement, no red balls appear. Then

$$P(B_0|A) = \frac{P(A|B_0)P(B_0)}{\sum_{i=0}^{n} P(A|B_i)P(B_i)} = \frac{1 \times \left(\frac{1}{2}\right)^n}{\sum_{i=0}^{n} \left(\frac{n-i}{n}\right)^k \binom{n}{i} \left(\frac{1}{2}\right)^n} = \frac{1}{\sum_{i=0}^{n} \binom{n}{i} \left(\frac{n-i}{n}\right)^k}.$$

34. Let E be the event that Albert's statement is the truth and F be the event that Donna tells the truth. Since Rose agrees with Donna and Rose always tells the truth, Donna is telling the truth as well. Therefore, the desired probability is $P(E \mid F) = P(EF)/P(F)$. To calculate $P(F)$, observe that for Rose to agree with Donna, none, two, or all four of Albert, Brenda, Charles, and Donna should have lied. Since these four people lie independently, this will happen with probability

$$\left(\frac{1}{3}\right)^4 + \binom{4}{2}\left(\frac{2}{3}\right)^2\left(\frac{1}{3}\right)^2 + \left(\frac{2}{3}\right)^4 = \frac{41}{81}.$$

To calculate $P(EF)$, note that EF is the event that Albert tells the truth and Rose agrees with Donna. This happens if all of them tell the truth, or Albert tells the truth but exactly two of Brenda, Charles and Donna lie. Hence

$$P(EF) = \left(\frac{1}{3}\right)^4 + \frac{1}{3} \cdot \binom{3}{2}\left(\frac{2}{3}\right)^2\left(\frac{1}{3}\right) = \frac{13}{81}.$$

Therefore,

$$P(E \mid F) = \frac{P(EF)}{P(F)} = \frac{13/81}{41/81} = \frac{13}{41} = 0.317.$$

5.2 POISSON RANDOM VARIABLES

1. $\lambda = (0.05)(60) = 3$; the answer is $1 - \dfrac{e^{-3}3^0}{0!} = 1 - e^{-3} = 0.9502$.

2. $\lambda = 1.8$; the answer is $\sum_{i=0}^{3} \dfrac{e^{-1.8}(1.8)^i}{i!} \approx 0.89$.

3. $\lambda = 0.025 \times 80 = 2$; the answer is $1 - \dfrac{e^{-2}2^0}{0!} - \dfrac{e^{-2}2^1}{1!} = 1 - 3e^{-2} = 0.594$.

4. $\lambda = (500)(0.0014) = 0.7$. The answer is $1 - \dfrac{e^{-0.7}(0.7)^0}{0!} - \dfrac{e^{-0.7}(0.7)^1}{1!} \approx 0.156$.

5. We call a room "success" if it is vacant next Saturday; we call it "failure" if it is occupied. Assuming that next Saturday is a random day, X, the number of vacant rooms on that day is approximately Poisson with rate $\lambda = 35$. Thus the desired probability is

$$1 - \sum_{i=0}^{29} \frac{e^{-35}(35)^i}{i!} = 0.823.$$

6. $\lambda = (3/10)35 = 10.5$. The probability of 10 misprints in a given chapter is $\dfrac{e^{-10.5}(10.5)^{10}}{10!} = 0.124$. Therefore, the desired probability is $(0.124)^2 = 0.0154$.

7. $P(X = 1) = P(X = 3)$ implies that $e^{-\lambda}\lambda = \dfrac{e^{-\lambda}\lambda^3}{3!}$ from which we get $\lambda = \sqrt{6}$. The answer is $\dfrac{e^{-\sqrt{6}}\left(\sqrt{6}\right)^5}{5!} = 0.063$.

8. The probability that a bun contains no raisins is $\dfrac{e^{-n/k}(n/k)^0}{0!} = e^{-n/k}$. So the answer is $\dbinom{4}{2}e^{-2n/k}(1 - e^{-n/k})^2$.

9. Let X be the number of times the randomly selected kid has hit the target. We are given that $P(X = 0) = 0.04$; this implies that $\dfrac{e^{-\lambda}2^0}{0!} = 0.04$ or $e^{-\lambda} = 0.04$. So $\lambda = -\ln 0.04 = 3.22$. Now

$$\begin{aligned} P(X \geq 2) &= 1 - P(X = 0) - P(X = 1) = 1 - 0.04 - \frac{e^{-\lambda}\lambda}{1!} \\ &= 1 - 0.04 - (0.04)(3.22) = 0.83. \end{aligned}$$

Therefore, 83% of the kids have hit the target at least twice.

10. First we calculate p_i's from binomial probability mass function with $n = 26$ and $p = 1/365$. Then we calculate them from Poisson probability mass function with parameter $\lambda = np = 26/365$. For different values of i, the results are as follows.

i	Binomial	Poisson
0	0.93115	0.93125
1	0.06651	0.06634
2	0.00228	0.00236
3	0.00005	0.00006.

Remark: In this example, since success is very rare, even for small n's Poisson gives good approximation for binomial. The following table demonstrates this fact for $n = 5$.

i	Binomial	Poisson
0	0.9874	0.9864
1	0.0136	0.0136
2	0.00007	0.00009.

11. Let $N(t)$ be the number of shooting stars observed up to time t. Let one minute be the unit of time. Then $\{N(t) : t \geq 0\}$ is a Poisson process with $\lambda = 1/12$. We have that

$$P\big(N(30) = 3\big) = \frac{e^{-30/12}(30/12)^3}{3!} = 0.21.$$

12. $P\big(N(2) = 0\big) = e^{-3(2)} = e^{-6} = 0.00248.$

13. Let $N(t)$ be the number of wrong calls up to t. If one day is taken as the time unit, it is reasonable to assume that $\{N(t) : t \geq 0\}$ is a Poisson process with $\lambda = 1/7$. By the independent increment property and stationarity, the desired probability is

$$P\big(N(1) = 0\big) = e^{-(1/7)\cdot 1} = 0.87.$$

14. Choose one month as the unit of time. Then $\lambda = 5$ and the probability of no crimes during any given month of a year is $P\big(N(1) = 0\big) = e^{-5} = 0.0067$. Hence the desired probability is

$$\binom{12}{2}(0.0067)^2(1 - 0.0067)^{10} = 0.0028.$$

15. Choose one day as the unit of time. Then $\lambda = 3$ and the probability of no accidents in one day is

$$P\Big(N(1) = 0\Big) = e^{-3} = 0.0498.$$

The number of days without any accidents in January is approximately another Poisson random variable with approximate rate $31(0.05) = 1.55$. Hence the desired probability is

$$\frac{e^{-1.55}(1.55)^3}{3!} \approx 0.13.$$

16. Choosing one hours as time unit, we have that $\lambda = 6$. Therefore, the desired probability is

$$\begin{aligned} P\big(N(0.5) = 1 \text{ and } N(2.5) = 10\big) &= P\big(N(0.5) = 1 \text{ and } N(2.5) - N(0.5) = 9\big) \\ &= P\big(N(0.5) = 1\big)P\big(N(2.5) - N(0.5) = 9\big) \\ &= P\big(N(0.5) = 1\big)P\big(N(2) = 9\big) \\ &= \frac{3^1 e^{-3}}{1!} \cdot \frac{12^9 e^{-12}}{9!} \approx 0.013. \end{aligned}$$

17. The expected number of fractures per meter is $\lambda = 1/60$. Let $N(t)$ be the number of fractures in t meters of wire. Then

$$P\big(N(t) = n\big) = \frac{e^{-t/60}(t/60)^n}{n!}, \quad n = 0, 1, 2, \dots .$$

In a ten minute period, the machine turns out 70 meters of wire. The desired probability, $P\big(N(70) > 1\big)$ is calculated as follows:

$$\begin{aligned} P\big(N(70) > 1\big) &= 1 - P\big(N(70) = 0\big) - P\big(N(70) = 1\big) \\ &= 1 - e^{-70/60} - \frac{70}{60}e^{-70/60} \approx 0.325. \end{aligned}$$

18. Let the epoch at which the traffic light for the left–turn lane turns red be labeled $t = 0$. Let $N(t)$ be the number of cars that arrive at the junction at or prior to t trying to turn left. Since cars arrive at the junction according to a Poisson process, clearly, $\big\{N(t) \colon t \geq 0\big\}$ is a stationary and orderly process which possesses independent increments. Therefore, $\big\{N(t) \colon t \geq 0\big\}$ is also a Poisson process. Its parameter is given by $\lambda = E\big[N(1)\big] = 4(0.22) = 0.88$. (For a rigorous proof, see the solution to Exercise 9, Section 12.2.) Thus

$$P\big(N(t) = n\big) = \frac{e^{-(0.88)t}\big[(0.88)t\big]^n}{n!},$$

and the desired probability is

$$P\big(N(3) \geq 4\big) = 1 - \sum_{n=0}^{3} \frac{e^{-(0.88)3}\big[(0.88)3\big]^n}{n!} \approx 0.273.$$

19. Let X be the number of earthquakes of magnitude 5.5 or higher on the Richter scale during the next 60 years. Clearly, X is a Poisson random variable with parameter $\lambda = 6(1.5) = 9$. Let A be the event that the earthquakes will not damage the bridge during the next 60 years. Since the events $\{X = i\}$, $i = 0, 1, 2, \dots$, are mutually exclusive and $\bigcup_{i=1}^{\infty}\{X = i\}$ is the sample space, by the Law of Total Probability (Theorem 3.4),

$$\begin{aligned} P(A) &= \sum_{i=0}^{\infty} P(A \mid X = i)P(X = i) = \sum_{i=0}^{\infty}(1 - 0.015)^i \, \frac{e^{-9}\, 9^i}{i!} \\ &= \sum_{i=0}^{\infty}(0.985)^i \frac{e^{-9}\, 9^i}{i!} = e^{-9}\sum_{i=0}^{\infty} \frac{\big[(0.985)(9)\big]^i}{i!} = e^{-9}e^{(0.985)(9)} = 0.873716. \end{aligned}$$

20. Let N be the total number of letter carriers in America. Let n be the total number of dog bites letter carriers sustain. Let X be the number of bites a randomly selected letter carrier, say Karl, sustains on a given year. Call a bite "success," if it is Karl that is bitten and failure if anyone but Karl is bitten. Since the letter carriers are bitten randomly, it is reasonable to assume that X is approximately a binomial random variable with parameters n and $p = 1/N$. Given that n is large (it was more than 7000 in 1983 and at least 2,795 in 1997), $1/N$ is small, and n/N is moderate, X can be approximated by a Poisson random variable with parameter $\lambda = n/N$. We know that $P(X = 0) = 0.94$. This implies that $(e^{-\lambda} \cdot \lambda^0)/0! = 0.94$. Thus $e^{-\lambda} = 0.94$, and hence $\lambda = -\ln 0.94 = 0.061875$. Therefore, X is a Poisson random variable with parameter 0.061875. Now

$$P\big(X > 1 \mid X \geq 1\big) = \frac{P(X > 1)}{P(X \geq 1)} = \frac{1 - P(X = 0) - P(X = 1)}{1 - P(X = 0)}$$

$$= \frac{1 - 0.94 - 0.0581625}{1 - 0.94} = 0.030625,$$

where

$$P(X = 1) = \frac{e^{-\lambda} \cdot \lambda^1}{1!} = \lambda e^{-\lambda} = (0.061875)(0.94) = 0.0581625.$$

Therefore, approximately 3.06% of the letter carriers who sustained one bite, will be bitten again.

21. We should find n so that $1 - \dfrac{e^{-nM/N}(nM/N)^0}{0!} \geq \alpha$. This gives $n \geq -N\ln(1-\alpha)/M$. The answer is the least integer greater than or equal to $-N\ln(1-\alpha)/M$.

22. **(a)** For each k-combination $n_1, n_2, \ldots, n_k$ of $1, 2, \ldots, n$, there are $(n-1)^{n-k}$ distributions with exactly k matches, where the matches occur at $n_1, n_2, \ldots, n_k$. This is because each of the remaining $n - k$ balls can be placed into any of the cells except the cell that has the same number as the ball. Since there are $\binom{n}{k}$ k-combinations $n_1, n_2, \ldots, n_k$ of $1, 2, \ldots, n$, the total number of ways we can place the n balls into the n cells so that there are exactly k matches is $\binom{n}{k}(n-1)^{n-k}$. Hence the desired probability is $\dfrac{\binom{n}{k}(n-1)^{n-k}}{n^n}$.

(b) Let X be the number of matches. We will show that $\lim_{n\to\infty} P(X = k) = e^{-1}/k!$; that is, X is Poisson with parameter 1. We have

$$\lim_{n\to\infty} P(X = k) = \lim_{n\to\infty} \frac{\binom{n}{k}(n-1)^{n-k}}{n^n} = \lim_{n\to\infty} \binom{n}{k}\left(\frac{n-1}{n}\right)^n (n-1)^{-k}$$

$$= \lim_{n\to\infty} \frac{1}{k!} \cdot \frac{n!}{(n-k)!} \cdot \left(1 - \frac{1}{n}\right)^n \cdot \frac{1}{(n-1)^k} = \frac{1}{k!} e^{-1}.$$

Note that $\lim_{n\to\infty}\left(1-\frac{1}{n}\right)^n = e^{-1}$, and $\lim_{n\to\infty}\frac{n!}{(n-k)!\,(n-1)^k} = 1$, since by Stirling's formula,

$$\begin{aligned}
\lim_{n\to\infty}\frac{n!}{(n-k)!\,(n-1)^k} &= \lim_{n\to\infty}\frac{\sqrt{2\pi n}\cdot n^n\cdot e^{-n}}{\sqrt{2\pi(n-k)}\cdot(n-k)^{n-k}\cdot e^{-(n-k)}\cdot(n-1)^k}\\
&= \lim_{n\to\infty}\sqrt{\frac{n}{n-k}}\cdot\frac{n^n}{(n-k)^n}\cdot\frac{(n-k)^k}{(n-1)^k}\cdot\frac{1}{e^k}\\
&= 1\cdot\cdot\,e^k\cdot 1\cdot\frac{1}{e^k} = 1,
\end{aligned}$$

where $\frac{n^n}{(n-k)^n}\to e^k$ because $\frac{(n-k)^n}{n^n} = \left(1-\frac{k}{n}\right)^n \to e^{-k}$.

23. **(a)** The probability of an even number of events in $(t, t+\alpha)$ is

$$\begin{aligned}
\sum_{n=0}^{\infty}\frac{e^{-\lambda\alpha}(\lambda\alpha)^{2n}}{(2n)!} &= e^{-\lambda\alpha}\sum_{n=0}^{\infty}\frac{(\lambda\alpha)^{2n}}{(2n)!} = e^{-\alpha\lambda}\Big[\frac{1}{2}\sum_{n=0}^{\infty}\frac{(\lambda\alpha)^n}{n!}+\frac{1}{2}\sum_{n=0}^{\infty}\frac{(-\lambda\alpha)^n}{n!}\Big]\\
&= e^{-\alpha\lambda}\Big[\frac{1}{2}e^{\lambda\alpha}+\frac{1}{2}e^{-\lambda\alpha}\Big] = \frac{1}{2}(1+e^{-2\lambda\alpha}).
\end{aligned}$$

(b) The probability of an odd number of events in $(t, t+\alpha)$ is

$$\begin{aligned}
\sum_{n=1}^{\infty}\frac{e^{-\lambda\alpha}(\lambda\alpha)^{2n-1}}{(2n-1)!} &= e^{-\lambda\alpha}\sum_{n=1}^{\infty}\frac{(\lambda\alpha)^{2n-1}}{(2n-1)!} = e^{-\lambda\alpha}\Big[\frac{1}{2}\sum_{n=0}^{\infty}\frac{(\lambda\alpha)^n}{n!}-\frac{1}{2}\sum_{n=0}^{\infty}\frac{(-\lambda\alpha)^n}{n!}\Big]\\
&= e^{-\lambda\alpha}\Big[\frac{1}{2}e^{\lambda\alpha}-\frac{1}{2}e^{-\lambda\alpha}\Big] = \frac{1}{2}\big(1-e^{-2\lambda\alpha}\big).
\end{aligned}$$

24. We have that

$$\begin{aligned}
&P\big(N_1(t)=n,\ N_2(t)=m\big)\\
&\quad= \sum_{i=0}^{\infty}P\big(N_1(t)=n,\ N_2(t)=m \mid N(t)=i\big)P\big(N(t)=i\big)\\
&\quad= P\big(N_1(t)=n,\ N_2(t)=m \mid N(t)=n+m\big)P\big(N(t)=n+m\big)\\
&\quad= \binom{n+m}{n}p^n(1-p)^m\cdot\frac{e^{-\lambda t}(\lambda t)^{n+m}}{(n+m)!}.
\end{aligned}$$

Therefore,

$$P\big(N_1(t)=n\big) = \sum_{m=0}^{\infty}P\big(N_1(t)=n,\ N_2(t)=m\big)$$

$$= \sum_{m=0}^{\infty} \binom{n+m}{n} p^n (1-p)^m \cdot \frac{e^{-\lambda t}(\lambda t)^{n+m}}{(n+m)!}$$
$$= \sum_{m=0}^{\infty} \frac{(n+m)!}{n!\,m!} p^n (1-p)^m \frac{e^{-\lambda tp} e^{-\lambda t(1-p)} (\lambda t)^n (\lambda t)^m}{(n+m)!}$$
$$= \sum_{m=0}^{\infty} \frac{e^{-\lambda tp} e^{-\lambda t(1-p)} (\lambda tp)^n \big[\lambda t(1-p)\big]^m}{n!\,m!}$$
$$= \frac{e^{-\lambda tp}(\lambda tp)^n}{n!} \sum_{m=0}^{\infty} \frac{e^{-\lambda t(1-p)}\big[\lambda t(1-p)\big]^m}{m!}$$
$$= \frac{e^{-\lambda tp}(\lambda tp)^n}{n!}.$$

It can easily be argued that the other properties of Poisson process are also satisfied for the process $\{N_1(t) \colon t \geq 0\}$. So $\{N_1(t) \colon t \geq 0\}$ is a Poisson process with rate λp. By symmetry, $\{N_2(t) \colon t \geq 0\}$ is a Poisson process with rate $\lambda(1-p)$.

25. Let $N(t)$ be the number of females entering the store between 0 and t. By Exercise 24, $\{N(t) \colon t \geq 0\}$ is a Poisson process with rate $1 \cdot (2/3) = 2/3$. Hence the desired probability is

$$P\big(N(15) = 15\big) = \frac{e^{-15(2/3)}\big[15(2/3)\big]^{15}}{15!} = 0.035.$$

26. **(a)** Let A be the region whose points have a (positive) distance d or less from the given tree. The desired probability is the probability of no trees in this region and is equal to

$$\frac{e^{-\lambda\pi d^2}(\lambda\pi d^2)^0}{0!} = e^{-\lambda\pi d^2}.$$

(b) We want to find the probability that the region A has at most $n-1$ trees. The desired quantity is

$$\sum_{i=0}^{n-1} \frac{e^{-\lambda\pi d^2}(\lambda\pi d^2)^i}{i!}.$$

27. $p(i) = (\lambda/i)p(i-1)$ implies that for $i < \lambda$, the function p is increasing and for $i > \lambda$ it is decreasing. Hence $i = [\lambda]$ is the maximum.

5.3 OTHER DISCRETE RANDOM VARIABLES

1. Let D denote a defective item drawn, and N denote a nondefective item drawn. The answer is $S = \{NNN, DNN, NDN, NND, NDD, DND, DDN\}$.

2. $S = \{ss,\ fss,\ sfs,\ sffs,\ ffss,\ fsfs,\ sfffs,\ fsffs,\ fffss,\ ffsfs,\ \dots\}$.

3. **(a)** $1/(1/12) = 12$. **(b)** $\left(\frac{11}{12}\right)^2\left(\frac{1}{12}\right) \approx 0.07$.

4. **(a)** $(1 - pq)^{r-1}pq$. **(b)** $1/pq$.

5. $\binom{7}{2}(0.2)^3(0.8)^5 \approx 0.055$.

6. **(a)** $(0.55)^5(0.45) \approx 0.023$. **(b)** $(0.55)^3(0.45)(0.55)^3(0.45) \approx 0.0056$.

7. $\left[\binom{5}{1}\binom{45}{7}\right]\Big/\binom{50}{8} = 0.42$.

8. The probability that at least n light bulbs are required is equal to the probability that the first $n - 1$ light bulbs are all defective. So the answer is p^{n-1}.

9. We have

$$\frac{P(N = n)}{P(X = x)} = \frac{\dbinom{n-1}{x-1}p^x(1-p)^{n-x}}{\dbinom{n}{x}p^x(1-p)^{n-x}} = \frac{x}{n}.$$

10. Let X be the number of the words the student had to spell until spelling a word correctly. The random variable X is geometric with parameter 0.70. The desired probability is given by

$$P(X \le 4) = \sum_{i=1}^{4}(0.30)^{i-1}(0.70) = 0.9919.$$

11. The average number of digits until the fifth 3 is $5/(1/10) = 50$. So the average number of digits before the fifth 3 is 49.

12. The probability that a random bridge hand has three aces is

$$p = \frac{\dbinom{4}{3}\dbinom{48}{10}}{\dbinom{52}{13}} = 0.0412.$$

Therefore, the average number of bridge hands until one has three aces is $1/p = 1/0.0412 = 24.27$.

13. Either the $(N + 1)$st success must occur on the $(N + M - m + 1)$st trial, or the $(M + 1)$st failure must occur on the $(N+M-m+1)$st trial. The answer is

$$\binom{N+M-m}{N}\Big(\frac{1}{2}\Big)^{N+M-m+1}+\binom{N+M-m}{M}\Big(\frac{1}{2}\Big)^{N+M-m+1}.$$

14. We have that $X+10$ is negative binomial with parameters $(10, 0.15)$. Therefore, $\forall i \geq 0$,

$$P(X=i)=P(X+10=i+10)=\binom{i+9}{9}(0.15)^{10}(0.85)^{i}.$$

15. Let X be the number of good diskettes in the sample. The desired probability is

$$P(X\geq 9)=P(X=9)+P(X=10)=\frac{\dbinom{10}{1}\dbinom{90}{9}}{\dbinom{100}{10}}+\frac{\dbinom{90}{10}\dbinom{10}{0}}{\dbinom{100}{10}}\approx 0.74.$$

16. We have that $560(0.35)=196$ persons make contributions. So the answer is

$$1-\frac{\dbinom{364}{15}}{\dbinom{560}{15}}-\frac{\dbinom{364}{14}\dbinom{196}{1}}{\dbinom{560}{15}}=0.987.$$

17. The transmission of a message takes more than t minutes, if the first $[t/2]+1$ times it is sent it will be garbled, where $[t/2]$ is the greatest integer less than or equal to $t/2$. The probability of this is $p^{[t/2]+1}$.

18. The probability that the sixth coin is accepted on the nth try is

$$\binom{n-1}{5}(0.10)^{6}(0.90)^{n-6}.$$

Therefore, the desired probability is

$$\sum_{n=50}^{\infty}\binom{n-1}{5}(0.10)^{6}(0.90)^{n-6}=1-\sum_{n=6}^{49}\binom{n-1}{5}(0.10)^{6}(0.90)^{n-6}=0.6346.$$

19. The probability that the station will successfully transmit or retransmit a message is $(1-p)^{N-1}$. This is because for the station to successfully transmit or retransmit its message, none of the other stations should transmit messages at the same instance. The number of transmissions and retransmissions of a message until the success is geometric with parameter $(1-p)^{N-1}$. Therefore, on average, the number of transmissions and retransmissions is $1/(1-p)^{N-1}$.

20. If the fifth tail occurs after the 14th trial, ten or more heads have occurred. Therefore, the fifth tail occurs before the tenth head if and only if the fifth tail occurs before or on the 14th flip. Calling tails success, X, the number of flips required to get the fifth tail is negative binomial with parameters 5 and $1/2$. The desired probability is given by

$$\sum_{n=5}^{14} P(X=n) = \sum_{n=5}^{14} \binom{n-1}{4} \Big(\frac{1}{2}\Big)^5 \Big(\frac{1}{2}\Big)^{n-5} \approx 0.91.$$

21. The probability of a straight is

$$\frac{10(4^5) - 40}{\binom{52}{5}} = 0.003924647.$$

Therefore, the expected number of poker hands required until the first straight is $1/0.003924647 = 254.80$.

22. **(a)** Since

$$\frac{P(X=n-1)}{P(X=n)} = \frac{1}{1-p} > 1,$$

$P(X=n)$ is a decreasing function of n; hence its maximum is at $n=1$.

(b) The probability that X is even is given by

$$\sum_{k=1}^{\infty} P(X=2k) = \sum_{k=1}^{\infty} p(1-p)^{2k-1} = \frac{p(1-p)}{1-(1-p)^2} = \frac{1-p}{2-p}.$$

(c) We want to show the following:

Let X be a discrete random variable with the set of possible values $\{1, 2, 3 \dots\}$. If for all positive integers n and m,

$$P(X > n+m \mid X > m) = P(X > n), \tag{17}$$

then X is a geometric random variable. That is, there exists a number p, $0 < p < 1$, such that

$$P(X=n) = p(1-p)^{n-1}. \tag{18}$$

To prove this, note that (17) implies that for all positive integers n and m,

$$\frac{P(X > n+m)}{P(X > m)} = P(X > n).$$

Therefore,

$$P(X > n+m) = P(X > n)P(X > m). \tag{19}$$

Let $p = P(X = 1)$; using induction, we prove that (18) is valid for all positive integers n. To show (18) for $n = 2$, note that (19) implies that

$$P(X > 2) = P(X > 1)P(X > 1).$$

Since $P(X > 1) = 1 - P(X = 1) = 1 - p$, this relation gives

$$1 - P(X = 1) - P(X = 2) = (1 - p)^2,$$

or

$$1 - p - P(X = 2) = (1 - p)^2,$$

which yields

$$P(X = 2) = p(1 - p),$$

so (18) is also true for $n = 2$. Now assume that (18) is valid for all positive integers i, $i \leq n$; that is, assume that

$$P(X = i) = p(1 - p)^{i-1}, \quad i \leq n. \tag{20}$$

We will show that (18) is true for $n + 1$. The induction hypothesis [relation (20)] implies that

$$P(X \leq n) = \sum_{i=1}^{n} P(X = i) = \sum_{i=1}^{n} p(1 - p)^{i-1} = p\frac{1 - (1 - p)^n}{1 - (1 - p)} = 1 - (1 - p)^n.$$

So $P(X > n) = (1 - p)^n$ and, similarly, $P(X > n - 1) = (1 - p)^{n-1}$. Now (19) yields

$$P(X > n + 1) = P(X > n)P(X > 1),$$

which implies that

$$1 - P(X \leq n) - P(X = n + 1) = (1 - p)^n(1 - p).$$

Substituting $P(X \leq n) = 1 - (1 - p)^n$ in this relation, we obtain

$$P(X = n + 1) = p(1 - p)^n,$$

which establishes (18) for $n + 1$. Therefore, we have what we wanted to show.

23. Consider a coin for which the probability of tails is $1 - p$ and the probability of heads is p. In successive and independent flips of the coin, let X_1 be the number of flips until the first head, X_2 be the total number of flips until the second head, X_3 be the total number of flips until the third head, and so on. Then the length of the first character of the message and X_1 are identically distributed. The total number of the bits forming the first two characters of the message and X_2 are identically distributed. The total number of the bits forming the first three characters of the message and X_3 are identically distributed, and so on. Therefore, the total number of the bits forming the message has the same distribution as X_k. This is negative binomial with parameters k and p.

24. Let X be the number of cartons to be opened before finding one without rotten eggs. X is *not* a geometric random variable because the number of cartons is limited, and one carton not having rotten eggs is *not* independent of another carton not having rotten eggs. However, it should be obvious that a geometric random variable with parameter $p = \binom{1000}{12} \Big/ \binom{1200}{12} = 0.1109$ is a good approximation for X. Therefore, we should expect *approximately* $1/p = 1/0.1109 = 9.015$ cartons to be opened before finding one without rotten eggs.

25. Either the Nth success should occur on the $(2N - M)$th trial or the Nth failure should occur on the $(2N - M)$th trial. By symmetry, the answer is

$$2 \cdot \binom{2N - M - 1}{N - 1} \Big(\frac{1}{2}\Big)^N \Big(\frac{1}{2}\Big)^{N-M} = \binom{2N - M - 1}{N - 1} \Big(\frac{1}{2}\Big)^{2N-M-1}.$$

26. The desired quantity is 2 times the probability of exactly N successes in $(2N - 1)$ trials and failures on the $(2N)$th and $(2N + 1)$st trials:

$$2\binom{2N - 1}{N} \Big(\frac{1}{2}\Big)^N \Big(1 - \frac{1}{2}\Big)^{(2N-1)-N} \cdot \Big(1 - \frac{1}{2}\Big)^2 = \binom{2N - 1}{N} \Big(\frac{1}{2}\Big)^{2N}.$$

27. Let X be the number of rolls until Adam gets a six. Let Y be the number of rolls of the die until Andrew rolls an odd number. Since the events $(X = i)$, $1 \le i < \infty$, form a partition of the sample space, by Theorem 3.4,

$$P\big(Y > X\big) = \sum_{i=1}^{\infty} P\big(Y > X \mid X = i\big) P\big(X = i\big) = \sum_{i=1}^{\infty} P\big(Y > i\big) P\big(X = i\big)$$

$$= \sum_{i=1}^{\infty} \Big(\frac{1}{2}\Big)^i \cdot \Big(\frac{5}{6}\Big)^{i-1} \frac{1}{6} = \frac{6}{5} \cdot \frac{1}{6} \sum_{i=1}^{\infty} \Big(\frac{5}{12}\Big)^i = \frac{1}{5} \cdot \frac{\frac{5}{12}}{1 - \frac{5}{12}} = \frac{1}{7},$$

where $P(Y > i) = (1/2)^i$ since for Y to be greater than i, Andrew must obtain an even number on each of the the first i rolls.

28. The probability of 4 tagged trout among the second 50 trout caught is

$$p_n = \frac{\binom{50}{4}\binom{n - 50}{46}}{\binom{n}{50}}.$$

It is logical to find the value of n for which p_n is maximum. (In statistics this value is called the *maximum likelihood estimate* for the number of trout in the lake.) To do this, note that

$$\frac{p_n}{p_{n-1}} = \frac{(n - 50)^2}{n(n - 96)}.$$

Now $p_n \geq p_{n-1}$ if and only if $(n-50)^2 \geq n(n-96)$, or $n \leq 625$. Therefore, $n = 625$ makes p_n maximum, and hence there are approximately 625 trout in the lake.

29. **(a)** Intuitively, it should be clear that the answer is D/N. To prove this, let E_j be the event of obtaining exactly j defective items among the first $(k-1)$ draws. Let A_k be the event that the kth item drawn is defective. We have

$$P(A_k) = \sum_{j=0}^{k-1} P(A_k \mid E_j)P(E_j) = \sum_{j=0}^{k-1} \frac{D-j}{N-k+1} \cdot \frac{\binom{D}{j}\binom{N-D}{k-1-j}}{\binom{N}{k-1}}.$$

Now

$$(D-j)\binom{D}{j} = D\binom{D-1}{j}$$

and

$$(N-k+1)\binom{N}{k-1} = N\binom{N-1}{k-1}.$$

Therefore,

$$P(A_k) = \sum_{j=0}^{k-1} \frac{D\binom{D-1}{j}\binom{N-D}{k-1-j}}{N\binom{N-1}{k-1}} = \frac{D}{N} \sum_{j=0}^{k-1} \frac{\binom{D-1}{j}\binom{N-D}{k-1-j}}{\binom{N-1}{k-1}} = \frac{D}{N},$$

where

$$\sum_{j=0}^{k-1} \frac{\binom{D-1}{j}\binom{N-D}{k-1-j}}{\binom{N-1}{k-1}} = 1$$

since $\dfrac{\binom{D-1}{j}\binom{N-D}{k-1-j}}{\binom{N-1}{k-1}}$ is the probability mass function of a hypergeometric random variable with parameters $N-1$, $D-1$, and $k-1$.

(b) Intuitively, it should be clear that the answer is $(D-1)/(N-1)$. To prove this, let A_k be as before and let F_j be the event of exactly j defective items among the first $(k-2)$ draws. Let B be the event that the $(k-1)$st and the kth items drawn are defective. We have

$$P(B) = \sum_{j=0}^{k-2} P(B \mid F_j)P(F_j)$$

$$= \sum_{j=0}^{k-2} \frac{(D-j)(D-j-1)}{(N-k+2)(N-k+1)} \cdot \frac{\binom{D}{j}\binom{N-D}{k-2-j}}{\binom{N}{k-2}}$$

$$= \sum_{j=0}^{k-2} \frac{D(D-1)\binom{D-2}{j}\binom{N-D}{k-2-j}}{N(N-1)\binom{N-2}{k-2}}$$

$$= \frac{D(D-1)}{N(N-1)} \sum_{j=0}^{k-2} \frac{\binom{D-2}{j}\binom{N-D}{k-2-j}}{\binom{N-2}{k-2}}$$

$$= \frac{D(D-1)}{N(N-1)}.$$

Using this, we have that the desired probability is

$$P(A_k \mid A_{k-1}) = \frac{P(A_k A_{k-1})}{P(A_{k-1})} = \frac{P(B)}{P(A_{k-1})} = \frac{\dfrac{D(D-1)}{N(N-1)}}{\dfrac{D}{N}} = \frac{D-1}{N-1}.$$

REVIEW PROBLEMS FOR CHAPTER 5

1. $\displaystyle\sum_{i=12}^{20} \binom{20}{i} (0.25)^i (0.75)^{20-i} = 0.0009.$

2. $N(t)$, the number of customers arriving at the post office at or prior to t is a Poisson process with $\lambda = 1/3$. Thus

$$P\big(N(30) \leq 6\big) = \sum_{i=0}^{6} P\big(N(30) = i\big) = \sum_{i=0}^{6} \frac{e^{-(1/3)30}\big[(1/3)30\big]^i}{i!} = 0.130141.$$

3. $4 \cdot \dfrac{8}{30} = 1.067.$

4. $\displaystyle\sum_{i=0}^{2} \binom{12}{i} (0.30)^i (0.70)^{12-i} = 0.253.$

5. $\binom{5}{2}(0.18)^2(0.82)^3 = 0.179.$

6. $\displaystyle\sum_{i=2}^{1999}\binom{i-1}{2-1}\Big(\frac{1}{1000}\Big)^2\Big(\frac{999}{1000}\Big)^{i-2} = 0.59386.$

7. $\displaystyle\sum_{i=7}^{12}\frac{\binom{160}{i}\binom{200}{12-i}}{\binom{360}{12}} = 0.244.$

8. Call a train that arrives between 10:15 A.M. and 10:28 A.M. a success. Then p, the probability of success is

$$p = \frac{28-15}{60} = \frac{13}{60}.$$

Therefore, the expected value and the variance of the number of trains that arrive in the given period are $10(13/60) = 2.167$ and $10(13/60)(47/60) = 1.697$, respectively.

9. The number of checks returned during the next two days is Poisson with $\lambda = 6$. The desired probability is

$$P(X \le 4) = \sum_{i=0}^{4}\frac{e^{-6}\,6^i}{i!} = 0.285.$$

10. Suppose that 5% of the items are defective. Under this hypothesis, there are $500(0.05) = 25$ defective items. The probability of two defective items among 30 items selected at random is

$$\frac{\binom{25}{2}\binom{475}{28}}{\binom{500}{30}} = 0.268.$$

Therefore, under the above hypothesis, having two defective items among 30 items selected at random is quite probable. The shipment should not be rejected.

11. N is a geometric random variable with $p = 1/2$. So $E(N) = 1/p = 2$, and $\text{Var}(N) = (1-p)/p^2 = \big[1-(1/2)\big]/(1/4) = 2.$

12. $\Big(\frac{5}{6}\Big)^5\Big(\frac{1}{6}\Big) = 0.067.$

13. The number of times a message is transmitted or retransmitted is geometric with parameter $1-p$. Therefore, the expected value of the number of transmissions and retransmissions of a message is $1/(1-p)$. Hence the expected number of retransmissions of a message is

$$\frac{1}{1-p} - 1 = \frac{p}{1-p}.$$

14. Call a customer a "success," if he or she will make a purchase using a credit card. Let E be the event that a customer entering the store will make a purchase. Let F be the event that the customer will use a credit card. To find p, the probability of success, we use the law of multiplication:

$$p = P(EF) = P(E)P\big(F \mid E\big) = (0.30)(0.85) = 0.255.$$

The random variable X is binomial with parameters 6 and 0.255. Hence

$$P\big(X = i\big) = \binom{6}{i}\big(0.255\big)^i\big(1 - 0.255\big)^{6-i}, \quad i = 0, 1, \ldots, 6.$$

Clearly, $E(X) = np = 6(0.255) = 1.53$ and

$$\text{Var}(X) = np(1-p) = 6(0.255)(1 - 0.255) = 1.13985.$$

15. $\displaystyle\sum_{i=3}^{5} \frac{\binom{18}{i}\binom{10}{5-i}}{\binom{28}{5}} = 0.772.$

16. By the formula for the expected value of a hypergeometric random variable, the desired quantity is $(5 \times 6)/16 = 1.875$.

17. We want to find the probability that at most 4 of the seeds do not germinate:

$$\sum_{i=0}^{4} \binom{40}{i}(0.06)^i(0.94)^{40-i} = 0.91.$$

18. $\displaystyle 1 - \sum_{i=0}^{2} \binom{20}{i}(0.06)^i(0.94)^{20-i} = 0.115.$

Let X be the number of requests for reservations at the end of the second day. It is reasonable to assume that X is Poisson with parameter $3 \times 3 \times 2 = 18$. Hence the desired probability is

$$P(X \geq 24) = 1 - \sum_{i=0}^{23} P(X = i) = 1 - \sum_{i=0}^{23} \frac{e^{-18}\,(18)^i}{i!} = 1 - 0.89889 = 0.10111.$$

19. Suppose that the company's claim is correct. Then the probability of 12 or less drivers using seat belts regularly is

$$\sum_{i=0}^{12} \binom{20}{i} (0.70)^i (0.30)^{20-i} \approx 0.228.$$

Therefore, under the assumption that the company's claim is true, it is quite likely that out of 20 randomly selected drivers, 12 use seat belts. This is not a reasonable evidence to conclude that the insurance company's claim is false.

20. **(a)** $(0.999)^{999}(0.001)^1 = 0.000368.$ **(b)** $\binom{2999}{2}(0.001)^3(0.999)^{2997} = 0.000224.$

21. Let X be the number of children having the disease. We have that the desired probability is

$$P(X = 3 \mid X \geq 1) = \frac{P(X = 3)}{P(X \geq 1)} = \frac{\binom{5}{3}(0.23)^3(0.77)^2}{1 - (0.77)^5} = 0.0989.$$

22. **(a)** $\left(\frac{w}{w+b}\right)^{n-1}\left(\frac{b}{w+b}\right).$ **(b)** $\left(\frac{w}{w+b}\right)^{n-1}.$

23. Let n be the desired number of seeds to be planted. Let X be the number of seeds which will germinate. We have that X is binomial with parameters n and 0.75. We want to find the smallest n for which

$$P(X \geq 5) \geq 0.90.$$

or, equivalently,

$$P(X < 5) \leq 0.10.$$

That is, we want to find the smallest n for which

$$\sum_{i=0}^{4} \binom{n}{i} (0.75)^i (.25)^{n-i} \leq 0.10.$$

By trial and error, as the following table shows, we find that the smallest n satisfying $P(X < 5) \leq 0.10$ is 9. So at least nine seeds is to be planted.

n	$\sum_{i=0}^{4} \binom{n}{i} (0.75)^i (.25)^{n-i}$
5	0.7627
6	0.4661
7	0.2436
8	0.1139
9	0.0489

24. Intuitively, it must be clear that the answer is k/n. To prove this, let B be the event that the ith baby born is blonde. Let A be the event that k of the n babies are blondes. We have

$$P(B \mid A) = \frac{P(AB)}{P(A)} = \frac{p \cdot \binom{n-1}{k-1} p^{k-1}(1-p)^{n-k}}{\binom{n}{k} p^k (1-p)^{n-k}} = \frac{\binom{n-1}{k-1}}{\binom{n}{k}} = \frac{k}{n}.$$

25. The size of a seed is a tiny fraction of the size of the area. Let us divide the area up into many small cells each about the size of a seed. Assume that, when the seeds are distributed, each of them will land in a single cell. Accordingly, the number of seeds distributed will equal the number of nonempty cells. Suppose that each cell has an equal chance of having a seed independent of other cells (this is only approximately true). Since λ is the average number of seeds per unit area, the expected number of seeds in the area, A, is λA. Let us call a cell in A a "success" if it is occupied by a seed. Let n be the total number of cells in A and p be the probability that a cell will contain a seed. Then X, the number of cells in A with seeds is a binomial random variable with parameters n and p. Using the formula for the expected number of successes in a binomial distribution ($= np$), we see that $np = \lambda A$ and $p = \lambda A/n$. As n goes to infinity, p approaches zero while np remains finite. Hence the number of seeds that fall on the area A is a Poisson random variable with parameter λA and

$$P(X = i) = \frac{e^{-\lambda A}(\lambda A)^i}{i!}.$$

26. Let $D/N \to p$, then by the Remark 5.2, for all n,

$$\frac{\binom{D}{x}\binom{N-D}{n-x}}{\binom{N}{n}} \approx \binom{n}{x} p^x (1-p)^{n-x}.$$

Now since $n \to \infty$ and $nD/N \to \lambda$, n is large and np is appreciable, thus

$$\binom{n}{x} p^x (1-p)^{n-x} \approx \frac{e^{-\lambda}\lambda^x}{x!}.$$

Chapter 6

Continuous Random Variables

6.1 PROBABILITY DENSITY FUNCTIONS

1. **(a)** $\int_0^\infty ce^{-3x}\,dx = 1 \Longrightarrow c = 3.$

(b) $P(0 < X \leq 1/2) = \int_0^{1/2} 3e^{-3x}\,dx = 1 - e^{-3/2} \approx 0.78.$

2. **(a)** $f(x) = \begin{cases} \dfrac{32}{x^3} & x \geq 4 \\ 0 & x < 4. \end{cases}$

(b) $P(X \leq 5) = 1 - (16/25) = 9/25,$
$P(X \geq 6) = 16/36 = 4/9,$
$P(5 \leq X \leq 7) = \big[1 - (16/49)\big] - \big[1 - (16/25)\big] = 0.313,$
$P(1 \leq X < 3.5) = 0 - 0 = 0.$

3. **(a)** $\int_1^2 c(x-1)(2-x)\,dx = 1 \Longrightarrow c\Big[-\frac{x^3}{3} + \frac{3x^2}{2} - 2x\Big]_1^2 = 1 \Longrightarrow c = 6.$

(b) $F(x) = \int_1^x 6(x-1)(2-x)\,dx,\quad 1 \leq x < 2.$ Thus

$$F(x) = \begin{cases} 0 & x < 1 \\ -2x^3 + 9x^2 - 12x + 5 & 1 \leq x < 2 \\ 1 & x \geq 2. \end{cases}$$

(c) $P(X < 5/4) = F(5/4) = 5/32,$
$P(3/2 \leq X \leq 2) = F(2) - F(3/2) = 1 - (1/2) = 1/2.$

4. **(a)** $P(X < 1.5) = \int_1^{1.5} \frac{2}{x^2}\,dx = \frac{2}{3}.$

(b) $$P(1 < X < 1.25 \mid X < 1.5) = \frac{\displaystyle\int_1^{1.25} \frac{2}{x^2}\,dx}{\displaystyle\int_1^{1.5} \frac{2}{x^2}\,dx} = \frac{2/5}{2/3} = \frac{3}{5}.$$

5. **(a)** $\displaystyle\int_{-1}^{1} \frac{c}{\sqrt{1-x^2}}\,dx = 1 \Longrightarrow \Big[c \cdot \arcsin x\Big]_{-1}^{1} = 1 \Longrightarrow c = 1/\pi.$

(b) For $-1 < x < 1$,

$$F(x) = \int_{-1}^{x} \frac{1}{\pi\sqrt{1-x^2}}\,dx = \frac{1}{\pi}\arcsin x + \frac{1}{2}.$$

Thus

$$F(x) = \begin{cases} 0 & x < -1 \\ \dfrac{1}{\pi}\arcsin x + \dfrac{1}{2} & -1 \le x < 1 \\ 1 & x \ge 1. \end{cases}$$

6. Since $h(x) \ge 0$ and

$$\int_{\alpha}^{\infty} \frac{f(x)}{1-F(\alpha)}\,dx = \frac{1}{1-F(\alpha)}\int_{\alpha}^{\infty} f(x)\,dx = \frac{1}{1-F(\alpha)}\big[1-F(\alpha)\big] = 1,$$

h is a probability density function.

7. **(a)** Let F be the distribution function of X. Then X is symmetric about α if and only if for all x, $1 - F(\alpha + x) = F(\alpha - x)$, or upon differentiation $f(\alpha + x) = f(\alpha - x)$.
(b) $f(\alpha + x) = f(\alpha - x)$ if and only if $(\alpha - x - 3)^2 = (\alpha + x - 3)^2$. This is true for all x, if and only if $\alpha - x - 3 = -(\alpha + x - 3)$ which gives $\alpha = 3$. A similar argument shows that g is symmetric about $\alpha = 1$.

8. **(a)** Since f is a probability density function, $\displaystyle\int_{-\infty}^{\infty} f(x)\,dx = 1$. But

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-1}^{0} k(2x - 3x^2)\,dx = k\int_{-1}^{0} (2x - 3x^2)\,dx = k\Big[x^2 - x^3\Big]_{-1}^{0} = -2k.$$

So $-2k = 1$ or $k = -1/2$.

(b) The loss is at most \$500 if and only if $X \ge -1/2$. Therefore, the desired probability is

$$P\Big(X \ge -\frac{1}{2}\Big) = \int_{-1/2}^{0} -\frac{1}{2}(2x - 3x^2)\,dx = -\frac{1}{2}\Big[x^2 - x^3\Big]_{-1/2}^{0} = \frac{3}{16}.$$

9. $P(X > 15) = \int_{15}^{\infty} \frac{1}{15} e^{-x/15}\,dx = \frac{1}{e}$. Thus the answer is

$$\sum_{i=4}^{8} \binom{8}{i} \Big(\frac{1}{e}\Big)^i \Big(1 - \frac{1}{e}\Big)^{8-i} = 0.3327.$$

10. Since $\alpha f + \beta g \geq 0$ and

$$\int_{-\infty}^{\infty} \big[\alpha f(x) + \beta g(x)\big]\,dx = \alpha \int_{-\infty}^{\infty} f(x)\,dx + \beta \int_{-\infty}^{\infty} g(x)\,dx = \alpha + \beta = 1,$$

$\alpha f + \beta g$ is also a probability density function.

11. Since $F(-\infty) = 0$ and $F(\infty) = 1$, We have that

$$\begin{cases} \alpha + \beta(-\pi/2) = 0 \\ \alpha + \beta(\pi/2) = 1. \end{cases}$$

Solving this system of two equations in two unknown, we obtain $\alpha = 1/2$ and $\beta = 1/\pi$. Thus

$$f(x) = F'(x) = \frac{2}{\pi(4 + x^2)}, \quad -\infty < x < \infty.$$

6.2 DENSITY FUNCTION OF A FUNCTION OF A RANDOM VARIABLE

1. Let G be the distribution function of Y; for $-8 < y < 8$,

$$G(y) = P(Y \leq y) = P(X^3 \leq y) = P(X \leq \sqrt[3]{y}\,) = \int_{-2}^{\sqrt[3]{y}} \frac{1}{4}\,dx = \frac{1}{4}\sqrt[3]{y} + \frac{1}{2}.$$

Therefore,

$$G(y) = \begin{cases} 0 & y < -8 \\ \frac{1}{4}\sqrt[3]{y} + \frac{1}{2} & -8 \leq y < 8 \\ 1 & y \geq 8. \end{cases}$$

This gives

$$g(y) = G'(y) = \begin{cases} \frac{1}{12} y^{-2/3} & -8 < y < 8 \\ 0 & \text{otherwise.} \end{cases}$$

Let H be the distribution function of Z; for $0 \le z < 16$,

$$H(z) = P(X^4 \le z) = P(-\sqrt[4]{z} \le x \le \sqrt[4]{z}\,) = \int_{-\sqrt[4]{z}}^{\sqrt[4]{z}} \frac{1}{4}\,dx = \frac{1}{2}\sqrt[4]{z}.$$

Thus

$$H(z) = \begin{cases} 0 & z < 0 \\ \dfrac{1}{2}\sqrt[4]{z} & 0 \le z < 16 \\ 1 & z \ge 16. \end{cases}$$

This gives

$$h(z) = H'(z) = \begin{cases} \dfrac{1}{8}z^{-3/4} & 0 < z < 16 \\ 0 & \text{otherwise.} \end{cases}$$

2. Let G be the probability distribution function of Y and g be its probability density function. For $t > 0$,

$$G(t) = P\big(e^X \le t\big) = P(X \le \ln t) = F(\ln t).$$

For $t \le 0$, $G(t) = 0$. Therefore,

$$g(t) = G'(t) = \begin{cases} \dfrac{1}{t} f(\ln t) & t > 0 \\ 0 & t \le 0. \end{cases}$$

3. The set of possible values of X is $A = (0, \infty)$. Let $h\colon (0, \infty) \to \mathbf{R}$ be defined by $h(x) = x\sqrt{x}$. The set of possible values of h is $B = (0, \infty)$. The inverse of h is g, where $g(y) = y^{2/3}$. Thus $g'(y) = 2/(3\sqrt[3]{y}\,)$ and hence

$$f_Y(y) = \frac{2}{3\sqrt[3]{y}} e^{-y^{2/3}}, \quad y \in (0, \infty).$$

To find the probability density function of e^{-X}, let $h\colon (0, \infty) \to \mathbf{R}$ be defined by $h(x) = e^{-x}$; h is an invertible function with the set of possible values $B = (0, 1)$. The inverse of h is $g(z) = -\ln z$. So $g'(z) = -1/z$. Therefore,

$$f_Z(z) = e^{-(-\ln z)}\left| -\frac{1}{z}\right| = z \cdot \frac{1}{z} = 1, \quad z \in (0, 1);$$

0, otherwise.

4. The set of possible values of X is $A = (0, \infty)$. Let $h\colon (0, \infty) \to \mathbf{R}$ be defined by $h(x) = \log_2 x$. The set of possible values of h is $B = (-\infty, \infty)$. h is invertible and its inverse is $g(y) = 2^y$, where $g'(y) = (\ln 2)2^y$. Thus

$$f_Y(y) = 3e^{-3\left(2^y\right)}\left|(\ln 2)2^y\right| = (3\ln 2)2^y e^{-3(2^y)}, \quad y \in (-\infty, \infty).$$

5. Let G and g be the probability distribution and the probability density functions of Y, respectively. Then

$$\begin{aligned} G(y) &= P(Y \le y) = P\left(\sqrt[3]{X^2} \le y\right) = P(X \le y\sqrt{y}\,) \\ &= \int_0^{y\sqrt{y}} \lambda e^{-\lambda x}\,dx = 1 - e^{-\lambda y\sqrt{y}}, \quad y \in [0, \infty). \end{aligned}$$

So

$$g(y) = G'(y) = \frac{3\lambda}{2}\sqrt{y}e^{-\lambda y\sqrt{y}}, \quad y \ge 0;$$

0, otherwise.

6. Let G and g be the probability distribution and density functions of X^2, respectively. For $t \ge 0$,

$$G(t) = P(X^2 \le t) = P(-\sqrt{t} < X < \sqrt{t}\,) = F(\sqrt{t}\,) - F(-\sqrt{t}\,).$$

Thus

$$g(t) = G'(t) = \frac{1}{2\sqrt{t}}\,f(\sqrt{t}\,) + \frac{1}{2\sqrt{t}}\,f(-\sqrt{t}\,) = \frac{1}{2\sqrt{t}}\Big[f(\sqrt{t}\,) + f(-\sqrt{t}\,)\Big], \quad t \ge 0.$$

For $t < 0$, $g(t) = 0$.

7. Let G and g be the distribution and density functions of Z, respectively. For $-\pi/2 < z < \pi/2$,

$$\begin{aligned} G(z) &= P(\arctan X \le z) = P(X \le \tan z) = \int_{-\infty}^{\tan z} \frac{1}{\pi(1 + x^2)}\,dx \\ &= \left[\frac{1}{\pi}\arctan x\right]_{-\infty}^{\tan z} = \frac{1}{\pi}z + \frac{1}{2}. \end{aligned}$$

Thus

$$g(z) = \begin{cases} \dfrac{1}{\pi} & -\dfrac{\pi}{2} < z < \dfrac{\pi}{2} \\ 0 & \text{elsewhere.} \end{cases}$$

8. Let G and g be distribution and density functions of Y, respectively. Then

$$\begin{aligned} G(t) &= P(Y \le t) = P(Y \le t \mid X \le 1)P(X \le 1) + P(Y \le t \mid X > 1)P(X > 1) \\ &= P(X \le t \mid X \le 1)P(X \le 1) + P\Big(X \ge \frac{1}{t} \;\Big|\; X > 1\Big)P(X > 1). \end{aligned}$$

For $t \geq 1$, this gives

$$G(t) = 1 \cdot \int_0^1 e^{-x}\,dx + 1 \cdot \int_1^\infty e^{-x}\,dx = 1.$$

For $0 < t < 1$, this gives

$$G(t) = P(X \leq t) + P\Big(X \geq \frac{1}{t}\Big) = \int_0^t e^{-x}\,dx + \int_{1/t}^\infty e^{-x}\,dx = 1 - e^{-t} + e^{-1/t}.$$

Hence

$$G(t) = \begin{cases} 0 & t \leq 0 \\ 1 - e^{-t} + e^{-1/t} & 0 < t < 1 \\ 1 & t \geq 1. \end{cases}$$

Therefore,

$$g(t) = G'(t) = \begin{cases} e^{-t} + \dfrac{1}{t^2} e^{-1/t} & 0 < t < 1 \\ 0 & \text{elsewhere.} \end{cases}$$

6.3 EXPECTATIONS AND VARIANCES

1. The probability density function of X is $f(x) = \begin{cases} 32/x^3 & x \geq 4 \\ 0 & x < 4. \end{cases}$ Thus

(a) $E(X) = \displaystyle\int_4^\infty \frac{32}{x^2}\,dx = 8.$

(b) $E(X^2) = \displaystyle\int_4^\infty \frac{32}{x}\,dx = \infty$; so $\text{Var}(X) = E(X^2) - \big[E(X)\big]^2$ does not exist.

2. **(a)** $E(X) = 6\displaystyle\int_1^2 (-x^3 + 3x^2 - 2x)\,dx = \frac{3}{2}.$

(b) $E(X^2) = 6\displaystyle\int_1^2 (-x^4 + 3x^3 - 2x^2)\,dx = \frac{23}{10}$; so $\text{Var}(X) = \dfrac{23}{10} - \dfrac{9}{4} = \dfrac{1}{20}$, and $\sigma_X = \dfrac{1}{\sqrt{20}}$.

3. The standardized value of the lifetime of a car muffler manufactured by company A is $(4.25-5)/2 = -0.375$. The corresponding value for company B is $(3.75-4)/1.5 = -0.167$. Therefore, the muffler of company B has performed relatively better.

4. $E\big(e^X\big) = \displaystyle\int_0^\infty e^x (3e^{-3x})\,dx = \int_0^\infty 3e^{-2x}\,dx = 3/2.$

5. $E(X) = \int_{-1}^{1} \frac{x}{\pi\sqrt{1-x^2}}\,dx = 0$, because the integrand is an odd function.

6. Let f be the probability density function of Y. Clearly,

$$f(y) = F^{'}(y) = \begin{cases} \frac{k}{A}e^{-k(\alpha-y)/A} & -\infty < y \leq \alpha \\ \\ 0 & y > \alpha. \end{cases}$$

Therefore,

$$E(Y) = \int_{-\infty}^{\alpha} \frac{k}{A} y e^{-k(\alpha-y)/A}\,dy = \frac{k}{A}e^{-k\alpha/A}\Big[\frac{A}{k} y e^{ky/A} - \frac{A^2}{k^2}e^{ky/A}\Big]_{-\infty}^{\alpha} = \alpha - \frac{A}{k}.$$

7. Let H be the distribution function of C; then

$$P(F \leq t) = P\Big(C \leq \frac{t-32}{1.8}\Big) = H\Big(\frac{t-32}{1.8}\Big).$$

Hence the probability density function of F is

$$\frac{d}{dt}P(F \leq t) = \frac{1}{1.8}h\Big(\frac{t-32}{1.8}\Big) = \frac{5}{9}h\Big(\frac{t-32}{1.8}\Big).$$

The expected value of F is given by

$$E(F) = 1.8E(C) + 32 = 1.8\int_{-\infty}^{\infty} xh(x)\,dx + 32.$$

8. $E(\ln X) = \int_1^2 \frac{2\ln x}{x^2}\,dx$. To calculate this integral, let $U = \ln x$, $dV = 1/x^2$, and use integration by parts:

$$\int_1^2 \frac{2\ln x}{x^2}\,dx = -\frac{2\ln x}{x}\Big|_1^2 - \int_1^2 -\frac{2}{x^2}\,dx = 1 - \ln 2 = 0.3069.$$

9. The expected value of the length of the other side is given by

$$E\big(\sqrt{81 - X^2}\big) = \int_2^4 \sqrt{81 - x^2} \cdot \frac{x}{6}\,dx.$$

Letting $u = 81 - x^2$, we get $du = -2x\,dx$ and

$$E\big(\sqrt{81 - X^2}\big) = \frac{1}{12}\int_{65}^{77} \sqrt{u}\,du \approx 8.4.$$

10. $E(X)=\int_{-\infty}^{\infty}\frac{1}{2}xe^{-|x|}\,dx=0$, because the integrand is an odd function. Now

$$E(X^2)=\int_{-\infty}^{\infty}\frac{1}{2}x^2e^{-|x|}\,dx=\int_0^{\infty}x^2e^{-x}\,dx$$

since the integrand is an even function; applying integration by parts to the last integral twice, we obtain $E(X^2)=2$. Hence $\text{Var}(X)=2-0^2=2$.

11. Note that

$$E\big(|X|^{\alpha}\big)=\int_{-\infty}^{\infty}\frac{|x|^{\alpha}}{\pi(1+x^2)}\,dx=\frac{2}{\pi}\int_0^{\infty}\frac{x^{\alpha}}{(1+x^2)}\,dx$$

since the integrand is an even function. Now for $0<\alpha<1$,

$$\int_0^{\infty}\frac{x^{\alpha}}{1+x^2}\,dx=\int_0^{1}\frac{x^{\alpha}}{1+x^2}\,dx+\int_1^{\infty}\frac{x^{\alpha}}{1+x^2}\,dx.$$

Clearly, the first integral in the right side is convergent. To show that the second one is also convergent, note that.

$$\frac{x^{\alpha}}{1+x^2}\le\frac{x^{\alpha}}{x^2}=\frac{1}{x^{2-\alpha}}.$$

Therefore,

$$\int_1^{\infty}\frac{x^{\alpha}}{1+x^2}\,dx\le\int_1^{\infty}\frac{1}{x^{2-\alpha}}\,dx=\Big[\frac{1}{(\alpha-1)x^{1-\alpha}}\Big]_1^{\infty}=\frac{1}{1-\alpha}<\infty.$$

For $\alpha\ge 1$,

$$\int_0^{\infty}\frac{x^{\alpha}}{1+x^2}\ge\int_1^{\infty}\frac{x^{\alpha}}{1+x^2}\,dx\ge\int_1^{\infty}\frac{x}{1+x^2}\,dx=\Big[\frac{1}{2}\ln(1+x^2)\Big]_1^{\infty}=\infty.$$

So $\int_0^{\infty}\frac{x^{\alpha}}{1+x^2}\,dx$ diverges.

12. By Remark 6.4,

$$E(X)=\int_0^{\infty}P(X>t)\,dt=\int_0^{\infty}(\alpha e^{-\lambda t}+\beta e^{-\mu t})\,dt=\frac{\alpha}{\lambda}+\frac{\beta}{\mu}.$$

13. **(a)** c_1 is an arbitrary positive number because $\forall c_1$, $\int_{c_1}^{\infty}\frac{c_1}{x^2}\,dx=1$. For $n>1$, $\int_{c_n}^{\infty}\frac{c_n}{x^{n+1}}\,dx=1$ implies that $c_n=n^{-1/(n-1)}$.

(b) $E(X_n)=\int_{c_n}^{\infty}\frac{c_n}{x^n}\,dx=\begin{cases}\infty & \text{if } n=1\\ n^{(n-2)/(n-1)}/(n-1) & \text{if } n>1.\end{cases}$

(c) $P(Z_n\le t)=P(\ln X_n\le t)=P(X_n\le e^t)=\int_{c_n}^{e^t}\frac{c_n}{x^{n+1}}\,dx=\frac{c_n}{n}\Big[\frac{1}{c_n^n}-\frac{1}{e^{nt}}\Big]$, where

$c_n = n^{-1/(n-1)}$. Let g_n be the probability density function of Z_n. Then $g_n(t) = c_n e^{-nt}$,

$t \geq \ln c_n$.

(d) $E(X_n^{m+1}) = \displaystyle\int_{c_n}^{\infty} \frac{c_n x^{m+1}}{x^{n+1}}\, dx$. This integral exists if and only if $m - n < -1$.

14. Using integration by parts twice, we obtain

$$\begin{aligned} E(X^{n+1}) &= \frac{1}{\pi}\int_0^{\pi} x^{n+2}\sin x\, dx = \pi^{n+1} + (n+2)\frac{1}{\pi}\int_0^{\pi} x^{n+1}\cos x\, dx \\ &= \pi^{n+1} + (n+2)\Big[-(n+1)\frac{1}{\pi}\int_0^{\pi} x^n \sin x\, dx\Big] \\ &= \pi^{n+1} + (n+2)\big[-(n+1)E(X^{n-1})\big]. \end{aligned}$$

Hence

$$E(X^{n+1}) + (n+1)(n+2)E(X^{n-1}) = \pi^{n+1}.$$

15. Since X is symmetric about α, for all $x \in (-\infty, \infty)$, $f(\alpha+x) = f(\alpha-x)$. Letting $y = x+\alpha$, we have

$$E(X) = \int_{-\infty}^{\infty} yf(y)\, dy = \int_{-\infty}^{\infty} (x+\alpha)f(x+\alpha)\, dx$$

$$= \int_{-\infty}^{\infty} xf(x+\alpha)\, dx + \alpha\int_{-\infty}^{\infty} f(x+\alpha)\, dx.$$

Now since f is symmetric about α, $xf(x+\alpha)$ is an odd function,

$$-xf(-x+\alpha) = -\big[xf(x+\alpha)\big].$$

Therefore, $\displaystyle\int_{-\infty}^{\infty} xf(x+\alpha) = 0$. Since $\displaystyle\int_{-\infty}^{\infty} f(x+\alpha)\, dx = \int_{-\infty}^{\infty} f(y)\, dy = 1$, we have $E(X) = 0 + \alpha \cdot 1 = \alpha$.

To show that the median of X is α, we will show that $P(X \leq \alpha) = P(X \geq \alpha)$. This also shows that the value of these two probabilities is 1/2. Letting $u = \alpha - x$, we have

$$P(X \leq \alpha) = \int_{-\infty}^{\alpha} f(x)\, dx = \int_0^{\infty} f(\alpha - u)\, du.$$

Letting $u = x - \alpha$, we have that

$$P(X \geq \alpha) = \int_{\alpha}^{\infty} f(x)\, dx = \int_0^{\infty} f(u+\alpha)\, du.$$

Since for all u,

$$f(\alpha - u) = f(\alpha + u),$$

we have that

$$P(X \le \alpha) = P(X \ge \alpha) = 1/2.$$

16. By Theorem 6.3,

$$E\big(|X-y|\big) = \int_{-\infty}^{\infty} |x-y| f(x)dx = \int_{-\infty}^{y} (y-x) f(x)\,dx + \int_{y}^{\infty} (x-y) f(x)\,dx$$

$$= y\int_{-\infty}^{y} f(x)\,dx - \int_{-\infty}^{y} xf(x)\,dx + \int_{y}^{\infty} xf(x)\,dx - y\int_{y}^{\infty} f(x)\,dx.$$

Hence

$$\frac{dE\big(|X-y|\big)}{dy} = \int_{-\infty}^{y} f(x)\,dx + yf(y) - yf(y) - yf(y) - \int_{y}^{\infty} f(x)\,dx + yf(y)$$

$$= \int_{-\infty}^{y} f(x)\,dx - \int_{y}^{\infty} f(x)\,dx.$$

Setting $\dfrac{dE\big(|X-y|\big)}{dy} = 0$, we obtain that y is the solution of the following equation:

$$\int_{-\infty}^{y} f(x)\,dx = \int_{y}^{\infty} f(x)\,dx.$$

By the definition of the median of a continuous random variable, the solution to this equation is $y = \text{median}(X)$. Hence $E\big(|X-y|\big)$ is minimum for $y = \text{median}(X)$.

17. **(a)** $\displaystyle\int_0^{\infty} I(t)\,dt = \int_0^{X} I(t)\,dt + \int_X^{\infty} I(t)\,dt = \int_0^{X} dt + \int_X^{\infty} 0\,dt = X.$

(Note that $\displaystyle\int_0^{\infty} I(t)\,dt$ is a random variable.)

(b) $\displaystyle E(X) = E\Big[\int_0^{\infty} I(t)\,dt\Big] = \int_0^{\infty} E\big[I(t)\big]\,dt = \int_0^{\infty} P(X > t)\,dt = \int_0^{\infty} \big[1 - F(t)\big]\,dt.$

(c) By part (b),

$$E(X^r) = \int_0^{\infty} P(X^r > t)\,dt = \int_0^{\infty} P\big(X > \sqrt[r]{t}\,\big)\,dt$$

$$= \int_0^{\infty} \Big[1 - F\big(\sqrt[r]{t}\,\big)\Big]\,dt = r\int_0^{\infty} y^{r-1}\big[1 - F(y)\big]\,dy,$$

where the last equality follows by the substitution $y = \sqrt[r]{t}$.

18. On the interval $[n, n+1)$,

$$P\big(|X| \geq n+1\big) \leq P\big(|X| > t\big) \leq P\big(|X| \geq n\big).$$

Therefore,

$$\int_n^{n+1} P\big(|X| \geq n+1\big)\,dt \leq \int_n^{n+1} P\big(|X| > t\big)\,dt \leq \int_n^{n+1} P\big(|X| \geq n\big)\,dt,$$

or

$$P\big(|X| \geq n+1\big) \leq \int_n^{n+1} P\big(|X| > t\big)\,dt \leq P\big(|X| \geq n\big).$$

So

$$\sum_{n=0}^{\infty} P\big(|X| \geq n+1\big) \leq \sum_{n=0}^{\infty} \int_n^{n+1} P\big(|X| > t\big)\,dt \leq \sum_{n=0}^{\infty} P\big(|X| > n\big),$$

and hence

$$\sum_{n=1}^{\infty} P\big(|X| \geq n\big) \leq E\big(|X|\big) \leq 1 + \sum_{n=1}^{\infty} P\big(|X| \geq n\big).$$

19. By Exercise 12,

$$E(X) = \frac{\alpha}{\lambda} + \frac{\beta}{\mu}.$$

Using Exercise 16, we obtain

$$E(X^2) = 2\int_0^{\infty} x(\alpha e^{-\lambda x} + \beta e^{-\mu x})\,dx = \frac{2\alpha}{\lambda^2} + \frac{2\beta}{\mu^2}.$$

Hence

$$\text{Var}(X) = \Big(\frac{2\alpha}{\lambda^2} + \frac{2\beta}{\mu^2}\Big) - \Big(\frac{\alpha}{\lambda} + \frac{\beta}{\mu}\Big)^2 = \frac{2\alpha - \alpha^2}{\lambda^2} + \frac{2\beta - \beta^2}{\mu^2} - \frac{2\alpha\beta}{\lambda\mu}.$$

20. $X \geq_{st} Y$ implies that for all t,

$$P(X > t) \;\geq\; P(Y > t). \tag{21}$$

Taking integrals of both sides of (21) yields,

$$\int_0^{\infty} P(X > t)\,dt \;\geq\; \int_0^{\infty} P(Y > t)\,dt. \tag{22}$$

Relation (21) also implies that

$$1 - P(X \leq t) \;\geq\; 1 - P(Y \leq t),$$

or, equivalently,

$$P(X \leq t) \;\leq\; P(Y \leq t)\cdot$$

Since this is true for all t, we have

$$P(X \le -t) \;\le\; P(Y \le -t)\cdot$$

Taking integrals of both sides of this inequality, we have

$$\int_0^\infty P(X \le -t) \;\le\; \int_0^\infty P(Y \le -t)\,dt,$$

or, equivalently,

$$-\int_0^\infty P(X \le -t) \;\ge\; -\int_0^\infty P(Y \le -t)\,dt. \tag{23}$$

Adding (22) and (23) yields

$$\int_0^\infty P(X > t)\,dt - \int_0^\infty P(X \le -t)\,dt \;\ge\; \int_0^\infty P(Y > t)\,dt - \int_0^\infty P(Y \le -t)\,dt\cdot$$

By Theorem 6.2, this gives $E(X) \ge E(Y)$. To show that the converse of this theorem is false, let X and Y be discrete random variables both with set of possible values $\{1, 2, 3\}$. Let the probability mass functions of X and Y be defined by

$$\begin{array}{lll} p_X(1) = 0.3 & p_X(2) = 0.4 & p_X(3) = 0.3 \\ p_Y(1) = 0.5 & p_Y(2) = 0.1 & p_Y(3) = 0.4 \end{array}$$

We have that $E(X) = 2 > E(Y) = 1.9$. However, since

$$P(X > 2) = 0.3 < P(Y > 2) = 0.4,$$

we see that X is not stochastically larger than Y.

21. First, we show that $\lim_{x\to-\infty} xP\big(X \le x\big) = 0$. To do so, since $x \to -\infty$, we concentrate on negative values of x. Letting $u = -t$, we have

$$xP\big(X \le x\big) = x\int_{-\infty}^{x} f(t)\,dt = x\int_{-x}^{\infty} f(-u)\,du = -\int_{-x}^{\infty} -xf(-u)\,du.$$

So it suffices to show that as $x \to -\infty$, $\int_{-x}^{\infty} -xf(-u)\,du \to 0$. Now

$$\int_{-x}^{\infty} -xf(-u)\,du \le \int_{-x}^{\infty} uf(-u)\,du.$$

Therefore, it remains to prove that $\int_{-x}^{\infty} uf(-u)\,du \to 0$ as $x \to -\infty$. But this is true because

$$\int_{-\infty}^{\infty} |u|f(-u)\,du = \int_{-\infty}^{\infty} |x|f(x)\,dx < \infty.$$

Next, we will show that $\lim_{x\to\infty} xP\big(X > x\big) = 0$. To do so, note that

$$\lim_{x\to\infty} xP\big(X > x\big) = \lim_{x\to\infty} x\int_x^\infty f(t)\,dt \leq \lim_{x\to\infty}\int_x^\infty tf(t)\,dt = 0$$

since $\int_{-\infty}^{\infty} |tf(t)|\,dt < \infty$.

REVIEW PROBLEMS FOR CHAPTER 6

1. Let F be the distribution function of Y. Clearly, $F(y) = 0$ if $y \leq 1$. For $y > 1$,

$$F(y) = P\Big(\frac{1}{X} \leq y\Big) = P\Big(X \geq \frac{1}{y}\Big) = \frac{1 - \dfrac{1}{y}}{1 - 0} = 1 - \frac{1}{y}.$$

So

$$f(y) = F'(y) = \begin{cases} 1/y^2 & y > 1 \\ 0 & \text{elsewhere.} \end{cases}$$

2. $E(X) = \displaystyle\int_1^\infty x \cdot \frac{2}{x^3}\,dx = \int_1^\infty \frac{2}{x^2}\,dx = -\frac{2}{x}\Big|_1^\infty = 2,$

$E(X^2) = \displaystyle\int_1^\infty x^2 \cdot \frac{2}{x^3}\,dx = 2\ln x\Big|_1^\infty = \infty$. So $\text{Var}(X)$ does not exist.

3. $E(X) = \displaystyle\int_0^1 (6x^2 - 6x^3)\,dx = \Big[2x^3 - \frac{6}{4}x^4\Big]_0^1 = \frac{1}{2},$

$E(X^2) = \displaystyle\int_0^1 (6x^3 - 6x^4)\,dx = \Big[\frac{6}{4}x^4 - \frac{6}{5}x^5\Big]_0^1 = \frac{3}{10},$

$\text{Var}(X) = \dfrac{3}{10} - \Big(\dfrac{1}{2}\Big)^2 = \dfrac{1}{20}, \quad \sigma_X = \dfrac{1}{2\sqrt{5}}.$

Therefore,

$$P\Big(\frac{1}{2} - \frac{2}{2\sqrt{5}} < X < \frac{1}{2} + \frac{2}{2\sqrt{5}}\Big) = \int_{\frac{1}{2}-\frac{1}{\sqrt{5}}}^{\frac{1}{2}+\frac{1}{\sqrt{5}}} (6x - 6x^2)\,dx$$

$$= \Big[3x^2 - 2x^3\Big]_{\frac{1}{2}-\frac{1}{\sqrt{5}}}^{\frac{1}{2}+\frac{1}{\sqrt{5}}} = \frac{11}{5\sqrt{5}}.$$

4. We have that

$$P(-2 < X < 1) = \int_{-2}^{1} \frac{e^{-|x|}}{2}\,dx = \frac{1}{2}\Big[\int_{-2}^{0} e^{x}\,dx + \int_{0}^{1} e^{-x}\,dx\Big]$$
$$= 1 - \frac{1}{2e} - \frac{1}{2e^2} = 0.748.$$

5. For all $c > 0$,

$$\int_0^\infty \frac{c}{1+x}\,dx = \Big[c\ln(1+x)\Big]_0^\infty = \infty.$$

So, for no value of c, $f(x)$ is a probability density function.

6. The set of possible values of X is $A = [1, 2]$. Let $h\colon [1, 2] \to \mathbf{R}$ be defined by $h(x) = e^x$. The set of possible values of e^X is $B = [e, e^2]$; the inverse of h is $g(y) = \ln y$, where $g'(y) = 1/y$. Therefore,

$$f_Y(y) = \frac{4(\ln y)^3}{15}|g'(y)| = \frac{4(\ln y)^3}{15y}, \quad y \in [e, e^2].$$

Applying the same procedure to Z and W, we obtain

$$f_Z(z) = \frac{4(\sqrt{z}\,)^3}{15}\Big|\frac{1}{2\sqrt{z}}\Big| = \frac{2z}{15}, \quad z \in [1, 4].$$

$$f_W(w) = \frac{2(1+\sqrt{w}\,)^3}{15\sqrt{w}} \quad w \in [0, 1].$$

7. The set of possible values of X is $A = (0, 1)$. Let $h\colon (0, 1) \to \mathbf{R}$ be defined by $h(x) = x^4$. The set of possible values of X^4 is $B = (0, 1)$. The inverse of $h(x) = x^4$ is $g(y) = \sqrt[4]{y}$. So $g'(y) = \dfrac{1}{4}y^{-3/4} = \dfrac{1}{4\sqrt{y}\sqrt[4]{y}}$. We have that

$$f_Y(y) = 30(\sqrt[4]{y}\,)^2(1-\sqrt[4]{y}\,)^2\Big|\frac{1}{4\sqrt[4]{y^3}}\Big| = 30\sqrt{y}(1-\sqrt[4]{y}\,)^2\frac{1}{4\sqrt{y}\sqrt[4]{y}}$$
$$= \frac{15(1-\sqrt[4]{y}\,)^2}{2\sqrt[4]{y}}, \quad y \in (0, 1).$$

8. We have that

$$f(x) = F'(x) = \begin{cases} \dfrac{1}{\pi\sqrt{1-x^2}} & -1 < x < 1 \\ 0 & \text{otherwise.} \end{cases}$$

Therefore,

$$E(X) = \int_{-1}^{1} \frac{x}{\pi\sqrt{1-x^2}}\,dx = 0$$

since the integrand is an odd function.

9. Clearly $\sum_{i=1}^{n} \alpha_i f_i \geq 0$. Since

$$\int_{-\infty}^{\infty} \Big(\sum_{i=1}^{n} \alpha_i f_i\Big)(x)\,dx = \sum_{i=1}^{n} \alpha_i \int_{-\infty}^{\infty} f_i(x)\,dx = \sum_{i=1}^{n} \alpha_i = 1,$$

$\sum_{i=1}^{n} \alpha_i f_i$ is a probability density function.

10. Let $U = x$ and $dV = f(x)dx$. Then $dU = dx$ and $V = F(x)$. Since $F(\alpha) = 1$,

$$\begin{aligned} E(X) &= \int_0^{\alpha} x f(x)\,dx = \Big[xF(x)\Big]_0^{\alpha} - \int_0^{\alpha} F(x)\,dx \\ &= \alpha F(\alpha) - \int_0^{\alpha} F(x)\,dx = \alpha - \int_0^{\alpha} F(x)\,dx \\ &= \int_0^{\alpha} dx - \int_0^{\alpha} F(x)\,dx = \int_0^{\alpha} \big[1 - F(x)\big]\,dx. \end{aligned}$$

11. Let X be the lifetime of a random light bulb. The probability that it lasts over 1000 hours is

$$P(X > 1000) = \int_{1000}^{\infty} \frac{5 \times 10^5}{x^3}\,dx = 5 \times 10^5 \Big[-\frac{1}{2x^2}\Big]_{1000}^{\infty} = \frac{1}{4}.$$

Thus the probability that out of six such light bulbs two last over 1000 hours is

$$\binom{6}{2}\Big(\frac{1}{4}\Big)^2\Big(\frac{3}{4}\Big)^4 \approx 0.3$$

12. Since $Y \geq 0$, $P(Y \leq t) = 0$ for $t < 0$. For $t \geq 0$,

$$\begin{aligned} P(Y \leq t) &= P\big(|X| \leq t\big) = P(-t \leq X \leq t) = P(X \leq t) - P(X < -t) \\ &= P(X \leq t) - P(X \leq -t) = F(t) - F(-t). \end{aligned}$$

Hence G, the probability distribution function of $|X|$ is given by

$$G(t) = \begin{cases} F(t) - F(-t) & \text{if } t \geq 0 \\ 0 & \text{if } t < 0; \end{cases}$$

g, the probability density function of $|X|$ is obtained by differentiating G:

$$g(t) = G'(t) = \begin{cases} f(t) + f(-t) & \text{if } t \geq 0 \\ 0 & \text{if } t < 0. \end{cases}$$

Chapter 7

Special Continuous Distributions

7.1 UNIFORM RANDOM VARIABLES

1. $(23-20)/(27-20) = 3/7$.

2. $15(1/4) = 3.75$.

3. Let 2:00 P.M. be the origin, then a and b satisfy the following system of two equations in two unknown.

$$\begin{cases} \dfrac{a+b}{2} = 0 \\ \dfrac{(b-a)^2}{12} = 12. \end{cases}$$

Solving this system, we obtain $a = -6$ and $b = 6$. So the bus arrives at a random time between 1:54 P.M. and 2:06 P.M.

4. $P(b^2 - 4 \geq 0) = P(b > 2 \text{ or } b < -2) = 2/6 = 1/3$.

5. The probability density function of R, the radius of the sphere is

$$f(r) = \begin{cases} \dfrac{1}{4-2} = \dfrac{1}{2} & 2 < r < 4 \\ 0 & \text{elsewhere.} \end{cases}$$

Thus

$$E(V) = \int_2^4 \Big(\frac{4}{3}\pi r^3\Big)\frac{1}{2}\,dr = 40\pi.$$

$$P\Big(\frac{4}{3}\pi R^3 < 36\pi\Big) = P(R^3 < 27) = P(R < 3) = \frac{1}{2}.$$

6. The problem is equivalent to choosing a random number X from $(0, \ell)$. The desired probability is

$$P\Big(X \leq \frac{\ell}{3}\Big) + P\Big(X \geq \frac{2\ell}{3}\Big) = \frac{\ell/3}{\ell} + \frac{\ell - (2\ell/3)}{\ell} = \frac{2}{3}.$$

7. Let X be a random number from $(0, \ell)$. The probability of the desired event is

$$P\Big(\min(X, \ell - X) \geq \frac{\ell}{3}\Big) = P\Big(X \geq \frac{\ell}{3},\ \ell - X \geq \frac{\ell}{3}\Big) = P\Big(\frac{\ell}{3} \leq X \leq \frac{2\ell}{3}\Big) = \frac{\dfrac{2\ell}{3} - \dfrac{\ell}{3}}{\ell} = \frac{1}{3}.$$

8. $\dfrac{180 - 90}{180 - 60} = \dfrac{3}{4}.$

9. Let X be a random point from $(0, b)$. A triangular pen is possible to construct if and only if the segments a, X, and $b - X$ are sides of a triangle. The probability of this is

$$\begin{aligned} P\big(a < X + (b - X),\ X < a + (b - X),\ b - X < a + X\big) &= P\Big(\frac{b-a}{2} < X < \frac{a+b}{2}\Big) \\ &= \frac{\dfrac{a+b}{2} - \dfrac{b-a}{2}}{b} = \frac{a}{b}. \end{aligned}$$

10. Let F be the probability distribution function and f be the probability density function of X. By definition,

$$\begin{aligned} F(x) &= P(X \leq x) = P(\tan\theta \leq x) = P(\theta \leq \arctan x) \\ &= \frac{\arctan x - \Big(-\dfrac{\pi}{2}\Big)}{\dfrac{\pi}{2} - \Big(-\dfrac{\pi}{2}\Big)} = \frac{1}{\pi}\arctan x + \frac{1}{2}, \quad -\infty < x < \infty. \end{aligned}$$

Thus

$$f(x) = F'(x) = \frac{1}{\pi(1 + x^2)}, \quad -\infty < x < \infty.$$

11. For $i = 0, 1, 2, \dots, n - 1$,

$$P\big([nX] = i\big) = P(i \leq nX < i + 1) = P\Big(\frac{i}{n} \leq X < \frac{i+1}{n}\Big) = \frac{\dfrac{i+1}{n} - \dfrac{i}{n}}{1 - 0} = \frac{1}{n}.$$

$P\big([nX] = i\big) = 0$, otherwise. Therefore, $[nX]$ is a random number from the set $\big\{0, 1, 2, \dots, n - 1\big\}$.

12. **(a)** Let G and g be the distribution and density functions of Y, respectively. Since $Y \geq 0$, $G(x) = 0$ if $x \leq 0$. If $x \geq 0$,

$$\begin{aligned} G(x) &= P(Y \leq x) = P\big(-\ln(1 - X) \leq x\big) = P\big(X \leq 1 - e^{-x}\big) \\ &= \frac{(1 - e^{-x}) - 0}{1 - 0} = 1 - e^{-x}. \end{aligned}$$

Thus

$$g(x) = G'(x) = \begin{cases} e^{-x} & x \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

(b) Let H and h be the probability distribution and probability density functions of Z, respectively. For $n > 0$, $H(x) = P(Z \leq x) = 0$, $x < 0$;

$$H(x) = P(Z \leq x) = P(X \leq \sqrt[n]{x}\,) = \sqrt[n]{x}, \quad 0 < x < 1;$$

$H(x) = 1$, if $x \geq 1$. Therefore,

$$h(x) = H'(x) = \begin{cases} \dfrac{1}{n} x^{\frac{1}{n}-1} & 0 < x < 1 \\ 0 & \text{elsewhere.} \end{cases}$$

For $n < 0$, $H(x) = P(X^n \leq x) = 0$, $x < 1$;

$$\begin{aligned} H(x) = P(X^n \leq x) &= P\Big(X^{-n} \geq \frac{1}{x}\Big) = P\Big(X \geq \Big(\frac{1}{x}\Big)^{-\frac{1}{n}}\Big) \\ &= P(X \geq x^{1/n}) = 1 - x^{1/n}, \quad x \geq 1. \end{aligned}$$

Therefore,

$$h(x) = \begin{cases} -\dfrac{1}{n} x^{\frac{1}{n}-1} & \text{if } x \geq 1 \\ 0 & \text{if } x < 1. \end{cases}$$

13. Cleary, $E(X) = (1 + \theta)/2$. This implies that $\theta = 2E(X) - 1$. Now

$$\text{Var}(X) = E\big(X^2\big) - \big[E(X)\big]^2 = \frac{(1 + \theta - 0)^2}{12}.$$

Therefore,

$$E\big(X^2\big) - \Big(\frac{1 + \theta}{2}\Big)^2 = \frac{1 + 2\theta + \theta^2}{12}.$$

This yields,

$$E\big(X^2\big) = \frac{\theta^2 + 2\theta + 1}{3}.$$

So

$$3E(X^2) - 2\theta - 1 = \theta^2.$$

But $\theta = 2E(X) - 1$; so

$$3E(X^2) - 2\big[2E(X) - 1\big] - 1 = \theta^2.$$

This implies that

$$E(3X^2 - 4X + 1) = \theta^2.$$

Therefore, one choice for $g(X)$ is $g(X) = 3X^2 - 4X + 1$.

14. Let S be the sample space over which X is defined. The functions $X\colon S \to \mathbf{R}$ and $F\colon \mathbf{R} \to [0, 1]$ can be composed to obtain the random variable $F(X)\colon S \to [0, 1]$. Clearly,

$$P\big(F(X) \le t\big) = \begin{cases} 1 & \text{if } t \ge 1 \\ 0 & \text{if } t \le 0. \end{cases}$$

Let $t \in (0, 1)$; it remains to prove that $P\big(F(X) \le t\big) = t$. To show this, note that since F is continuous, $F(-\infty) = 0$, and $F(\infty) = 1$, the inverse image of t, $F^{-1}\big(\{t\}\big)$, is nonempty. We know that F is nondecreasing; since F is not necessarily strictly increasing, $F^{-1}\big(\{t\}\big)$ might have more than one element. For example, if F is the constant t on some internal $(a, b) \subseteq (0, 1)$, then $F(x) = t$ for all $x \in (a, b)$, implying that (a, b) is contained in $F^{-1}\big(\{t\}\big)$. Let

$$x_0 = \inf\big\{x\colon F(x) > t\big\}.$$

Then $F(x_0) = t$ and

$$F(x) \le t \quad \text{if and only if} \quad x \le x_0.$$

Therefore,

$$P\big(F(X) \le t\big) = P\big(X \le x_0\big) = F(x_0) = t.$$

We have shown that

$$P\big(F(X) \le t\big) = \begin{cases} 0 & \text{if } t \le 0 \\ t & \text{if } 0 \le t \le 1 \\ 1 & \text{if } t \ge 1, \end{cases}$$

meaning that $F(X)$ is uniform over $(0, 1)$.

15. We are given that Y is a uniform random variable. First we show that Y is uniform over the interval $(0, 1)$. To do this, it suffices to show that $P(Y \le 1) = 1$ and $P(Y < 0) = 0$. These are obvious implications of the fact that g is nonnegative and $\displaystyle\int_{-\infty}^{\infty} g(x)\,dx = 1$:

$$P(Y \le 1) = P\Big(\int_{-\infty}^{X} g(t)\,dt \le 1\Big) = 1.$$

$$P(Y < 0) = P\Big(\int_{-\infty}^{X} g(t)\,dt < 0\Big) = 0,$$

The following relation shows that the probability density function of X is g.

$$\frac{d}{du}P(X \le u) = \frac{d}{du}P\Big(Y \le \int_{-\infty}^{u} g(t)\,dt\Big) = \frac{d}{du}\left(\frac{\displaystyle\int_{-\infty}^{u} g(t)\,dt - 0}{1 - 0}\right) = g(u),$$

where the last equality follows from the fundamental theorem of calculus.

16. Let F be the distribution function of X, then $F(t) = P(X \leq t)$ is 0 for $t < -1$ and is 1 for $t \geq 4$. Let $-1 \leq t < 4$; we have that

$$F(t) = P(X \leq t) = P(5\omega - 1 \leq t) = P\Big(\omega \leq \frac{t+1}{5}\Big)$$
$$= P\Big(\omega \in \Big(0, \frac{t+1}{5}\Big)\Big) = \int_0^{(t+1)/5} dx = \frac{t+1}{5}.$$

Therefore,

$$F(t) = \begin{cases} 0 & t < -1 \\ \dfrac{t+1}{5} & -1 \leq t < 4 \\ 1 & t \geq 4. \end{cases}$$

This is the distribution function of a uniform random variable over $(-1, 4)$.

17. We have that $X = n$ if and only if $\sqrt{Y} = 0.y_1ny_3y_4y_5\cdots$, or, equivalently, if and only if, $10\sqrt{Y} = y_1.ny_3y_4y_5\cdots$. Therefore, $X = n$ if and only if for some $k \in \{0, 1, 2, \ldots, 9\}$,

$$k + \frac{n}{10} \leq 10\sqrt{Y} < k + \frac{n+1}{10}.$$

This is equivalent to

$$\frac{1}{100}\Big(k + \frac{n}{10}\Big)^2 \leq Y < \frac{1}{100}\Big(k + \frac{n+1}{10}\Big)^2.$$

Therefore, the desired probability is

$$\sum_{k=0}^{9} P\Big(\frac{1}{100}\Big(k + \frac{n}{10}\Big)^2 \leq Y < \frac{1}{100}\Big(k + \frac{n+1}{10}\Big)^2\Big)$$
$$= \sum_{k=0}^{9}\Big[\frac{1}{100}\Big(k + \frac{n+1}{10}\Big)^2 - \frac{1}{100}\Big(k + \frac{n}{10}\Big)^2\Big]$$
$$= \sum_{k=0}^{9} \frac{20k + 2n + 1}{10,000} = 0.091 + 0.002n.$$

We see that this quantity increases as n does.

7.2 NORMAL RANDOM VARIABLES

1. Since $np = (0.90)(50) = 45$ and $\sqrt{np(1-p)} = 2.12$,

$$P(X \geq 44.5) = P\Big(Z \geq \frac{44.5 - 45}{2.12}\Big) = P(Z \geq -0.24)$$
$$= 1 - \Phi(-0.24) = \Phi(0.24) = 0.5948.$$

2. $np = 1095/365 = 3$ and $\sqrt{np(1-p)} = \sqrt{3\Big(\dfrac{364}{365}\Big)} = 1.73$. Therefore,

$$P(X \geq 5.5) = P\Big(Z \geq \frac{5.5 - 3}{1.73}\Big) = 1 - \Phi(1.45) = 0.0735.$$

3. We have that

$$P(|Z|) \leq x) = P(-x \leq Z \leq x) = \Phi(x) - \Phi(-x)$$
$$= \Phi(x) - \big[1 - \Phi(x)\big] = 2\Phi(x) - 1 = \Psi(x).$$

4. Let

$$g(x) = P(x < Z < x + \alpha) = \frac{1}{\sqrt{2\pi}} \int_x^{x+\alpha} e^{-y^2/2}\, dy.$$

The number x that maximizes $P(x < Z < x + \alpha)$ is the root of $g'(x) = 0$; that is, it is the solution of

$$g'(x) = \frac{1}{\sqrt{2\pi}}\big[e^{-(x+\alpha)^2/2} - e^{-x^2/2}\big] = 0,$$

which is $x = -\alpha/2$.

5. $E(X \cos X)$, $E(\sin X)$, and $E\Big(\dfrac{X}{1+X^2}\Big)$ are, respectively, $\dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{-\infty}^{\infty} (x \cos x) e^{-x^2/2}\, dx$, $\dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{-\infty}^{\infty} (\sin x) e^{-x^2/2}\, dx$, and $\dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{-\infty}^{\infty} \dfrac{x}{1+x^2} e^{-x^2/2}\, dx$. Since these are integrals of *odd functions* from $-\infty$ to ∞, all three of them are 0.

6. **(a)** $P(X > 35.5) = P\Big(\dfrac{X - 35.5}{4.8} > \dfrac{35.5 - 35.5}{4.8}\Big) = 1 - \Phi(0) = 0.5.$

(b) The desired probability is given by

$$P(30 < X < 40) = P\Big(\frac{30 - 35.5}{4.8} < X < \frac{40 - 35.5}{4.8}\Big) = \Phi(0.94) - \Phi(-1.15)$$
$$= \Phi(0.94) + \Phi(1.15) - 1 = 0.8264 + 0.8749 - 1 = 0.701.$$

7. Let X be the grade of a randomly selected student;

$$P(X \geq 90) = P\Big(Z \geq \frac{90-67}{8}\Big) = 1 - \Phi(2.88) = 1 - 0.9980 = 0.002,$$

$$P(80 \leq X < 90) = P\Big(\frac{80-67}{8} \leq Z < \frac{90-67}{8}\Big) = \Phi(2.88) - \Phi(1.63)$$

$$= 0.9980 - 0.9484 = 0.0496.$$

Similarly, $P(70 \leq X < 80) = 0.3004$, $P(60 \leq X < 70) = 0.4586$, and $P(X < 60) = 0.1894$. Therefore, approximately 0.2%, 4.96%, 30.04%, 45.86%, and 18.94% get A, B, C, D, and F, respectively.

8. Let X be the blood pressure of a randomly selected person;

$$P(89 < X < 96) = P\Big(\frac{89-80}{7} < Z < \frac{96-80}{7}\Big) = P(1.29 < Z < 2.29) = 0.0875,$$

$$P(X > 95) = P\Big(Z > \frac{95-80}{7}\Big) = 0.016.$$

Therefore, 8.75% have mild hypertension while 1.6% are hypertensive.

9. $P(74.5 < X < 75.8) = P(-0.5 < Z < 0.8) = \Phi(0.8) - \big[1 - \Phi(0.5)\big] = 0.4796.$

10. We must find x so that $P(110 - x < X < 110 + x) = 0.50$, or, equivalently,

$$P\Big(-\frac{x}{20} < \frac{X-110}{20} < \frac{x}{20}\Big) = 0.50.$$

Therefore, we must find the value of x which satisfies $P\big(-x/20 < Z < x/20\big) = 0.50$ or $\Phi(x/20) - \Phi(-x/20) = 0.50$. Since $\Phi(-x/20) = 1 - \Phi(x/20)$, x satisfies $2\Phi(x/20) = 1.50$ or $\Phi(x/20) = 0.75$. Using Table 1 of the appendix, we get $x/20 = 0.67$ or $x = 13.4$ So the desired interval is $(110 - 13.4, 110 + 13.4) = (96.6, 123.4)$.

11. Let X be the amount of cereal in a box. We want to have $P(X \geq 16) \geq 0.90$. This gives

$$P\Big(Z \geq \frac{16-16.5}{\sigma}\Big) \geq 0.90,$$

or $\Phi(0.5/\sigma) \geq 0.90$. The smallest value for $0.5/\sigma$ satisfying this inequality is 1.29; so the largest value for σ is obtained from $0.5/\sigma = 1.29$. This gives $\sigma = 0.388$.

12. Let X be the score of a randomly selected individual;

$$P(X \geq 14) = P\Big(Z \geq \frac{14-12}{3}\Big) = P(Z \geq 0.67) = 0.2514.$$

Therefore, the probability that none of the eight individuals make a score less than 14 is $(0.2514)^8 = 0.000016$.

13. We want to find t so that $P(X \le t) = 1/2$. This implies that

$$P\Big(\frac{X-\mu}{\sigma} \le \frac{t-\mu}{\sigma}\Big) = \frac{1}{2},$$

or $\Phi\Big(\dfrac{t-\mu}{\sigma}\Big) = \dfrac{1}{2}$; so $\dfrac{t-\mu}{\sigma} = 0$ which gives $t = \mu$.

14. We have that

$$\begin{aligned} P(|X-\mu| > k\sigma) &= P(X-\mu > k\sigma) + P(X-\mu < -k\sigma) = P(Z > k) + P(Z < -k) \\ &= \big[1-\Phi(k)\big] + \big[1-\Phi(k)\big] = 2\big[1-\Phi(k)\big]. \end{aligned}$$

This shows that $P(|X-\mu| > k\sigma)$ does not depend on μ or σ.

15. Let X be the lifetime of a randomly selected light bulb.

$$P(X \ge 900) = P\Big(Z \ge \frac{900-1000}{100}\Big) = 1 - \Phi(-1) = \Phi(1) = 0.8413.$$

Hence the company's claim is false.

16. Let X be the lifetime of the light bulb manufactured by the first company. Let Y be the lifetime of the light bulb manufactured by the second company. Assuming that X and Y are independent, the desired probability, $P\big(\max(X,Y) \ge 980\big)$, is calculated as follows.

$$\begin{aligned} P\big(\max(X,Y) \ge 980\big) &= 1 - P\big(\max(X,Y) < 980\big) = 1 - P(X < 980, Y < 980) \\ &= 1 - P(X < 980)\,P(Y < 980) \\ &= 1 - P\Big(Z < \frac{980-1000}{100}\Big)P\Big(Z < \frac{980-900}{150}\Big) \\ &= 1 - P(Z < -0.2)P(Z < 0.53) = 1 - \big[1-\Phi(0.2)\big]\Phi(0.53) \\ &= 1 - (1-0.5793)(0.7019) = 0.7047. \end{aligned}$$

17. Let r be the rate of return of this stock; r is a normal random variable with mean $\mu = 0.12$ and standard deviation $\sigma = 0.06$. Let n be the number of shares Mrs. Lovotti should purchase. We want to find the smallest n for which the probability of profit in one year is at least \$1000. Let X be the current price of the total shares of the stock that Mrs. Lovotti buys this year, and Y be the total price of the shares next year. We want to find the smallest n for which $P(Y - X \ge 1000)$. We have

$$\begin{aligned} P(Y - X \ge 1000) &= P\Big(\frac{Y-X}{X} \ge \frac{1000}{X}\Big) = P\Big(r \ge \frac{1000}{X}\Big) \\ &= P\Big(r \ge \frac{1000}{35n}\Big) = P\left(Z \ge \frac{\dfrac{1000}{35n} - 0.12}{0.06}\right) \ge 0.90. \end{aligned}$$

Therefore, we want to find the smallest n for which

$$P\left(Z \leq \frac{\dfrac{1000}{35n} - 0.12}{0.06}\right) \leq 0.10.$$

By Table 1 of the Appendix, this is satisfied if

$$\frac{\dfrac{1000}{35n} - 0.12}{0.06} \leq -1.29.$$

This gives $n \geq 670.69$. Therefore, Mrs. Lovotti should buy 671 shares of the stock.

18. We have that

$$f(x) = \frac{1}{\sqrt{1/2}\sqrt{\pi}} \exp\Big[-\frac{(x-1)^2}{1/2}\Big] = \frac{1}{(1/2)\sqrt{2\pi}} \exp\Big[-\frac{(x-1)^2}{2(1/4)}\Big].$$

This shows that f is the probability density function of a normal random variable with mean 1 and standard deviation $1/2$ (variance $1/4$).

19. Let F be the distribution function of $|X-\mu|$. $F(t) = 0$ if $t < 0$; for $t \geq 0$,

$$\begin{aligned}
F(t) &= P\big(|X-\mu| \leq t\big) = P(-t \leq X-\mu \leq t) \\
&= P(\mu - t \leq X \leq \mu + t) = P\Big(-\frac{t}{\sigma} \leq \frac{X-\mu}{\sigma} \leq \frac{t}{\sigma}\Big) \\
&= \Phi\Big(\frac{t}{\sigma}\Big) - \Phi\Big(-\frac{t}{\sigma}\Big) = \Phi\Big(\frac{t}{\sigma}\Big) - \Big[1 - \Phi\Big(\frac{t}{\sigma}\Big)\Big] = 2\Phi\Big(\frac{t}{\sigma}\Big) - 1.
\end{aligned}$$

Therefore,

$$F(t) = \begin{cases} 2\Phi\Big(\dfrac{t}{\sigma}\Big) - 1 & t \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

This gives

$$F'(t) = \frac{2}{\sigma}\Phi'\Big(\frac{t}{\sigma}\Big) \quad t \geq 0.$$

Hence

$$E\big(|X-\mu|\big) = \int_0^\infty t\frac{2}{\sigma}\Phi'\Big(\frac{t}{\sigma}\Big)\,dt.$$

substituting $u = t/\sigma$, we obtain

$$\begin{aligned}
E(|X-\mu|) &= 2\sigma \int_0^\infty u\Phi'(u)\,du = \frac{2\sigma}{\sqrt{2\pi}} \int_0^\infty ue^{-u^2/2}\,du \\
&= \frac{2\sigma}{\sqrt{2\pi}}\Big[-e^{-u^2/2}\Big]_0^\infty = \frac{2\sigma}{\sqrt{2\pi}} = \sigma\sqrt{\frac{2}{\pi}}.
\end{aligned}$$

20. The general form of the probability density function of a normal random variable is

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\Big[-\frac{(x-\mu)^2}{2\sigma^2}\Big] = \frac{1}{\sigma\sqrt{2\pi}} \exp\Big(-\frac{1}{2\sigma^2}x^2 + \frac{\mu}{\sigma^2}x - \frac{\mu^2}{2\sigma^2}\Big).$$

Comparing this with the given probability density function, we see that

$$\begin{cases} \sqrt{k} = \dfrac{1}{\sigma\sqrt{2\pi}} \\ k^2 = \dfrac{1}{2\sigma^2} \\ 2k = -\dfrac{\mu}{\sigma^2} \\ \dfrac{\mu^2}{2\sigma^2} = 1. \end{cases}$$

Solving the first two equations for k and σ, we obtain $k = \pi$ and $\sigma = 1/(\pi\sqrt{2})$. These and the third equation give $\mu = -1/\pi$ which satisfy the fourth equation. So $k = \pi$ and f is the probability density function of $N\Big(-\dfrac{1}{\pi}, \dfrac{1}{2\pi^2}\Big)$.

21. Let X be the viscosity of the given brand. We must find the smallest x for which $P(X \le x) \ge 0.90$ or $P\Big(Z \le \dfrac{x-37}{10}\Big) \ge 0.90$. This gives $\Phi\Big(\dfrac{x-37}{10}\Big) \ge 0.90$ or $(x-37)/10 = 1.29$; so $x = 49.9$.

22. Let X be the length of the residence of a family selected at random from this town. Since

$$P(X \ge 96) = P\Big(Z \ge \frac{96-80}{30}\Big) = 0.298,$$

using binomial distribution, the desired probability is

$$1 - \sum_{i=0}^{2} \binom{12}{i} (0.298)^i (1-0.298)^{12-i} = 0.742.$$

23. We have

$$\begin{aligned} E(e^{\alpha Z}) &= \int_{-\infty}^{\infty} e^{\alpha x} \cdot \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\,dx \\ &= e^{\alpha^2/2} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\alpha^2 + \alpha x - \frac{1}{2}x^2}\,dx \\ &= e^{\alpha^2/2} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x-\alpha)^2}\,dx = e^{\alpha^2/2}, \end{aligned}$$

where $\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x-\alpha)^2}\,dx = 1$, since $\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x-\alpha)^2}$ is the probability density function of a normal random variable with mean α and variance 1.

24. For $t \geq 0$,

$$P(Y \leq t) = P\big(-\sqrt{t} \leq X \leq \sqrt{t}\big) = P\Big(-\frac{\sqrt{t}}{\sigma} \leq Z \leq \frac{\sqrt{t}}{\sigma}\Big) = 2\Phi\Big(\frac{\sqrt{t}}{\sigma}\Big) - 1.$$

Let f be the probability density function of Y. Then

$$f(t) = \frac{d}{dt} P(Y \leq t) = 2\frac{1}{2\sigma\sqrt{t}}\Phi'\Big(\frac{\sqrt{t}}{\sigma}\Big), \quad t \geq 0.$$

So

$$f(t) = \begin{cases} \dfrac{1}{\sigma\sqrt{2\pi t}} \exp\Big(-\dfrac{t}{2\sigma^2}\Big) & t \geq 0 \\ 0 & t \leq 0. \end{cases}$$

25. For $t \geq 0$,

$$P(Y \leq t) = P\big(e^X \leq t\big) = P(X \leq \ln t) = P\Big(Z \leq \frac{\ln t - \mu}{\sigma}\Big) = \Phi\Big(\frac{\ln t - \mu}{\sigma}\Big).$$

Let f be the probability density function of Y. We have

$$f(t) = \frac{d}{dt} P(Y \leq t) = \frac{1}{\sigma t}\Phi'\Big(\frac{\ln t - \mu}{\sigma}\Big), \quad t \geq 0.$$

So

$$f(t) = \begin{cases} \dfrac{1}{\sigma t\sqrt{2\pi}} \exp\Big[-\dfrac{(\ln t - \mu)^2}{2\sigma^2}\Big] & t \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

26. Let f be the probability density function of Y. Since for $t \geq 0$,

$$P(Y \leq t) = P\big(\sqrt{|X|} \leq t\big) = P\big(|X| \leq t^2\big) = P\big(-t^2 \leq X \leq t^2\big) = 2\Phi(t^2) - 1,$$

we have that

$$f(t) = \frac{d}{dt} P(Y \leq t) = \begin{cases} 4t\dfrac{1}{\sqrt{2\pi}} e^{-t^4/2} & t \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

27. Suppose that X is the number of books sold in a month. The random variable X is binomial with parameters $n = (800)(30) = 24,000$ and $p = 1/5001$. Moreover, $E(X) = np = 4.8$ and $\sigma_X = \sqrt{np(1-p)} = 2.19$. Let k be the number of copies of the bestseller to be ordered every month. We want to have $P(X < k) > 0.98$ or $P(X \leq k-1) > 0.98$. Using De Moivre-Laplace theorem and making correction for continuity, this inequality is valid if

$$P\Big(\frac{X-4.8}{2.19} < \frac{k-1+0.5-4.8}{2.19}\Big) > 0.98.$$

From Table 1 of the appendix, we have $(k-1+0.5-4.8)/2.19 = 2.06$, or $k = 9.81$. Therefore, the store should order 10 copies a month.

28. Let X be the number of light bulbs of type I. We want to calculate $P(18 \leq X \leq 22)$. Since the number of light bulbs is large and half of the light bulbs are type I, we can assume that X is *approximately* binomial with parameters 40 and 1/2. Note that $np = 20$ and $\sqrt{np(1-p)} = \sqrt{10}$. Using De Moivre-Laplace theorem and making correction for continuity, we have

$$\begin{aligned} P(17.5 \leq X \leq 22.5) &= P\Big(\frac{17.5-20}{\sqrt{10}} \leq \frac{X-20}{\sqrt{10}} \leq \frac{22.5-20}{\sqrt{10}}\Big) \\ &= \Phi(0.79) - \Phi(-0.79) = 2\Phi(0.79) - 1 = 0.5704. \end{aligned}$$

Remark: Using binomial distribution, the solution to this problem is

$$\sum_{i=18}^{22} \binom{40}{i} \Big(\frac{1}{2}\Big)^{i} \Big(\frac{1}{2}\Big)^{40-i} = 0.5704.$$

As we see, up to at least 4 decimal places, this solution gives the same answer as obtained above. This indicates the importance of correction for continuity; if it is ignored, we obtain 0.4714, an answer which is almost 10% lower than the actual answer.

29. Let X be the number of 1's selected; X is binomial with parameters $100,000$ and $1/40$. Thus $np = 2500$ and $\sqrt{np(1-p)} = 49.37$. So

$$P(X \geq 3500) \approx P\Big(Z \geq \frac{3499.50 - 2500}{49.37}\Big) = 1 - \Phi(20.25) = 0.$$

Hence it is fair to say that the algorithm is not accurate.

30. Note that

$$ka^{-x^2} = k \exp\big(-x^2 \ln a\big) = k \exp\Big(-\frac{x^2}{1/\ln a}\Big).$$

Comparing this with the probability density function of a normal random variable with parameters μ and σ, we see that $\mu = 0$ and $2\sigma^2 = 1/\ln a$. Thus $\sigma = \sqrt{1/(2\ln a)}$, and hence

$$k = \frac{1}{\sigma\sqrt{2\pi}} = \sqrt{\frac{\ln a}{\pi}}.$$

So, for this value of k, the function f is the probability density function a normal random variable with mean 0 and standard deviation $\sqrt{1/(2\ln a)}$.

31. **(a)** The derivation of these inequalities from the hint is straightforward.

(b) By part (a),

$$1 - \frac{1}{x^2} < \frac{1 - \Phi(x)}{\big[1/(x\sqrt{2\pi})\big]e^{-x^2/2}} < 1.$$

Thus

$$1 \leq \lim_{x\to\infty} \frac{1 - \Phi(x)}{\big[1/(x\sqrt{2\pi})\big]e^{-x^2/2}} \leq 1,$$

from which (b) follows.

32. By part (b) of Exercise 31,

$$\lim_{t\to\infty} P\Big(Z > t + \frac{x}{t} \;\Big|\; Z \geq t\Big) = \lim_{t\to\infty} \frac{P\Big(Z > t + \dfrac{x}{t}\Big)}{P(Z \geq t)}$$

$$= \lim_{t\to\infty} \frac{\dfrac{1}{\Big(t + \dfrac{x}{t}\Big)\sqrt{2\pi}} \exp\Big[-\Big(t + \dfrac{x}{t}\Big)^2\Big/2\Big]}{\dfrac{1}{t\sqrt{2\pi}} e^{-t^2/2}}$$

$$= \lim_{t\to\infty} \frac{t^2}{t^2 + x} \exp\Big(-x - \frac{x^2}{2t^2}\Big) = e^{-x}.$$

33. Let X be the amount of soft drink in a random bottle. We are given that $P(X < 15.5) = 0.07$ and $P(X > 16.3) = 0.10$. These imply that $\Phi\Big(\dfrac{15.5 - \mu}{\sigma}\Big) = 0.07$ and and $\Phi\Big(\dfrac{16.3 - \mu}{\sigma}\Big) = 0.90$. Using Tables 1 and 2 of the appendix, we obtain

$$\begin{cases} \dfrac{15.5 - \mu}{\sigma} = -1.48 \\ \dfrac{16.3 - \mu}{\sigma} = 1.28. \end{cases}$$

Solving these two equations in two unknowns, we obtain $\mu = 15.93$ and $\sigma = 0.29$.

34. Let X be the height of a randomly selected skeleton from group 1. Then

$$P(X > 185) = P\Big(Z > \frac{185 - 172}{9}\Big) = P(Z > 1.44) = 0.0749.$$

Now suppose that the skeleton's of the second group belong to the family of the first group. The probability of finding three or more skeleton's with heights above 185 centimeters is

$$\sum_{i=3}^{5} \binom{5}{i} (0.0749)^i (0.9251)^{5-i} = 0.0037.$$

Since the chance of this event is very low, it is reasonable to assume that the second group is not part of the first one. However, we must be careful that in reality, this observation is not sufficient to make a judgment. In the lack of other information, if a decision is to be made solely based on this observation, then we must reject the hypothesis that the second group is part of the first one.

35. For $t \in (0, \infty)$, let A be the region whose points have a (positive) distance t or less from the given tree. The area of A is πt^2. Let X be the distance from the given tree to its nearest tree. We have that

$$P(X > t) = P(\text{no trees in } A) = \frac{e^{-\lambda\pi t^2}(\lambda\pi t^2)^0}{0!} = e^{-\lambda\pi t^2}.$$

Now by Remark 6.4,

$$E(X) = \int_0^\infty P(X > t)\,dt = \int_0^\infty e^{-\lambda\pi t^2}\,dt.$$

Letting $u = \big(\sqrt{2\lambda\pi}\,\big)t$, we obtain

$$E(X) = \frac{1}{\sqrt{\lambda}}\frac{1}{\sqrt{2\pi}}\int_0^\infty e^{-u^2/2}\,du = \frac{1}{\sqrt{\lambda}}\frac{1}{2} = \frac{1}{2\sqrt{\lambda}}.$$

36. Note that $dy = x\,ds$; so

$$\begin{aligned} I^2 &= \int_0^\infty \Big[\int_0^\infty e^{-(x^2+x^2s^2)/2}\,x\,ds\Big]\,dx = \int_0^\infty \Big[\int_0^\infty e^{-x^2(1+s^2)/2}\,x\,dx\Big]\,ds \qquad (\text{let } u = x^2)\\ &= \int_0^\infty \Big[\int_0^\infty e^{-u(1+s^2)/2}\,\frac{1}{2}\,du\Big]\,ds = \frac{1}{2}\int_0^\infty \Big[-\frac{2}{1+s^2}e^{-u(1+s^2)/2}\Big]_0^\infty\,ds\\ &= \int_0^\infty \frac{1}{1+s^2}\,ds = \Big[\arctan s\Big]_0^\infty = \frac{\pi}{2}. \end{aligned}$$

7.3 EXPONENTIAL RANDOM VARIABLES

1. Let X be the time until the next customer arrives; X is exponential with parameter $\lambda = 3$. Hence $P(X > x) = e^{-\lambda x}$, and $P(X > 3) = e^{-9} = 0.0001234$.

2. Let m be the median of an exponential random variable with rate λ. Then $P(X > m) = 1/2$; thus $e^{-\lambda m} = 1/2$ or $m = \dfrac{\ln 2}{\lambda}$.

3. For $-\infty < y < \infty$,

$$P(Y \le y) = P(-\ln X \le y) = P\big(X \ge e^{-y}\big) = e^{-e^{-y}}.$$

Thus $g(y)$, the probability density function of Y is given by

$$g(y) = \frac{d}{dy} P(Y \le y) = e^{-y} \cdot e^{-e^{-y}} = e^{-y - e^{-y}}.$$

4. Let X be the time between the first and second heart attacks. We are given that $P(X \le 5) = 1/2$. Since exponential is memoryless, the probability that a person who had one heart attack five years ago will not have another one during the next five years is still $P(X > 5)$ which is $1 - P(X \le 5) = 1/2$.

5. **(a)** Suppose that the next customer arrives in X minutes. By the memoryless property, the desired probability is

$$P\Big(X < \frac{1}{30}\Big) = 1 - e^{-5(1/30)} = 0.1535.$$

(b) Let Y be the time between the arrival times of the 10th and 11th customers; Y is exponential with $\lambda = 5$. So the answer is

$$P\Big(Y \le \frac{1}{30}\Big) = 1 - e^{-5(1/30)} = 0.1535.$$

6.

$$\begin{aligned}
P\big(|X - E(X)| \ge 2\sigma_X\big) &= P\Big(\Big|X - \frac{1}{\lambda}\Big| \ge \frac{2}{\lambda}\Big) \\
&= P\Big(X - \frac{1}{\lambda} \ge \frac{2}{\lambda}\Big) + P\Big(X - \frac{1}{\lambda} \le -\frac{2}{\lambda}\Big) \\
&= P\Big(X \ge \frac{3}{\lambda}\Big) + P\Big(X \le -\frac{1}{\lambda}\Big) \\
&= e^{-\lambda(3/\lambda)} + 0 = e^{-3} = 0.049787.
\end{aligned}$$

7. **(a)** $P(X > t) = e^{-\lambda t}$.

(b) $P(t \le X \le s) = \big(1 - e^{-\lambda s}\big) - \big(1 - e^{-\lambda t}\big) = e^{-\lambda t} - e^{-\lambda s}$.

8. The number of documents typed by the secretary on a given eight-hour working day is Poisson with parameter $\lambda = 8$. So the answer is

$$\sum_{i=12}^{\infty} \frac{e^{-8} 8^i}{i!} = 1 - \sum_{i=0}^{11} \frac{e^{-8} 8^i}{i!} = 1 - 0.888 = 0.112.$$

9. The answer is

$$E\big[350 - 40N(12)\big] = 350 - 40\Big(\frac{1}{18}\cdot 12\Big) = 323.33.$$

10. Mr. Jones makes his phone calls when either A or B is finished his call. At that time the remaining phone call of A or B, whichever is not finished, and the duration of the call of Mr. Jones both have the same distribution due to the memoryless property of the exponential distribution. Hence, by symmetry, the probability that Mr. Jones finishes his call sooner than the other one is $1/2$.

11. Let $N(t)$ be the number of change-of-states occurring in $[0, t]$. Let X_1 be the time until the machine breaks down for the first time. Let X_2 be the time it will take to repair the machine, X_3 be the time since the machine was fixed until it breaks down again, and so on. Clearly, X_1, $X_2, \dots$ are the times between consecutive change of states. Since $\{X_1, X_2, \dots\}$ is a sequence of independent and identically distributed exponential random variables with mean $1/\lambda$, by Remark 7.2, $\big\{N(t)\colon t \ge 0\big\}$ is a Poisson process with rate λ. Therefore, $N(t)$ is a Poisson random variable with parameter λt.

12. The probability mass function of L is given by

$$P(L = n) = (1-p)^{n-1}p, \quad n = 1, 2, 3, \dots .$$

Hence

$$P(L > n) = (1-p)^n, \quad n = 0, 1, 2, \dots .$$

Therefore,

$$\begin{aligned} P(T \le x) &= P(L \le 1000x) = 1 - P(L > 1000x) = 1 - (1-p)^{1000x} \\ &= 1 - e^{1000x\ln(1-p)} = 1 - e^{-x[-1000\ln(1-p)]}, \quad x > 0. \end{aligned}$$

This shows that T is exponential with parameter $\lambda = -1000\ln(1-p)$.

13. **(a)** We must have $\int_{-\infty}^{\infty} ce^{-|x|}\,dx = 1$; thus

$$c = \frac{1}{\int_{-\infty}^{\infty} e^{-|x|}\,dx} = \frac{1}{2\int_0^{\infty} e^{-x}\,dx} = \frac{1}{2}.$$

(b) $E(X^{2n+1}) = \int_{-\infty}^{\infty} \frac{1}{2}x^{2n+1}e^{-|x|}\,dx = 0$, because the integrand is an odd function.

$$E(X^{2n}) = \int_{-\infty}^{\infty} \frac{1}{2}x^{2n}e^{-|x|}\,dx = \int_0^{\infty} x^{2n}e^{-x}\,dx,$$

because the integrand is an even function. We now use induction to prove that $\int_0^{\infty} x^n e^{-x}\,dx = n!$. For $n = 1$, the integral is the expected value of an exponential random variable with parameter 1; so it equals to $1 = 1!$. Assume that the identity is valid for $n-1$. Using integration by parts, we show it for n.

$$\int_0^\infty x^n e^{-x}\,dx = -\Big[-x^n e^{-x}\Big]_0^\infty + \int_0^\infty n x^{n-1} e^{-x}\,dx = 0 + n(n-1)! = n!.$$

Hence $E(X^{2n}) = (2n)!$.

14. $P\big([X]=n\big) = P(n \le X < n+1) = \int_n^{n+1} \lambda e^{-\lambda x}\,dx = -e^{-\lambda x}\Big|_n^{n+1} = \big(e^{-\lambda}\big)^n\big(1-e^{-\lambda}\big)$. This is the probability mass function of a geometric random variable with parameter $p = 1 - e^{-\lambda}$.

15. Let that $G(t) = P(X > t) = 1 - F(t)$. By the memoryless property of X,

$$P(X > s + t \mid X > t) = P(X > s),$$

for all $s \ge 0$ and $t \ge 0$. This implies that

$$P(X > s + t) = P(X > s)P(X > t),$$

or

$$G(s+t) = G(s)G(t), \quad t \ge 0,\ s \ge 0. \tag{24}$$

Now for arbitrary positive integers n and m, (24) gives that

$$G\Big(\frac{2}{n}\Big) = G\Big(\frac{1}{n} + \frac{1}{n}\Big) = G\Big(\frac{1}{n}\Big)G\Big(\frac{1}{n}\Big) = \Big[G\Big(\frac{1}{n}\Big)\Big]^2,$$

$$G\Big(\frac{3}{n}\Big) = G\Big(\frac{2}{n} + \frac{1}{n}\Big) = G\Big(\frac{2}{n}\Big)G\Big(\frac{1}{n}\Big) = \Big[G\Big(\frac{1}{n}\Big)\Big]^2 G\Big(\frac{1}{n}\Big) = G\Big(\frac{1}{n}\Big)^3,$$

$$\vdots$$

$$G\Big(\frac{m}{n}\Big) = \Big[G\Big(\frac{1}{n}\Big)\Big]^m.$$

Also

$$G(1) = G\Big(\underbrace{\frac{1}{n} + \frac{1}{n} + \cdots + \frac{1}{n}}_{n \text{ terms}}\Big) = \Big[G\Big(\frac{1}{n}\Big)\Big]^n$$

yields

$$G\Big(\frac{1}{n}\Big) = \big[G(1)\big]^{1/n}. \tag{25}$$

Hence

$$G(m/n) = \big[G(1)\big]^{m/n}. \tag{26}$$

Now we show that $G(1) > 0$. If not, $G(1) = 0$ and by (25), $G(1/n) = 0$ for all positive integer n. This and right continuity of G imply that

$$P(X \leq 0) = F(0) = 1 - G(0) = 1 - G\Big(\lim_{n\to\infty} \frac{1}{n}\Big)$$
$$= 1 - \lim_{n\to\infty} G\Big(\frac{1}{n}\Big) = 1 - 0 = 1,$$

which is a contradiction to the given fact that X is a positive random variable. Thus $G(1) > 0$ and we can define $\lambda = -\ln\big[G(1)\big]$. This gives

$$G(1) = e^{-\lambda},$$

and by (26),

$$G(m/n) = e^{-\lambda(m/n)}.$$

Thus far, we have proved that for any positive rational t,

$$G(t) = e^{-\lambda t}. \tag{27}$$

To prove the same relation for a positive irrational number t, recall from calculus that for each positive integer n, there exists a rational number t_n in $\Big(t,\ t + \dfrac{1}{n}\Big)$. Since $t < t_n < t + \dfrac{1}{n}$, $\lim_{n\to\infty} t_n$ exists and is t. On the other hand because F is right continuous, $G = 1 - F$ is also right continuous and so

$$G(t) = \lim_{n\to\infty} G(t_n).$$

But since t_n is rational, (27) implies that, $G(t_n) = e^{-\lambda t_n}$. Hence

$$G(t) = \lim_{n\to\infty} e^{-\lambda t_n} = e^{-\lambda t}.$$

Thus $F(t) = 1 - e^{-\lambda t}$ for all t, and X is exponential.

Remark: If X is memoryless, then $P(X \leq 0) = 0$. To see this, note that $P(X > s + t \mid X > t) = P(X > s)$ implies $P(X \leq s + t \mid X > t) = P(X \leq s)$. Letting $s = t = 0$, we get $P(X \leq 0 \mid X > 0) = P(X \leq 0)$. But $P(X \leq 0 \mid X > 0) = 0$; therefore $P(X \leq 0) = 0$. This shows that the memoryless property cannot be defined for random variables possessing nonpositive values with positive probability.

7.4 GAMMA DISTRIBUTIONS

1. Let f be the probability density function of a gamma random variable with parameters r and λ. Then

$$f(x)=\frac{\lambda^r x^{r-1}e^{-\lambda x}}{\Gamma(r)}.$$

Therefore,

$$f'(x)=\frac{\lambda^r}{\Gamma(r)}\Big[-\lambda e^{-\lambda x}x^{r-1}+e^{-\lambda x}(r-1)x^{r-2}\Big]=-\frac{\lambda^{r+1}}{\Gamma(r)}\,x^{r-2}e^{-\lambda x}\Big(x-\frac{r-1}{\lambda}\Big).$$

This relation implies that the function f is increasing if $x<(r-1)/\lambda$, it is decreasing if $x>(r-1)/\lambda$, and $f'(x)=0$ if $x=(r-1)/\lambda$. Therefore, $x=(r-1)/\lambda$ is a maximum of the function f. Moreover, since f' has only one root, the point $x=(r-1)/\lambda$ is the only maximum of f.

2. We have that

$$\begin{aligned}P(cX\le t)=P(X\le t/c)&=\int_0^{t/c}\frac{(\lambda e^{-\lambda x})(\lambda x)^{r-1}}{\Gamma(r)}\,dx \qquad (\text{let } u=cx)\\&=\int_0^{t}\frac{\lambda e^{-\lambda u/c}(\lambda u/c)^{r-1}}{\Gamma(r)}(1/c)\,du\\&=\int_0^{t}\frac{(\lambda/c)e^{-\lambda u/c}(\lambda u/c)^{r-1}}{\Gamma(r)}\,du.\end{aligned}$$

This shows that cX is gamma with parameters r and λ/c.

3. Let $N(t)$ be the number of babies born at or prior to t. $\big\{N(t)\colon t\ge 0\big\}$ is a Poisson process with $\lambda=12$. Let X be the time it takes before the next three babies are born. The random variable X is gamma with parameters 3 and 12. The desired probability is

$$P(X\ge 7/24)=\int_{7/24}^{\infty}\frac{12e^{-12x}(12x)^2}{\Gamma(3)}\,dx=864\int_{7/24}^{\infty}x^2e^{-12x}\,dx.$$

Applying integration by parts twice, we get

$$\int x^2e^{-12x}\,dx=-\frac{1}{12}x^2e^{-12x}-\frac{1}{72}xe^{-12x}-\frac{1}{864}e^{-12x}+c.$$

Thus

$$P\Big(X\ge\frac{7}{24}\Big)=864\Big[-\frac{1}{12}x^2e^{-12x}-\frac{1}{72}xe^{-12x}-\frac{1}{864}e^{-12x}\Big]_{7/24}^{\infty}=0.3208.$$

Remark: A simpler way to do this problem is to avoid gamma random variables and use the properties of Poisson processes:

$$P\Big(N\Big(\frac{7}{24}\Big)\le 2\Big)=\sum_{i=0}^{2}P\Big(N\Big(\frac{7}{24}\Big)=i\Big)=\sum_{i=0}^{2}\frac{e^{-(7/24)12}\big[(7/24)12\big]^i}{i!}=0.3208.$$

4.

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_0^{\infty} \frac{\lambda e^{-\lambda x}(\lambda x)^{r-1}}{\Gamma(r)}\,dx = \frac{\lambda^r}{\Gamma(r)} \int_0^{\infty} e^{-\lambda x} x^{r-1}\,dx.$$

Let $t = \lambda x$; then $dt = \lambda dx$, so

$$\begin{aligned}\int_{-\infty}^{\infty} f(x)\,dx &= \frac{\lambda^r}{\Gamma(r)} \int_0^{\infty} e^{-t} \cdot \frac{t^{r-1}}{\lambda^{r-1}} \cdot \frac{1}{\lambda}\,dx \\ &= \frac{1}{\Gamma(r)} \int_0^{\infty} e^{-t} t^{r-1}\,dt = \frac{1}{\Gamma(r)}\,\Gamma(r) = 1.\end{aligned}$$

5. Let X be the time until the restaurant starts to make profit; X is a gamma random variable with parameters 31 and 12. Thus $E(X) = 31/12$; that is, two hours and 35 minutes.

6. By the method of Example 5.17, the number of defective light bulbs produced is a Poisson process at the rate of $(200)(0.015) = 3$ per hour. Therefore, X, the time until 25 defective light bulbs are produced is gamma with parameters $\lambda = 3$ and $r = 25$. Hence

$$E(X) = \frac{r}{\lambda} = \frac{25}{3} = 8.33.$$

That is, it will take, on average, 8 hours and 20 minutes to fill up the can.

7.

$$\Gamma\Big(\frac{1}{2}\Big) = \int_0^{\infty} t^{-1/2} e^{-t}\,dt.$$

Making the substitution $t = y^2/2$, we get

$$\begin{aligned}\Gamma\Big(\frac{1}{2}\Big) &= \sqrt{2} \int_0^{\infty} e^{-y^2/2}\,dy = \frac{\sqrt{2}}{2} \int_{-\infty}^{\infty} e^{-y^2/2}\,dy \\ &= \sqrt{\pi} \cdot \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-y^2/2}\,dy = \sqrt{\pi}.\end{aligned}$$

Hence

$$\Gamma\Big(\frac{3}{2}\Big) = \frac{1}{2}\Gamma\Big(\frac{1}{2}\Big) = \frac{1}{2}\cdot\sqrt{\pi},$$

$$\Gamma\Big(\frac{5}{2}\Big) = \frac{3}{2}\Gamma\Big(\frac{3}{2}\Big) = \frac{3}{2}\cdot\frac{1}{2}\cdot\sqrt{\pi},$$

$$\Gamma\Big(\frac{7}{2}\Big) = \frac{5}{2}\Gamma\Big(\frac{5}{2}\Big) = \frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\cdot\sqrt{\pi},$$

$$\vdots$$

$$\Gamma\Big(n+\frac{1}{2}\Big) = \Gamma\Big(\frac{2n+1}{2}\Big) = \frac{2n-1}{2}\cdot\frac{2n-3}{2}\cdots\frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\cdot\sqrt{\pi}$$

$$= \frac{(2n)!}{2^{2n}\big[(2n)\cdots 6\cdot 4\cdot 2\big]}\sqrt{\pi}$$

$$= \frac{(2n)!\sqrt{\pi}}{2^n\cdot 2^n\cdot n!} = \frac{(2n)!\sqrt{\pi}}{4^n\cdot n!}.$$

8. **(a)** Let F be the probability distribution function of Y. For $t \le 0$, $F(t) = P(Z^2 \le t) = 0$. For $t > 0$,

$$F(t) = P(Y \le t) = P\big(Z^2 \le t\big) = P\Big(-\sqrt{t} \le Z \le \sqrt{t}\Big)$$
$$= \Phi\big(\sqrt{t}\big) - \Phi\big(-\sqrt{t}\big) = \Phi\big(\sqrt{t}\big) - \Big[1 - \Phi\big(\sqrt{t}\big)\Big] = 2\Phi\big(\sqrt{t}\big) - 1.$$

Let f be the probability density function of Y. For $t \le 0$, $f(t) = 0$. For $t > 0$,

$$f(t) = F'(t) = 2\cdot\frac{1}{2\sqrt{t}}\,\Phi'\big(\sqrt{t}\big) = \frac{1}{\sqrt{t}}\cdot\frac{1}{\sqrt{2\pi}}e^{-t/2} = \frac{1}{\sqrt{2\pi t}}e^{-t/2} = \frac{\frac{1}{2}e^{-t/2}\Big(\frac{1}{2}t\Big)^{-1/2}}{\Gamma(1/2)},$$

where by the previous exercise, $\sqrt{\pi} = \Gamma(1/2)$. This shows that Y is gamma with parameters $\lambda = 1/2$ and $r = 1/2$.

(b) Since $(X-\mu)/\sigma$ is standard normal, by part (a), W is gamma with parameters $\lambda = 1/2$ and $r = 1/2$.

9. The following solution is an intuitive one. A rigorous mathematical solution would have to consider the sum of two random variables, each being the minimum of n exponential random variables; so it would require material from joint distributions. However, the intuitive solution has its own merits and it is important for students to understand it.

Let the time Howard enters the bank be the origin and let $N(t)$ be the number of customers served by time t. As long as all of the servers are busy, due to the memoryless property of the exponential distribution, $\big\{N(t)\colon t \geq 0\big\}$ is a Poisson process with rate $n\lambda$. This follows because if one server serves at the rate λ, n servers will serve at the rate $n\lambda$. For the Poisson process $\big\{N(t)\colon t \geq 0\big\}$, every time a customer is served and leaves, an "event" has occurred. Therefore, again because of the memoryless property, the service time of the person ahead of Howard begins when the first "event" occurs and Howard's service time begins when the second "event" occurs. Therefore, Howard's waiting time in the queue is the time of the second event of the Poisson process $\big\{N(t), t \geq 0\big\}$. This period, as we know, has a gamma distribution with parameters 2 and $n\lambda$.

10. Since the lengths of the characters are independent of each other and identically distributed, for any two intervals Δ_1 and Δ_2 with the same length, the probability that n characters are emitted during Δ_1 is equal to the probability that n characters are emitted in Δ_2. Moreover, for $s > 0$, the number of characters being emitted during $(t, t+s]$ is independent of the number of characters that have been emitted in $[0, t]$. Clearly, characters are not emitted simultaneously. Therefore, $\big\{N(t)\colon t \geq 0\big\}$ is stationary, possesses independent increments, and is orderly. So it is a Poisson process. By Exercise 11, Section 7.3, the time until the first character is emitted is exponential with parameter $\lambda = -1000 \ln(1-p)$. Thus $\big\{N(t)\colon t \geq 0\big\}$ is a Poisson process with parameter $\lambda = -1000 \ln(1-p)$. Knowing this, we have that the time until the message is emitted, that is, the time until the kth character is emitted is gamma with parameters k and $\lambda = -1000 \ln(1-p)$.

7.5 BETA DISTRIBUTIONS

1. Yes, it is a probability density function of a beta random variable with parameters $\alpha = 2$ and $\beta = 3$. Note that $\dfrac{1}{B(2,3)} = \dfrac{4!}{1!\,2!} = 12$. We have

$$E(X) = \frac{2}{5}, \qquad \text{Var}X = \frac{6}{6(5^2)} = \frac{1}{25}.$$

2. No, it is not because, for $\alpha = 3$ and $\beta = 5$, we have

$$\frac{1}{B(3,5)} = \frac{7!}{2!\,4!} = 105 \neq 120.$$

3. Let $\alpha = 5$ and $\beta = 6$. Then f is the probability density function of a beta random variable with parameters 5 and 6 for

$$c = \frac{1}{B(5,6)} = \frac{10!}{4!\,5!} = 1260.$$

For this value of c,

$$E(X) = \frac{5}{11}, \qquad \text{Var}X = \frac{30}{12(11^2)} = \frac{5}{242}.$$

4. The answer is

$$P(p \geq 0.60) = \int_{0.60}^{1} \frac{1}{B(20, 13)} x^{19} (1-x)^{12}\, dx$$
$$= \frac{32!}{19!\, 12!} \int_{0.60}^{1} x^{19} (1-x)^{12}\, dx = 0.538.$$

5. Let X be the proportion of resistors the procurement office purchases from this vendor. We know that X is beta. Let α and β be the parameters of the density function of X. Then

$$\begin{cases} \dfrac{\alpha}{\alpha+\beta} = \dfrac{1}{3} \\[2ex] \dfrac{\alpha\beta}{(\alpha+\beta+1)(\alpha+\beta)^2} = \dfrac{1}{18}. \end{cases}$$

Solving this system of 2 equations in 2 unknowns, we obtain $\alpha = 1$ and $\beta = 2$. The desired probability is

$$P(X \geq 7/12) = \int_{7/12}^{1} \frac{1}{B(1, 2)} x^{1-1} (1-x)^{2-1}\, dx = 2\int_{7/12}^{1} (1-x)\, dx = \frac{50}{288} \approx 0.17.$$

6. Let X be the median of the fractions for the 13 sections of the course; X is a beta random variable with parameters 7 and 7. Let Y be a binomial random variable with parameters 13 and 0.40. By Theorem 7.2,

$$P(X \leq 0.40) = P(Y \geq 7).$$

Therefore,

$$P(X \geq 0.40) = P(Y \leq 6) = \sum_{i=0}^{6} \binom{13}{i} (0.40)^i (0.60)^{13-i} = 0.771156.$$

7. Let Y be a binomial random variable with parameters 25 and 0.25; by Theorem 7.2,

$$P(X \leq 0.25) = P(Y \geq 5).$$

Therefore,

$$P(X \geq 0.25) = P(Y < 5) = \sum_{i=0}^{4} \binom{25}{i} (0.25)^i (0.75)^{25-i} = 0.214.$$

8. **(a)** Clearly,

$$E(Y) = a + (b-a)E(X) = a + (b-a)\frac{\alpha}{\alpha+\beta},$$

$$\text{Var}(X) = (b-a)^2\,\text{Var}(X) = \frac{(b-a)^2\alpha\beta}{(\alpha+\beta+1)(\alpha+\beta)^2}.$$

(b) Note that $0 < X < 1$ implies that $a < Y < b$. Let $a < t < b$; then

$$P(Y \le t) = P\big(a + (b-a)X \le t\big) = P\Big(X \le \frac{t-a}{b-a}\Big)$$
$$= \int_0^{(t-a)/(b-a)} \frac{1}{B(\alpha,\beta)}\, x^{\alpha-1}\,(1-x)^{\beta-1}\,dx.$$

Let $y = (b-a)x + a$; we have

$$P(Y \le t) = \int_a^t \frac{1}{B(\alpha,\beta)}\Big(\frac{y-a}{b-a}\Big)^{\alpha-1}\Big(1 - \frac{y-a}{b-a}\Big)^{\beta-1} \cdot \frac{1}{b-a}\,dy$$
$$= \int_a^t \frac{1}{b-a} \cdot \frac{1}{B(\alpha,\beta)}\Big(\frac{y-a}{b-a}\Big)^{\alpha-1}\Big(\frac{b-y}{b-a}\Big)^{\beta-1}\,dy.$$

This shows that the probability density function of Y is

$$f(y) = \frac{1}{b-a} \cdot \frac{1}{B(\alpha,\beta)}\Big(\frac{y-a}{b-a}\Big)^{\alpha-1}\Big(\frac{b-y}{b-a}\Big)^{\beta-1}, \quad a < y < b.$$

(c) Note that $a = 2$, $b = 6$. Hence

$$P(Y < 3) = \int_2^3 \frac{1}{4} \cdot \frac{4!}{1!\,2!}\Big(\frac{y-2}{4}\Big)\Big(\frac{6-y}{4}\Big)^2\,dy$$
$$= \frac{3}{64}\int_2^3 (y-2)(6-y)^2\,dy = \frac{3}{64} \cdot \frac{67}{12} = \frac{67}{256} \approx 0.26.$$

9. Suppose that

$$f(x) = \frac{1}{B(\alpha,\beta)}\, x^{\alpha-1}(1-x)^{\beta-1}, \quad 0 < x < 1,$$

is symmetric about a point a. Then $f(a-x) = f(a+x)$. That is, for $0 < x < \min(a, 1-a)$,

$$(a-x)^{\alpha-1}(1-a+x)^{\beta-1} = (a+x)^{\alpha-1}(1-a-x)^{\beta-1}. \tag{28}$$

Since α and β are not necessarily integers, for $(a-x)^{\alpha-1}$ and $(1-a-x)^{\beta-1}$ to be well-defined, we need to restrict ourselves to the range $0 < x < \min(a, 1-a)$. Now, if $a < 1-a$, then, by continuity, (28) is valid for $x = a$. Substituting a for x in (28), we obtain

$$(2a)^{\alpha-1}(1-2a)^{\beta-1} = 0.$$

Since $a \neq 0$, this implies that $a = 1/2$. If $1 - a < a$, then, by continuity, (28) is valid for $x = 1 - a$. Substituting $1 - a$ for x in (28), we obtain

$$(2a - 1)^{\alpha-1}(2 - 2a)^{\beta-1} = 0.$$

Since $a \neq 1$, this implies that $a = 1/2$. Therefore, in either case $a = 1/2$. In (28), substituting $a = 1/2$, and taking $x = 1/4$, say, we get

$$(1/4)^{\alpha-1}(3/4)^{\beta-1} = (3/4)^{\alpha-1}(1/4)^{\beta-1}.$$

This gives $3^{\beta-\alpha} = 0$, which can only hold for $\alpha = \beta$. Therefore, only beta density functions with $\alpha = \beta$ are symmetric, and they are symmetric about $a = 1/2$.

10. $t = 0$ gives $x = 0$; $t = \infty$ gives $x = 1$. Since $dx = \dfrac{2t}{(1+t^2)^2}\,dt$, we have

$$B(\alpha, \beta) = \int_0^\infty \Big(\frac{t^2}{1+t^2}\Big)^{\alpha-1}\Big(\frac{1}{1+t^2}\Big)^{\beta-1} \cdot \frac{2t}{(1+t^2)^2}\,dt = 2\int_0^\infty t^{2\alpha-1}(1+t^2)^{-(\alpha+\beta)}\,dt.$$

11. We have that

$$B(\alpha, \beta) = \int_0^1 x^{\alpha-1}(1-x)^{\beta-1}\,dx.$$

Let $x = \cos^2\theta$ to obtain

$$B(\alpha, \beta) = 2\int_0^{\pi/2} (\cos\theta)^{2\alpha-1}(\sin\theta)^{2\beta-1}\,d\theta.$$

Now

$$\Gamma(\alpha) = \int_0^\infty t^{\alpha-1}e^{-t}\,dt.$$

Use the substitution $t = y^2$ to obtain

$$\Gamma(\alpha) = 2\int_0^\infty y^{2\alpha-1}e^{-y^2}\,dy.$$

This implies that

$$\Gamma(\alpha)\Gamma(\beta) = 4\int_0^\infty\int_0^\infty x^{2\alpha-1}y^{2\beta-1}e^{-(x^2+y^2)}\,dx\,dy.$$

Now we evaluate this double integral by means of a change of variables to polar coordinates: $y = r\sin\theta$, $x = r\cos\theta$; we obtain

$$\begin{aligned}\Gamma(\alpha)\Gamma(\beta) &= 4\int_0^\infty\int_0^{\pi/2} r^{2(\alpha+\beta)-1}(\cos\theta)^{2\alpha-1}(\sin\theta)^{2\beta-1}e^{-r^2}\,d\theta\,dr\\ &= 2B(\alpha,\beta)\int_0^\infty r^{2(\alpha+\beta)-1}e^{-r^2}\,dr = B(\alpha,\beta)\int_0^\infty u^{\alpha+\beta-1}e^{-u}\,du \quad (\text{let } u = r^2)\\ &= B(\alpha,\beta)\Gamma(\alpha+\beta).\end{aligned}$$

Thus

$$B(\alpha, \beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha + \beta)}.$$

12. We will show that $E(X^2) = n/(n-2)$. Since $E(X^2) < \infty$, by Remark 6.6, $E(X) < \infty$. Since $E(X)$ exists and $xf(x)$ is an odd function, we have

$$E(X) = \int_{-\infty}^{\infty} xf(x)\,dx = 0.$$

Consequently,

$$\text{Var}(X) = E(X^2) - \big[E(X)\big]^2 = \frac{n}{n-2}.$$

Therefore, all we need to find is $E(X^2)$. By Theorem 6.3,

$$E(X^2) = \frac{\Gamma\Big(\dfrac{n+1}{2}\Big)}{\sqrt{n\pi}\,\Gamma\Big(\dfrac{n}{2}\Big)} \int_{-\infty}^{\infty} x^2\Big(1 + \frac{x^2}{n}\Big)^{-(n+1)/2}\,dx.$$

Substituting $x = (\sqrt{n}\,)t$ in this integral yields

$$E(X^2) = \frac{\Gamma\Big(\dfrac{n+1}{2}\Big)}{\sqrt{n\pi}\,\Gamma\Big(\dfrac{n}{2}\Big)} \int_{-\infty}^{\infty} (nt^2)(1+t^2)^{-(n+1)/2}\sqrt{n}\,dt$$

$$= \frac{\Gamma\Big(\dfrac{n+1}{2}\Big)}{\sqrt{\pi}\,\Gamma\Big(\dfrac{n}{2}\Big)} \cdot 2n\int_0^{\infty} t^2(1+t^2)^{-(n+1)/2}\,dt.$$

By the previous two exercises,

$$2\int_0^{\infty} t^2(1+t^2)^{-(n+1)/2}\,dt = B\Big(\frac{3}{2}, \frac{n-2}{2}\Big) = \frac{\Gamma\Big(\dfrac{3}{2}\Big)\Gamma\Big(\dfrac{n-2}{2}\Big)}{\Gamma\Big(\dfrac{n+1}{2}\Big)}.$$

Therefore,

$$E(X^2) = \frac{\Gamma\Big(\dfrac{n+1}{2}\Big)}{\sqrt{\pi}\cdot\Gamma\Big(\dfrac{n}{2}\Big)} \cdot n \cdot \frac{\Gamma\Big(\dfrac{3}{2}\Big)\Gamma\Big(\dfrac{n-2}{2}\Big)}{\Gamma\Big(\dfrac{n+1}{2}\Big)} = \frac{n\Gamma\Big(\dfrac{3}{2}\Big)\Gamma\Big(\dfrac{n-2}{2}\Big)}{\sqrt{\pi}\,\Gamma\Big(\dfrac{n}{2}\Big)}.$$

By the solution to Exercise 7, Section 7.4, $\Gamma(1/2) = \sqrt{\pi}$. Using the identity $\Gamma(r+1) = r\Gamma(r)$, we have

$$\Gamma\Big(\frac{3}{2}\Big) = \frac{1}{2}\Gamma\Big(\frac{1}{2}\Big) = \frac{\sqrt{\pi}}{2};$$

$$\Gamma\Big(\frac{n}{2}\Big) = \Gamma\Big(\frac{n-2}{2} + 1\Big) = \frac{n-2}{2}\Gamma\Big(\frac{n-2}{2}\Big).$$

Consequently,

$$E(X^2) = \frac{n\dfrac{\sqrt{\pi}}{2}\Gamma\Big(\dfrac{n-2}{2}\Big)}{\sqrt{\pi}\cdot\dfrac{n-2}{2}\Gamma\Big(\dfrac{n-2}{2}\Big)} = \frac{n}{n-2}.$$

7.6 SURVIVAL ANALYSIS AND HAZARD FUNCTIONS

1. Let X be the lifetime of the electrical component, F be its probability distribution function, and $\lambda(t)$ be its failure rate. For some constants α and β, we are given that

$$\lambda(t) = \alpha t + \beta.$$

Since $\lambda(48) = 0.10$ and $\lambda(72) = 0.15$,

$$\begin{cases} 48\alpha + \beta = 0.10 \\ 72\alpha + \beta = 0.15. \end{cases}$$

Solving this system of two equations in two unknowns gives $\alpha = 1/480$ and $\beta = 0$. Hence $\lambda(t) = t/480$. By (7.6), for $t > 0$,

$$P(X > t) = \bar{F}(t) = \exp\Big(-\int_0^t \frac{u}{480}\,du\Big) = e^{-t^2/960}.$$

Let f be the probability density function of X. This also gives

$$f(t) = -\frac{d}{dt}\bar{F}(t) = \frac{t}{480}e^{-t^2/960}.$$

The answer to part (a) is

$$P(X > 30) = e^{-900/960} = e^{-0.9375} = 0.392.$$

The exact value for part (b) is

$$\begin{aligned} P(X < 31 \mid X > 30) &= \frac{P(30 < X < 31)}{P(X > 30)} \\ &= \frac{1}{0.392}\int_{30}^{31}(t/480)e^{-t^2/960}\,dt = \frac{0.02411}{0.392} = 0.0615. \end{aligned}$$

Note that for small Δ_t, $\lambda(t)\Delta_t$ is approximately the probability that the component fails within Δ_t hours after t, given that it has not yet failed by time t. Letting $\Delta_t = 1$, for $t = 30$, $\lambda(t)\Delta_t \approx 0.0625$ which is relatively close to the exact value of 0.0615. This is interesting because $\Delta_t = 1$ is not that small, and one may not expect close approximations anyway.

2. Let $\bar{F}$ be the survival function of a Weibull random variable. We have

$$\bar{F}(t) = \int_t^\infty \alpha x^{\alpha-1} e^{-x^\alpha}\, dx.$$

Letting $u = x^\alpha$, we have $du = \alpha x^{\alpha-1}\, dx$. Thus

$$\bar{F}(t) = \int_{t^\alpha}^\infty e^{-u}\, du = -e^{-u}\Big|_{t^\alpha}^\infty = e^{-t^\alpha}.$$

Therefore,

$$\lambda(t) = \frac{\alpha t^{\alpha-1} e^{-t^\alpha}}{e^{-t^\alpha}} = \alpha t^{\alpha-1}.$$

$\lambda(t) = 1$, for $\alpha = 1$; so the Weibull in this case is exponential with parameter 1. Clearly, for $\alpha < 1$, $\lambda'(t) < 0$; so $\lambda(t)$ is decreasing. For $\alpha > 1$, $\lambda'(t) > 0$; so $\lambda(t)$ is increasing. Note that for $\alpha = 2$, the failure rate is the straight line $\lambda(t) = 2t$.

REVIEW PROBLEMS FOR CHAPTER 7

1. $\dfrac{30 - 25}{37 - 25} = \dfrac{5}{12}$.

2. Let X be the weight of a randomly selected women from this community. The desired quantity is

$$P(X > 170 \mid X > 140) = \frac{P(X > 170)}{P(X > 140)} = \frac{P\Big(Z > \dfrac{170 - 130}{20}\Big)}{P\Big(Z > \dfrac{140 - 130}{20}\Big)}$$

$$= \frac{P(Z > 2)}{P(Z > 0.5)} = \frac{1 - \Phi(2)}{1 - \Phi(0.5)} = \frac{1 - 0.9772}{1 - 0.6915} = 0.074.$$

3. Let X be the number of times the digit 5 is generated; X is binomial with parameters $n = 1000$ and $p = 1/10$. Thus $np = 100$ and $\sqrt{np(1-p)} = \sqrt{90} = 9.49$. Using normal approximation and making correction for continuity,

$$P(X \le 93.5) = P\Big(Z \le \frac{93.5 - 100}{9.49}\Big) = P(Z \le -0.68) = 1 - \Phi(0.68) = 0.248.$$

4. The given relation implies that

$$1 - e^{-2\lambda} = 2\big[(1 - e^{-3\lambda}) - (1 - e^{-2\lambda})\big].$$

This is equivalent to

$$3e^{-2\lambda} - 2e^{-3\lambda} - 1 = 0,$$

or, equivalently,

$$\big(e^{-\lambda} - 1\big)^2\big(2e^{-\lambda} + 1\big) = 0.$$

The only root of this equation is $\lambda = 0$ which is not acceptable. Therefore, it is not possible that X satisfy the given relation.

5. Let X be the lifetime of a random light bulb. Then

$$P(X < 1700) = 1 - e^{-(1/1700)\cdot 1700} = 1 - e^{-1}.$$

The desired probability is

$$\begin{aligned} &1 - P(\text{none fails}) - P(\text{one fails}) \\ &\quad = 1 - \binom{20}{0}(1 - e^{-1})^0(e^{-1})^{20} - \binom{20}{1}\big(1 - e^{-1}\big)\big(e^{-1}\big)^{19} = 0.999999927. \end{aligned}$$

6. Note that $\lim_{x\to 0} x \ln x = 0$; so

$$E(-\ln X) = \int_0^1 (-\ln x)\,dx = \Big[x - x\ln x\Big]_0^1 = 1.$$

7. Let X be the diameter of the randomly chosen disk in inches. We are given that $X \sim N(4, 1)$. We want to find the distribution function of $2.5X$; we have

$$P(2.5X \le x) = P(X \le x/2.5) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x/2.5} e^{-(t-4)^2/2}\,dt.$$

8. If $\alpha < 0$, then $\alpha + \beta < \beta$; therefore,

$$P(\alpha \le X \le \alpha + \beta) = P(0 \le X \le \alpha + \beta) \le P(0 \le X \le \beta).$$

If $\alpha > 0$, then $e^{-\lambda\alpha} < 1$. Thus

$$\begin{aligned} P(\alpha \le X \le \alpha + \beta) &= \big[1 - e^{-\lambda(\alpha+\beta)}\big] - \big(1 - e^{-\lambda\alpha}\big) \\ &= e^{-\lambda\alpha}\big(1 - e^{-\lambda\beta}\big) < 1 - e^{-\lambda\beta} = P(0 \le X \le \beta). \end{aligned}$$

9. We are given that $1/\lambda = 1.25$; so $\lambda = 0.8$. Let X be the time it takes for a random student to complete the test. Since $P(X > 1) = e^{-(0.8)1} = e^{-0.8}$, the desired probability is

$$1 - \big(e^{-0.8}\big)^{10} = 1 - e^{-8} = 0.99966.$$

10. Note that

$$f(x) = ke^{-[x-(3/2)]^2+17/4} = ke^{17/4} \cdot e^{-[x-(3/2)]^2}.$$

Comparing this with the probability density function of a normal random variable with mean 3/2, we see that $\sigma^2 = 1/2$ and $ke^{17/4} = 1/(\sigma\sqrt{2\pi})$. Therefore,

$$k = \frac{1}{\sigma\sqrt{2\pi}} e^{-17/4} = \frac{1}{\pi} e^{-17/4}.$$

11. Let X be the grade of a randomly selected student.

$$P(X \geq 90) = P\Big(Z \geq \frac{90-72}{7}\Big) = 1 - \Phi(2.57) = 0.0051.$$

Similarly,

$$\begin{aligned} P(80 \leq X < 90) &= P(1.14 \leq Z < 2.57) = 0.122, \\ P(70 \leq X < 80) &= P(-0.29 \leq Z < 1.14) = 0.487, \\ P(60 \leq X < 70) &= P(-1.71 \leq Z < -0.29) = 0.3423, \\ P(X < 60) &= P(Z < -1.71) = 0.0436. \end{aligned}$$

Therefore, approximately 0.51% will get A, 12.2% will get B, 48.7% will get C, 34.23% D, and 4.36% F.

12. Since $E(X) = 1/\lambda$,

$$P\big(X > E(X)\big) = e^{-\lambda(1/\lambda)} = e^{-1} = 0.36788.$$

13. Round off error to the nearest integer is uniform over $(-0.5, 0.5)$; round off error to the nearest 1st decimal place is uniform over $(-0.05, 0.05)$; round off error to the nearest 2nd decimal place is uniform over $(-0.005, 0.005)$, and so on. In general, round off error to the nearest k decimal places is uniform over $(-5/10^{k+1},\ 5/10^{k+1})$.

14. We want to find the smallest a for which $P(X \leq a) \geq 0.90$. This implies

$$P\Big(Z \leq \frac{a-175}{22}\Big) \geq 0.90.$$

Using Table 1 of the appendix, we see that $(a-175)/22 = 1.29$ or $a = 203.38$.

15. Let X be the breaking strength of the yarn under consideration. Clearly,

$$P(X \geq 100) = P\Big(Z \geq \frac{100-95}{11}\Big) = 1 - \Phi(0.45) = 0.33.$$

So the desired probability is

$$1 - \binom{10}{0}(0.33)^0(0.67)^{10} - \binom{10}{1}(0.33)^1(0.67)^9 = 0.89.$$

16. Let X be the time until the 91st call is received. X is a gamma random variable with parameters $r = 91$ and $\lambda = 23$. The desired probability is

$$P(X \geq 4) = \int_4^\infty \frac{23e^{-23x}\,(23x)^{91-1}}{\Gamma(91)}\,dx$$
$$= 1 - \int_0^4 \frac{23e^{-23x}\,(23x)^{91-1}}{90!}\,dx$$
$$= 1 - \frac{23^{91}}{90!}\int_0^4 x^{90}e^{-23x}\,dx = 1 - 0.55542 = 0.44458.$$

17. Clearly,

$$E(X) = \frac{(1-\theta)+(1+\theta)}{2} = 1,$$
$$\text{Var}(X) = \frac{(1+\theta-1+\theta)^2}{12} = \frac{\theta^2}{3}.$$

Now

$$E\big(X^2\big) - \big[E(X)\big]^2 = \frac{\theta^2}{3}$$

implies that

$$E\big(X^2\big) = \frac{\theta^2}{3} + 1,$$

which yeilds $3E(X^2) - 1 = \theta^2$, or, equivalently, $E(3X^2 - 1) = \theta^2$. Therefore, one choice for $g(X)$ is $g(X) = 3X^2 - 1$.

18. Let α and β be the parameters of the density function of X/ℓ. Solving the following two equations in two unknowns,

$$E(X/\ell) = \frac{\alpha}{\alpha+\beta} = \frac{3}{7},$$
$$\text{Var}(X/\ell) = \frac{\alpha\beta}{(\alpha+\beta+1)(\alpha+\beta)^2} = \frac{3}{98},$$

we obtain $\alpha = 3$ and $\beta = 4$. Therefore, X/ℓ is beta with parameters 3 and 4. The desired probability is

$$P(\ell/7 < X < \ell/3) = P(1/7 < X/\ell < 1/3) = \int_{1/7}^{1/3} \frac{1}{B(3,4)}x^2(1-x)^3\,dx$$
$$= 60\int_{1/7}^{1/3} x^2(1-x)^3\,dx = 0.278.$$

Chapter 8

Bivariate Distributions

8.1 JOINT DISTRIBUTIONS OF TWO RANDOM VARIABLES

1. **(a)** $\sum_{x=1}^{2}\sum_{y=1}^{2} k(x/y) = 1$ implies that $k = 2/9$.

(b) $p_X(x) = \sum_{y=1}^{2}\big[(2x)/(9y)\big] = x/3, \quad x = 1, 2.$

$p_Y(y) = \sum_{x=1}^{2}\big[(2x)/(9y)\big] = 2/(3y), \quad y = 1, 2.$

(c) $P(X > 1 \mid Y = 1) = \dfrac{p(2,1)}{p_Y(1)} = \dfrac{4/9}{2/3} = \dfrac{2}{3}.$

(d) $E(X) = \displaystyle\sum_{y=1}^{2}\sum_{x=1}^{2} x \cdot \frac{2}{9}\Big(\frac{x}{y}\Big) = \frac{5}{3}; \qquad E(Y) = \sum_{y=1}^{2}\sum_{x=1}^{2} y \cdot \frac{2}{9}\Big(\frac{x}{y}\Big) = \frac{4}{3}.$

2. **(a)** $\sum_{x=1}^{3}\sum_{y=1}^{2} c(x+y) = 1$ implies that $c = 1/21$.

(b) $p_X(x) = \sum_{y=1}^{2}(1/21)(x+y) = (2x+3)/21. \quad x = 1, 2, 3.$

$p_Y(y) = \sum_{x=1}^{3}(1/21)(x+y) = (6+3y)/21. \quad y = 1, 2.$

(c) $P(X \ge 2 \mid Y = 1) = \dfrac{p(2,1) + p(3,1)}{p_Y(1)} = \dfrac{7/21}{9/21} = \dfrac{7}{9}.$

(d) $E(X) = \displaystyle\sum_{x=1}^{3}\sum_{y=1}^{2} \frac{1}{21} x(x+y) = \frac{46}{21}; \quad E(Y) = \sum_{x=1}^{3}\sum_{y=1}^{2} \frac{1}{21} y(x+y) = \frac{11}{7}.$

3. **(a)** $k(1+1+1+9+4+9) = 1$ implies that $k = 1/25$.

(b)
$p_X(1) = p(1,1) + p(1,3) = 12/25, \qquad p_X(2) = p(2,3) = 13/25;$
$p_Y(1) = p(1,1) = 2/25, \qquad p_Y(3) = p(1,3) + p(2,3) = 23/25.$

Therefore,

$$p_X(x) = \begin{cases} 12/25 & \text{if } x = 1 \\ 13/25 & \text{if } x = 2, \end{cases} \qquad p_Y(y) = \begin{cases} 2/25 & \text{if } y = 1 \\ 23/25 & \text{if } y = 3. \end{cases}$$

(c) $E(X) = 1 \cdot \dfrac{12}{25} + 2 \cdot \dfrac{13}{25} = \dfrac{38}{25}$; $\quad E(Y) = 1 \cdot \dfrac{2}{25} + 3 \cdot \dfrac{23}{25} = \dfrac{71}{25}$.

4. $P(X > Y) = p(1,0) + p(2,0) + p(2,1) = 2/5$,
$P(X + Y \le 2) = p(1,0) + p(1,1) + p(2,0) = 7/25$,
$P(X + Y = 2) = p(1,1) + p(2,0) = 6/25$.

5. Let X be the number of sheep stolen; let Y be the number of goats stolen. Let $p(x, y)$ be the joint probability mass function of X and Y. Then, for $0 \le x \le 4$, $0 \le y \le 4$, $0 \le x + y \le 4$,

$$p(x, y) = \frac{\dbinom{7}{x}\dbinom{8}{y}\dbinom{5}{4 - x - y}}{\dbinom{20}{4}};$$

$p(x, y) = 0$, for other values of x and y.

6. The following table gives $p(x, y)$, the joint probability mass function of X and Y; $p_X(x)$, the marginal probability mass function of X; and $p_Y(y)$, the marginal probability mass function of Y.

	y						
x	0	1	2	3	4	5	$p_X(x)$
2	1/36	0	0	0	0	0	1/36
3	0	2/36	0	0	0	0	2/36
4	1/36	0	2/36	0	0	0	3/36
5	0	2/36	0	2/36	0	0	4/36
6	1/36	0	2/36	0	2/36	0	5/36
7	0	2/36	0	2/36	0	2/36	6/36
8	1/36	0	2/36	0	2/36	0	5/36
9	0	2/36	0	2/36	0	0	4/36
10	1/36	0	2/36	0	0	0	3/36
11	0	2/36	0	0	0	0	2/36
12	1/36	0	0	0	0	0	1/36
$p_Y(y)$	6/36	10/36	8/36	6/36	4/36	2/36	

7. $p(1,1) = 0$, $p(1,0) = 0.30$, $p(0,1) = 0.50$, $p(0,0) = 0.20$.

8. **(a)** For $0 \le x \le 7$, $0 \le y \le 7$, $0 \le x + y \le 7$,

$$p(x, y) = \frac{\dbinom{13}{x}\dbinom{13}{y}\dbinom{26}{7-x-y}}{\dbinom{52}{7}}.$$

For all other values of x and y, $p(x, y) = 0$.

(b) $P(X \ge Y) = \sum_{y=0}^{3} \sum_{x=y}^{7-y} p(x, y) = 0.61107$.

9. **(a)** $f_X(x) = \displaystyle\int_0^x 2\,dy = 2x, \quad 0 \le x \le 1; \quad f_Y(y) = \int_y^1 2\,dx = 2(1-y), \quad 0 \le y \le 1.$

(b) $$E(X) = \int_0^1 x f_X(x)\,dx = \int_0^1 x(2x)\,dx = 2/3;$$

$$E(Y) = \int_0^1 y f_Y(y)\,dy = \int_0^1 2y(1-y)\,dy = 1/3.$$

(c) $$P\Big(X < \frac{1}{2}\Big) = \int_0^{1/2} f_X(x)\,dx = \int_0^{1/2} 2x\,dx = \frac{1}{4},$$

$$P(X < 2Y) = \int_0^1 \int_{x/2}^x 2\,dy\,dx = \frac{1}{2},$$

$$P(X = Y) = 0.$$

10. **(a)** $$f_X(x) = \int_0^x 8xy\,dy = 4x^3, \quad 0 \le x \le 1,$$

$$f_Y(y) = \int_y^1 8xy\,dx = 4y(1-y^2), \quad 0 \le y \le 1.$$

(b) $$E(X) = \int_0^1 x f_X(x)\,dx = \int_0^1 x \cdot 4x^3\,dx = 4/5;$$

$$E(Y) = \int_0^1 y f_Y(y)\,dy = \int_0^1 y \cdot 4y(1-y^2)\,dy = 8/15.$$

11. $$f_X(x) = \int_0^2 \frac{1}{2} y e^{-x}\,dy = e^{-x}, \quad x > 0; \quad f_Y(y) = \int_0^\infty \frac{1}{2} y e^{-x}\,dx = \frac{1}{2} y, \quad 0 < y < 2.$$

12. Let $R = \big\{(x, y) \colon 0 \le x \le 1,\ 0 \le y \le 1\big\}$. Since area$(R) = 1$, $P(X + Y \le 1/2)$ is the area of the region $\big\{(x, y) \in R \colon x + y \le 1/2\big\}$ which is 1/8. Similarly, $P(X - Y \le 1/2)$ is the area of the region $\{(x, y) \in R\colon x - y \leq 1/2\}$ which is 7/8. $P(X^2 + Y^2 \leq 1)$ is the area of the region $\{(x, y) \in R\colon x^2 + y^2 \leq 1\}$ which is $\pi/4$. $P(XY \leq 1/4)$ is the sum of the area of the region $\{(x, y)\colon 0 \leq x \leq 1/4,\ 0 \leq y \leq 1\}$ which is 1/4 and the area of the region under the curve $y = 1/(4x)$ from 1/4 to 1. (Draw a figure.) Therefore,

$$P(XY \leq 1/4) = \frac{1}{4} + \int_{1/4}^{1} \frac{1}{4x}\, dx \approx 0.597.$$

13. **(a)** The area of R is $\displaystyle\int_0^1 (x - x^2)\, dx = \frac{1}{6}$; so

$$f(x, y) = \begin{cases} 6 & \text{if } (x, y) \in R \\ 0 & \text{elsewhere.} \end{cases}$$

(b) $\displaystyle f_X(x) = \int_{x^2}^{x} f(x, y)\, dy = \int_{x^2}^{x} 6\, dy = 6x(1 - x), \quad 0 < x < 1;$

$$f_Y(y) = \int_{y}^{\sqrt{y}} f(x, y)\, dx = \int_{y}^{\sqrt{y}} 6\, dx = 6(\sqrt{y} - y), \quad 0 < y < 1.$$

(c) $\displaystyle E(X) = \int_0^1 x f_X(x)\, dx = \int_0^1 6x^2(1 - x)\, dx = 1/2;$

$$E(Y) = \int_0^1 y f_Y(y)\, dy = \int_0^1 6y(\sqrt{y} - y)\, dy = 2/5.$$

14. Let X and Y be the minutes past 11:30 A.M. that the man and his fiancée arrive at the lobby, respectively. We have that X and Y are uniformly distributed over $(0, 30)$. Let $S = \{(x, y)\colon 0 \leq x \leq 30,\ 0 \leq y \leq 30\}$, and $R = \{(x, y) \in S\colon y \leq x - 12 \text{ or } y \geq x + 12\}$. The desired probability is the area of R divided by the area of S: $324/900 = 0.36$. (Draw a figure.)

15. Let X and Y be two randomly selected points from the interval $(0, \ell)$. We are interested in $E(|X - Y|)$. Since the joint probability density function of X and Y is

$$f(x, y) = \begin{cases} \dfrac{1}{\ell^2} & 0 < x < \ell,\ 0 < y < \ell \\ 0 & \text{elsewhere,} \end{cases}$$

$$\begin{aligned}E\big(|X-Y|\big) &= \int_0^\ell\int_0^\ell |x-y|\frac{1}{\ell^2}\,dx\,dy\\ &= \frac{1}{\ell^2}\int_0^\ell\Big[\int_0^y (y-x)\,dx\Big]dy + \frac{1}{\ell^2}\int_0^\ell\Big[\int_y^\ell (x-y)\,dx\Big]dy\\ &= \frac{\ell}{6}+\frac{\ell}{6}=\frac{\ell}{3}.\end{aligned}$$

16. The problem is equivalent to the following: Two random numbers X and Y are selected at random and independently from $(0, \ell)$. What is the probability that $|X - Y| < X$? Let $S = \big\{(x, y)\colon 0 < x < \ell,\ 0 < y < \ell\big\}$ and

$$R = \big\{(x, y) \in S\colon |x-y| < x\big\} = \big\{(x, y) \in S\colon y < 2x\big\}.$$

The desired probability is the area of R which is $3\ell^2/4$ divided by ℓ^2. So the answer is 3/4. (Draw a figure.)

17. Let $S = \big\{(x, y)\colon 0 < x < 1,\ 0 < y < 1\big\}$ and $R = \big\{(x, y) \in S\colon y \le x \text{ and } x^2 + y^2 \le 1\big\}$. The desired probability is the area of R which is $\pi/8$ divided by the area of S which is 1. So the answer is $\pi/8$.

18. We prove this for the case in which X and Y are continuous random variables with joint probability density function f. For discrete random variables the proof is similar. The relation $P(X \le Y) = 1$, implies that $f(x, y) = 0$ if $x > y$. Hence by Theorem 8.2,

$$\begin{aligned}E(X) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xf(x,y)\,dx\,dy\\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{y} xf(x,y)\,dx\,dy\\ &\le \int_{-\infty}^{\infty}\int_{-\infty}^{y} yf(x,y)\,dx\,dy\\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} yf(x,y)\,dx\,dy = E(Y).\end{aligned}$$

19. Let H be the distribution function of a random variable with probability density function h. That is, let $H(x) = \displaystyle\int_{-\infty}^{x} h(y)\,dy$. Then

$$\begin{aligned}P(X \ge Y) &= \int_{-\infty}^{\infty}\int_{-\infty}^{x} h(x)h(y)\,dy\,dx = \int_{-\infty}^{\infty} h(x)\Big[\int_{-\infty}^{x} h(y)\,dy\Big]dx\\ &= \int_{-\infty}^{\infty} h(x)H(x)\,dx = \frac{1}{2}\big[H(x)\big]^2\Big|_{-\infty}^{\infty} = \frac{1}{2}(1^2 - 0^2) = \frac{1}{2}.\end{aligned}$$

20. Since $0 \le 2G(x) - 1 \le 1$, $0 \le 2H(y) - 1 \le 1$, and $-1 \le \alpha \le 1$, we have that

$$-1 \le \alpha\big[2G(x) - 1\big]\big[2H(y) - 1\big] \le 1.$$

So

$$0 \le 1 + \alpha\big[2G(x) - 1\big]\big[2H(y) - 1\big] \le 2.$$

This and $g(x) \ge 0$, $h(y) \ge 0$ imply that $f(x, y) \ge 0$. To prove that f is a joint probability density function, it remains to show that $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x, y)\,dx\,dy = 1$.

$$\begin{aligned}
&\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x, y)\,dx\,dy \\
&\quad = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x)h(y)\,dx\,dy + \alpha\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x)h(y)\big[2G(x) - 1\big]\big[2H(y) - 1\big]\,dx\,dy \\
&\quad = 1 + \alpha\Big\{\int_{-\infty}^{\infty} h(y)\big[2H(y) - 1\big]\,dy\Big\}\Big\{\int_{-\infty}^{\infty} g(x)\big[2G(x) - 1\big]\,dx\Big\} \\
&\quad = 1 + \alpha\frac{1}{4}\big[2H(y) - 1\big]^2\Big|_{-\infty}^{\infty}\ \frac{1}{4}\big[2G(x) - 1\big]^2\Big|_{-\infty}^{\infty} = 1 + \alpha\cdot 0\cdot 0 = 1.
\end{aligned}$$

Now we calculate the marginals.

$$\begin{aligned}
f_X(x) &= \int_{-\infty}^{\infty} g(x)h(y)\Big\{1 + \alpha\big[2G(x) - 1\big]\big[2H(y) - 1\big]\Big\}\,dy \\
&= \int_{-\infty}^{\infty} g(x)h(y)\,dy + \alpha\int_{-\infty}^{\infty} g(x)h(y)\big[2G(x) - 1\big]\big[2H(y) - 1\big]\,dy \\
&= g(x)\int_{-\infty}^{\infty} h(y)\,dy + \alpha g(x)\big[2G(x) - 1\big]\int_{-\infty}^{\infty} h(y)\big[2H(y) - 1\big]\,dy \\
&= g(x) + \alpha g(x)\big[2G(x) - 1\big]\frac{1}{4}\big[2H(y) - 1\big]^2\Big|_{-\infty}^{\infty} \\
&= g(x) + \alpha g(x)\big[2G(x) - 1\big]\cdot 0 = g(x) + 0 = g(x).
\end{aligned}$$

Similarly, $f_Y(y) = h(y)$.

21. Orient the circle counterclockwise and let X be the length of the arc NM and Y be length of the arc NL. Let R be the radius of the circle; clearly, $0 \le X \le 2\pi R$ and $0 \le Y \le 2\pi R$. The angle MNL is acute if and only if $|Y - X| < \pi R$. Therefore, the sample space of this experiment is

$$S = \big\{(x, y) \colon 0 \le x \le 2\pi R,\ 0 \le y \le 2\pi R\big\}$$

and the desired event is

$$E = \big\{(x, y) \in S \colon |y - x| < \pi R\big\}.$$

The probability that $\angle MNL$ is acute is the area of E which is $3\pi^2 R^2$ divided by the area of S which is $4\pi^2 R^2$; that is, $3/4$.

22. Let

$$S = \big\{(x, y) \in \mathbf{R^2} \colon 0 \le x \le 1,\ 0 \le y \le 1\big\}, \qquad A = \big\{(x, y) \in S \colon 0 < x + y < 0.5\big\},$$

$$B = \big\{(x, y) \in S \colon 0.5 < x + y < 1.5\big\}, \qquad C = \big\{(x, y) \in S \colon x + y > 1.5\big\}.$$

The probability that the integer nearest to $x+y$ is 0 is $\dfrac{\text{area}(A)}{\text{area }(S)}=\dfrac{1}{8}$, The probability that the integer nearest to $x+y$ is 1 is $\dfrac{\text{area}(B)}{\text{area}(S)}=\dfrac{3}{4}$, and the probability that the nearest integer to $x+y$ is 2 is $\dfrac{\text{area }(C)}{\text{area}(S)}=\dfrac{1}{8}$.

23. Let X be a random number from $(0,a)$ and Y be a random number from $(0,b)$. In $\dbinom{4}{3}=4$ ways we can select three of X, $a-X$, Y, and $b-Y$. If X, $a-X$, and Y are selected, a triangular pen is possible to make if and only if $X<(a-X)+Y$, $a-X<X+Y$, and $Y<X+(a-X)$. The probability of this event is the area of

$$\big\{(x,y)\in\mathbf{R^2}\colon 0<x<a,\ 0<y<b,\ 2x-y<a,\ 2x+y>a,\ y<a\big\}$$

which is $a^2/2$ divided by the area of

$$S=\big\{(x,y)\in\mathbf{R^2}\colon 0<x<a,\ 0<y<b\big\}$$

which is ab: $(a^2/2)/ab=a/(2b)$. Similarly, for each of the other three 3-combinations of X, $a-x$, Y, and $b-Y$ also the probability that the three segments can be used to form a triangular pen is $a/(2b)$. Thus the desired probability is

$$\frac{1}{4}\cdot\frac{a}{2b}+\frac{1}{4}\cdot\frac{a}{2b}+\frac{1}{4}\cdot\frac{a}{2b}+\frac{1}{4}\cdot\frac{a}{2b}=\frac{a}{2b}.$$

24. Let X and Y be the two points that are placed on the segment. Let E be the event that the length of none of the three parts exceeds the given value α. Clearly, $P(E\mid X<Y)=P(E\mid Y<X)$ and $P(X<Y)=P(Y<X)=1/2$. Therefore,

$$\begin{aligned}P(E)&=P(E\mid X<Y)P(X<Y)+P(E\mid Y<X)P(Y<X)\\&=P(E\mid X<Y)\frac{1}{2}+P(E\mid X<Y)\frac{1}{2}=P(E\mid X<Y).\end{aligned}$$

This shows that for calculation of $P(E)$, we may reduce the sample space to the case where $X<Y$. The reduced sample space is

$$S=\big\{(x,y)\colon x<y,\ 0<x<\ell,\ 0<y<\ell\big\}.$$

The desired probability is the area of

$$R=\big\{(x,y)\in S\colon x<\alpha,\ y-x<\alpha,\ y>\ell-\alpha\big\}$$

divided by area$(S)=\ell^2/2$. But

$$\text{area}(R)=\begin{cases}\dfrac{(3\alpha-\ell)^2}{2} & \text{if } \dfrac{\ell}{3}\le\alpha\le\dfrac{\ell}{2}\\[2ex] \dfrac{\ell^2}{2}-\dfrac{3\ell^2}{2}\Big(1-\dfrac{\alpha}{\ell}\Big)^2 & \text{if } \dfrac{\ell}{2}\le\alpha\le\ell.\end{cases}$$

Hence the desired probability is

$$P(E) = \begin{cases} \left(\dfrac{3\alpha}{\ell} - 1\right)^2 & \text{if } \dfrac{\ell}{3} \le \alpha \le \dfrac{\ell}{2} \\ 1 - 3\left(1 - \dfrac{\alpha}{\ell}\right)^2 & \text{if } \dfrac{\ell}{2} \le \alpha \le \ell. \end{cases}$$

25. R is the square bounded by the lines $x + y = 1$, $-x + y = 1$, $-x - y = 1$, and $x - y = 1$; its area is 2. To find the probability density function of X, the x-coordinate of the point selected at random from R, first we calculate $P(X \le t)$, $\forall t$. For $-1 \le t < 0$, $P(X \le t)$ is the area of the triangle bound by the lines $-x + y = 1$, $-x - y = 1$, and $x = t$ which is $(1+t)^2$ divided by area$(R) = 2$. (Draw a figure.) For $0 \le t < 1$, $P(X \le t)$ is the area inside R to the left of the line $x = t$ which is $2 - (1-t)^2$ divided by area$(R) = 2$. Therefore,

$$P(X \le t) = \begin{cases} 0 & t < -1 \\ \dfrac{(1+t)^2}{2} & -1 \le t < 0 \\ \dfrac{2 - (1-t)^2}{2} & 0 \le t < 1 \\ 1 & t \ge 1, \end{cases}$$

and hence

$$\frac{d}{dt} P(X \le t) = \begin{cases} 1 + t & -1 \le t < 0 \\ 1 - t & 0 \le t < 1 \\ 0 & \text{otherwise.} \end{cases}$$

This shows that $f_X(t)$, the probability density function of X is given by $f_X(t) = 1 - |t|$, $-1 \le t \le 1$; 0, elsewhere.

26. Clearly,

$$P(Z \le z) = \iint\limits_{\{(x,y):\, y/x \le z\}} f(x, y)\, dx\, dy.$$

Now for $x > 0$, $y/x \le z$ if and only if $y \le xz$; for $x < 0$, $y/x \le z$ if and only if $y \ge xz$. Therefore, integration region is

$$\Big\{(x, y) \colon x < 0,\ y \ge xz\Big\} \cup \Big\{(x, y) \colon x > 0,\ y \le xz\Big\}.$$

Thus

$$P(Z \le z) = \int_{-\infty}^{0} \left(\int_{xz}^{\infty} f(x, y)\, dy \right) dx + \int_{0}^{\infty} \left(\int_{-\infty}^{xz} f(x, y)\, dy \right) dx.$$

Using the substitution $y = tx$, we get

$$\begin{aligned}
P(Z \le z) &= \int_{-\infty}^{0} \Big(\int_{z}^{-\infty} x f(x, tx)\, dt \Big)\, dx + \int_{0}^{\infty} \Big(\int_{-\infty}^{z} x f(x, tx)\, dt \Big)\, dx \\
&= \int_{-\infty}^{0} \Big(\int_{-\infty}^{z} -x f(x, tx)\, dt \Big)\, dx + \int_{0}^{\infty} \Big(\int_{-\infty}^{z} x f(x, tx)\, dt \Big)\, dx \\
&= \int_{-\infty}^{0} \Big(\int_{-\infty}^{z} |x| f(x, tx)\, dt \Big)\, dx + \int_{0}^{\infty} \Big(\int_{-\infty}^{z} |x| f(x, tx)\, du \Big)\, dx \\
&= \int_{-\infty}^{\infty} \Big(\int_{-\infty}^{z} |x| f(x, tx)\, dt \Big)\, dx = \int_{-\infty}^{z} \Big(\int_{-\infty}^{\infty} |x| f(x, tx)\, dx \Big)\, dt.
\end{aligned}$$

Differentiating with respect to z, Fundamental Theorem of Calculus implies that,

$$f_Z(z) = \frac{d}{dz} P(Z \le z) = \int_{-\infty}^{\infty} |x| f(x, xz)\, dx.$$

27. Note that there are exactly n such closed semicircular disks because the probability that the diameter through P_i contains any other point P_j is 0. (Draw a figure.) Let E be the event that all the points are contained in a closed semicircular disk. Let E_i be the event that the points are all in D_i. Clearly, $E = \cup_{i=1}^{n} E_i$. Since there is at most one D_i, $1 \le i \le n$, that contains all the P_i's, the events $E_1, E_2, \ldots, E_n$ are mutually exclusive. Hence

$$P(E) = P\Big(\bigcup_{i=1}^{n} E_i \Big) = \sum_{i=1}^{n} P(E_i) = \sum_{i=1}^{n} \Big(\frac{1}{2}\Big)^{n-1} = n \Big(\frac{1}{2}\Big)^{n-1},$$

where the next-to-the-last equality follows because $P(E_i)$ is the probability that P_1, P_2, $\ldots$, P_{i-1}, P_{i+1}, $\ldots$, P_n fall inside D_i. The probability that any of these falls inside D_i is (area of D_i)/(area of the disk) $= 1/2$ independently of the others. Hence the probability that all of them fall inside D_i is $(1/2)^{n-1}$.

28. We have that

$$\begin{aligned}
f_X(x) &= \frac{\Gamma(\alpha+\beta+\gamma)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)} \int_0^{1-x} x^{\alpha-1} y^{\beta-1} (1-x-y)^{\gamma-1}\, dy \\
&= \frac{1}{B(\alpha, \beta+\gamma) B(\beta, \gamma)} x^{\alpha-1} \int_0^{1-x} y^{\beta-1} (1-x-y)^{\gamma-1}\, dy.
\end{aligned}$$

Let $z = y/(1-x)$; then $dy = (1-x)\, dz$, and

$$\int_0^{1-x} y^{\beta-1}(1-x-y)^{\gamma-1}\, dy = (1-x)^{\beta+\gamma-1} \int_0^1 z^{\beta-1}(1-z)^{\gamma-1}\, dz = (1-x)^{\beta+\gamma-1} B(\beta, \gamma).$$

So

$$\begin{aligned}
f_X(x) &= \frac{1}{B(\alpha, \beta+\gamma) B(\beta, \gamma)} x^{\alpha-1} (1-x)^{\beta+\gamma-1} B(\beta, \gamma) \\
&= \frac{1}{B(\alpha, \beta+\gamma)} x^{\alpha-1} (1-x)^{\beta+\gamma-1}.
\end{aligned}$$

This shows that X is beta with parameters $(\alpha, \beta + \gamma)$. A similar argument shows that Y is beta with parameters $(\beta, \gamma + \alpha)$.

29. It is straightforward to check that $f(x, y) \geq 0$, f is continuous and

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x, y)\,dx\,dy = 1.$$

Therefore, f is a continuous probability density function. We will show that $\dfrac{\partial F}{\partial x}$ does not exist at (0, 0). Similarly, one can show that $\dfrac{\partial F}{\partial x}$ does not exist at any point on the y-axis. Note that for small $\Delta x > 0$,

$$\begin{aligned}F(\Delta x, 0) - F(0, 0) &= P(X \leq \Delta x\,,\ Y \leq 0) - P(X \leq 0\,,\ Y \leq 0)\\ &= P(0 \leq X \leq \Delta x\,,\ Y \leq 0) = \int_{-\infty}^{0}\int_{0}^{\Delta x} f(x, y)\,dx\,dy.\end{aligned}$$

Now, from the definition of $f(x, y)$, we must have $\Delta x < (1/2)e^{y}$ or, equivalently, $y > \ln(2\Delta x)$. Thus, for small $\Delta x > 0$,

$$F(\Delta x, 0) - F(0, 0) = \int_{\ln(2\Delta x)}^{0}\int_{0}^{\Delta x}(1 - 2xe^{-y})\,dx\,dy = (\Delta x)^2 - \Big[(\Delta x)\ln(2\Delta x) + \frac{\Delta x}{2}\Big].$$

This implies that

$$\lim_{\Delta x \to 0+}\frac{F(\Delta x, 0) - F(0, 0)}{\Delta x} = \lim_{\Delta x \to 0+}\Big[\Delta x - \ln(2\Delta x) - \frac{1}{2}\Big] = \infty,$$

showing that $\dfrac{\partial F}{\partial x}$ does not exist at (0, 0).

8.2 INDEPENDENT RANDOM VARIABLES

1. Note that $p_X(x) = (1/25)(3x^2 + 5)$, $p_Y(y) = (1/25)(2y^2 + 5)$. Now $p_X(1) = 8/25$, $p_Y(0) = 5/25$, and $p(1, 0) = 1/25$. Since $p(1, 0) \neq p_X(1)p_Y(0)$, X and Y are dependent.

2. Note that

$$\begin{aligned}p(1, 1) &= \frac{1}{7},\\ p_X(1) &= p(1, 1) + p(1, 2) = \frac{1}{7} + \frac{2}{7} = \frac{3}{7},\\ p_Y(1) &= p(1, 1) + p(2, 1) = \frac{1}{7} + \frac{5}{7} = \frac{6}{7}.\end{aligned}$$

Since $p(1, 1) \neq p_X(1)p_Y(1)$, X and Y are dependent.

3. By the independence of X and Y,

$$P(X=1,\ Y=3)=P(X=1)P(Y=3)=\frac{1}{2}\Big(\frac{2}{3}\Big)\cdot\frac{1}{2}\Big(\frac{2}{3}\Big)^3=\frac{4}{81}.$$

$$\begin{aligned}P(X+Y=3)&=P(X=1,\ Y=2)+P(X=2,\ Y=1)\\&=\frac{1}{2}\Big(\frac{2}{3}\Big)\cdot\frac{1}{2}\Big(\frac{2}{3}\Big)^2+\frac{1}{2}\Big(\frac{2}{3}\Big)^2\cdot\frac{1}{2}\Big(\frac{2}{3}\Big)=\frac{4}{27}.\end{aligned}$$

4. No, they are not independent because, for example, $P(X=0\mid Y=8)=1$ but

$$P(X=0)=\frac{\dbinom{39}{8}}{\dbinom{52}{8}}=0.08175\neq 1,$$

showing that $P(X=0\mid Y=8)\neq P(X=0)$.

5. The answer is

$$\binom{7}{2}\Big(\frac{1}{2}\Big)^2\Big(\frac{1}{2}\Big)^5\cdot\binom{8}{2}\Big(\frac{1}{2}\Big)^2\Big(\frac{1}{2}\Big)^6=0.0179.$$

6. We have that

$$P\big(\max(X,Y)\le t\big)=P(X\le t,\ Y\le t)=P(X\le t)P(Y\le t)=F(t)G(t).$$

$$\begin{aligned}P\big(\min(X,Y)\le t\big)&=1-P\big(\min(X,Y)>t\big)\\&=1-P(X>t,\ Y>t)=1-P(X>t)P(Y>t)\\&=1-\big[1-F(t)\big]\big[1-G(t)\big]=F(t)+G(t)-F(t)G(t).\end{aligned}$$

7. Let X and Y be the number of heads obtained by Adam and Andrew, respectively. The desired probability is

$$\begin{aligned}\sum_{i=0}^{n}P(X=i,\ Y=i)&=\sum_{i=0}^{n}P(X=i)P(Y=i)\\&=\sum_{i=0}^{n}\binom{n}{i}\Big(\frac{1}{2}\Big)^i\Big(\frac{1}{2}\Big)^{n-i}\cdot\binom{n}{i}\Big(\frac{1}{2}\Big)^i\Big(\frac{1}{2}\Big)^{n-i}\\&=\Big(\frac{1}{2}\Big)^{2n}\sum_{i=0}^{n}\binom{n}{i}^2=\Big(\frac{1}{2}\Big)^{2n}\binom{2n}{n},\end{aligned}$$

where the last equality follows by Example 2.28.

An Intuitive Solution: Let Z be the number of tails obtained by Andrew. The desired probability is

$$\sum_{i=0}^{n} P(X=i,\ Y=i) = \sum_{i=0}^{n} P(X=i,\ Z=i) = \sum_{i=0}^{n} P(X=i,\ Y=n-i)$$
$$= P(\text{Adam and Andrew get a total of } n \text{ heads})$$
$$= P(\ n \text{ heads in } 2n \text{ flips of a fair coin}) = \Big(\frac{1}{2}\Big)^{2n}\binom{2n}{n}.$$

8. For $i,\ j \in \{0, 1, 2, 3\}$, the sum of the numbers in the ith row is $p_X(i)$ and the sum of the numbers in the jth row is $p_Y(j)$. We have that

$$p_X(0) = 0.41, \quad p_X(1) = 0.44, \quad p_X(2) = 0.14, \quad p_X(3) = 0.01;$$
$$p_Y(0) = 0.41, \quad p_Y(1) = 0.44, \quad p_Y(2) = 0.14, \quad p_Y(3) = 0.01.$$

Since for all $x,\ y \in \{0, 1, 2, 3\}$, $p(x, y) = p_X(x)p_Y(y)$, X and Y are independent.

9. They are not independent because

$$f_X(x) = \int_0^x 2\,dy = 2x, \quad 0 \le x \le 1;$$
$$f_Y(y) = \int_y^1 2\,dx = 2(1-y), \quad 0 \le y \le 1;$$

and so $f(x, y) \neq f_X(x)f_Y(y)$.

10. Let X and Y be the amount of cholesterol in the first and in the second sandwiches, respectively. Since X and Y are continuous random variables, $P(X = Y) = 0$ regardless of what the probability density functions of X and Y are.

11. We have that

$$f_X(x) = \int_0^\infty x^2 e^{-x(y+1)}\,dy = xe^{-x}, \quad x \ge 0;$$
$$f_Y(y) = \int_0^\infty x^2 e^{-x(y+1)}\,dx = \frac{2}{(y+1)^3}, \quad y \ge 0,$$

where the second integral is calculated by applying integration by parts twice. Now since $f(x, y) \neq f_X(x)f_Y(y)$, X and Y are not independent.

12. Clearly,

$$E(XY) = \int_0^1 \int_x^1 (xy)(8xy)\,dy\,dx = \int_0^1 \Big(\int_x^1 8y^2\,dy\Big)x^2\,dx = \frac{4}{9},$$
$$E(X) = \int_0^1 \int_x^1 x(8xy)\,dy\,dx = \frac{8}{15},$$
$$E(Y) = \int_0^1 \int_x^1 y(8xy)\,dy\,dx = \frac{4}{5}.$$

So $E(XY) \neq E(X)E(Y)$.

13. Since

$$f(x, y) = e^{-x} \cdot 2e^{-2y} = f_X(x) f_Y(y),$$

X and Y are independent exponential random variables with parameters 1 and 2, respectively. Therefore,

$$E(X^2Y) = E(X^2)E(Y) = 2 \cdot \frac{1}{2} = 1.$$

14. The joint probability density function of X and Y is given by

$$f(x, y) = \begin{cases} e^{-(x+y)} & x > 0,\ y > 0 \\ 0 & \text{elsewhere.} \end{cases}$$

Let G be the probability distribution function, and g be the probability density function of X/Y. For $t > 0$,

$$G(t) = P\Big(\frac{X}{Y} \leq t\Big) = P(X \leq tY)$$
$$= \int_0^\infty \Big(\int_0^{ty} e^{-(x+y)}\,dx\Big)\,dy = \frac{t}{1+t}.$$

Therefore, for $t > 0$,

$$g(t) = G'(t) = \frac{1}{(1+t)^2}.$$

Note that $G'(t) = 0$ for $t < 0$; $G'(0)$ does not exist.

15. Let F and f be the probability distribution and probability density functions of $\max(X, Y)$, respectively. Clearly,

$$F(t) = P\big(\max(X, Y) \leq t\big) = P(X \leq t,\ Y \leq t) = (1 - e^{-t})^2, \quad t \geq 0.$$

Thus

$$f(t) = F'(t) = 2e^{-t}(1 - e^{-t}) = 2e^{-t} - 2e^{-2t}.$$

Hence

$$E\big[\max(X,Y)\big] = 2\int_0^\infty te^{-t}\,dt - \int_0^\infty 2te^{-2t}\,dt = 2 - \frac{1}{2} = \frac{3}{2}.$$

Note that $\int_0^\infty te^{-t}\,dt$ is the expected value of an exponential random variable with parameter 1, thus it is 1. Also, $\int_0^\infty 2te^{-2t}\,dt$ is the expected value of an exponential random variable with parameter 2, thus it is 1/2.

16. Let F and f be the probability distribution and probability density functions of $\max(X,Y)$. For $-1 < t < 1$,

$$F(t) = P\big(\max(X,Y) \le t\big) = P(X \le t,\ Y \le t) = P(X \le t)P(Y \le t) = \Big(\frac{t+1}{2}\Big)^2.$$

Thus

$$f(t) = F'(t) = \frac{t+1}{2}, \quad -1 < t < 1.$$

Therefore,

$$E(X) = \int_{-1}^{1} t\Big(\frac{t+1}{2}\Big)\,dt = \frac{1}{3}.$$

17. Let F and f be the probability distribution and probability density functions of XY, respectively. Clearly, for $t \le 0$, $F(t) = 0$ and for $t \ge 1$, $F(t) = 1$. For $0 < t < 1$,

$$F(t) = P(XY \le t) = 1 - P(XY > t) = 1 - \int_t^1 \int_{t/x}^1 dy\,dx = t - t\ln t.$$

Hence

$$f(t) = F'(t) = \begin{cases} -\ln t & 0 < t < 1 \\ 0 & \text{elsewhere.} \end{cases}$$

18. The joint probability density function of X and Y is given by

$$f(x,y) = \begin{cases} \dfrac{1}{\text{area }(R)} = \dfrac{1}{\pi} & \text{if } (x,y) \in R \\ 0 & \text{otherwise.} \end{cases}$$

Now

$$f_X(x) = \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{1}{\pi}\,dy = \frac{2}{\pi}\sqrt{1-x^2},$$

$$f_Y(y) = \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \frac{1}{\pi}\,dx = \frac{2}{\pi}\sqrt{1-y^2}.$$

Since $f(x,y) \ne f_X(x)f_Y(y)$, the random variables X and Y are not independent.

19. Let X be the number of adults and Y be the number of children who get sick. The desired probability is

$$\sum_{i=0}^{5}\sum_{j=i+1}^{6} P(Y=i,\ X=j) = \sum_{i=0}^{5}\sum_{j=i+1}^{6} P(Y=i)P(X=j)$$
$$= \sum_{i=0}^{5}\sum_{j=i+1}^{6} \binom{6}{i}(0.30)^i(0.70)^{6-i}\cdot\binom{6}{j}(0.2)^j(0.8)^{6-j} = 0.22638565.$$

20. Let X be the lifetime of the muffler Elizabeth buys from company A and Y be the lifetime of the muffler she buys from company B. The joint probability density function of X and Y is $h(x, y) = f(x)g(y)$, $x > 0$, $y > 0$. So the desired probability is

$$P(Y > X) = \int_0^{\infty}\Big[\int_x^{\infty}\frac{2}{11}e^{-(2y)/11}\,dy\Big]\frac{1}{6}e^{-x/6}\,dx = \frac{11}{23}.$$

21. If I_A and I_B are independent, then

$$P(I_A = 1,\ I_B = 1) = P(I_A = 1)P(I_B = 1).$$

This is equivalent to $P(AB) = P(A)P(B)$ which shows that A and B are independent. On the other hand, if $\{A, B\}$ is an independent set, so are the following: $\{A, B^c\}$, $\{A^c, B\}$, and $\{A^c, B^c\}$. Therefore,

$$P(AB) = P(A)P(B), \qquad P(AB^c) = P(A)P(B^c),$$
$$P(A^cB) = P(A^c)P(B), \qquad P(A^cB^c) = P(A^c)P(B^c).$$

These relations, respectively, imply that

$$P(I_A = 1,\ I_B = 1) = P(I_A = 1)P(I_B = 1),$$
$$P(I_A = 1,\ I_B = 0) = P(I_A = 1)P(I_B = 0),$$
$$P(I_A = 0,\ I_B = 1) = P(I_A = 0)P(I_B = 1),$$
$$P(I_A = 0,\ I_B = 0) = P(I_A = 0)P(I_B = 0).$$

These four relations show that I_A and I_B are independent random variables.

22. The joint probability density function of B and C is

$$f(b, c) = \begin{cases}\dfrac{9b^2c^2}{676} & 1 < b < 3,\ 1 < c < 3\\ 0 & \text{otherwise.}\end{cases}$$

For $X^2 + BX + C$ to have two real roots we must have $B^2 - 4C > 0$, or, equivalently, $B^2 > 4C$. Let

$$E = \{(b, c)\colon 1 < b < 3,\ 1 < c < 3,\ b^2 > 4c\};$$

the desired probability is

$$\iint_E \frac{9b^2c^2}{676}\,db\,dc = \int_2^3 \Big(\int_1^{b^2/4} \frac{9b^2c^2}{676}\,dc\Big)\,db \approx 0.12.$$

(Draw a figure to verify the region of integration.)

23. Note that

$$f_X(x) = \int_{-\infty}^{\infty} g(x)h(y)\,dy = g(x)\int_{-\infty}^{\infty} h(y)\,dy,$$
$$f_Y(y) = \int_{-\infty}^{\infty} g(x)h(y)\,dx = h(y)\int_{-\infty}^{\infty} g(x)\,dx.$$

Now

$$\begin{aligned} f_X(x)f_Y(y) &= g(x)h(y)\int_{-\infty}^{\infty} h(y)\,dy \int_{-\infty}^{\infty} g(x)\,dx \\ &= f(x,y)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(y)g(x)\,dy\,dx \\ &= f(x,y)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dy\,dx = f(x,y). \end{aligned}$$

This relation shows that X and Y are independent.

24. Let G and g be the probability distribution and probability density functions of $\max(X, Y)\big/\min(X, Y)$. Then $G(t) = 0$ if $t < 1$. For $t \geq 1$,

$$\begin{aligned} G(t) &= P\Big(\frac{\max(X,Y)}{\min(X,Y)} \leq t\Big) = P\Big(\max(X,Y) \leq t\min(X,Y)\Big) \\ &= P\Big(X \leq t\min(X,Y),\ Y \leq t\min(X,Y)\Big) \\ &= P\Big(\min(X,Y) \geq \frac{X}{t},\ \min(X,Y) \geq \frac{Y}{t}\Big) \\ &= P\Big(X \geq \frac{X}{t},\ Y \geq \frac{X}{t},\ X \geq \frac{Y}{t},\ Y \geq \frac{Y}{t}\Big) \\ &= P\Big(Y \geq \frac{X}{t},\ X \geq \frac{Y}{t}\Big) = P\Big(\frac{X}{t} \leq Y \leq tX\Big). \end{aligned}$$

This quantity is the area of the region

$$\Big\{(x,y)\colon 0 < x < 1,\ 0 < y < 1,\ \frac{x}{t} \leq y \leq tx\Big\}$$

which is equal to $(t-1)/t$. Hence

$$G(t) = \begin{cases} 0 & t < 1 \\ \dfrac{t-1}{t} & t \geq 1, \end{cases}$$

and therefore,

$$g(t) = G'(t) = \begin{cases} \dfrac{1}{t^2} & t \geq 1 \\ 0 & \text{elsewhere.} \end{cases}$$

25. Let F be the distribution function of $X/(X+Y)$. Since $X/(X+Y) \in (0, 1)$, we have that

$$F(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geq 1. \end{cases}$$

For $0 \leq t < 1$,

$$P\Big(\frac{X}{X+Y} \leq t\Big) = P\Big(Y \geq \frac{1-t}{t}X\Big) = \lambda^2 \int_0^\infty \int_{[(1-t)x]/t}^\infty e^{-\lambda x} e^{-\lambda y}\, dy\, dx$$
$$= \lambda \int_0^\infty e^{-\lambda x} e^{-[\lambda(1-t)x]/t}\, dx = \lambda \int_0^\infty e^{-\lambda x/t}\, dt = t.$$

Therefore,

$$F(t) = \begin{cases} 0 & t < 0 \\ t & 0 \leq t < 1 \\ 1 & t \geq 1. \end{cases}$$

This shows that $X/(X+Y)$ is uniform over $(0, 1)$.

26. The fact that if X and Y are both normal with mean 0 and equal variance implies that $f(x, y)$ is circularly symmetrical is straightforward. We prove the converse; suppose that f is circularly symmetrical, then there exists a function φ so that

$$f_X(x) f_Y(y) = \varphi\big(\sqrt{x^2+y^2}\,\big).$$

Differentiating this relation with respect to x and using

$$f_Y(y) = \frac{f_X(x) f_Y(y)}{f_X(x)} = \varphi\big(\sqrt{x^2+y^2}\,\big)/f_X(x)$$

yields

$$\frac{\varphi'\big(\sqrt{x^2+y^2}\,\big)}{\varphi\big(\sqrt{x^2+y^2}\,\big)\sqrt{x^2+y^2}} = \frac{f_X'(x)}{x f_X(x)}.$$

Now the right side of this equation is a function of x while its left side is a function of $\sqrt{x^2+y^2}$. This implies that $f_X'(x)/\big[xf_X(x)\big]$ is constant. To prove this, we show that for any given x_1 and x_2,

$$\frac{f_X'(x_1)}{x_1 f_X(x_1)} = \frac{f_X'(x_2)}{x_2 f_X(x_2)}.$$

Let $y_1 = x_2$ and $y_2 = x_1$; then $x_1^2 + y_1^2 = x_2^2 + y_2^2$ and we have

$$\frac{f_X'(x_1)}{x_1 f_X(x_1)} = \frac{\varphi'\Big(\sqrt{x_1^2+y_1^2}\Big)}{\varphi\Big(\sqrt{x_1^2+y_1^2}\Big)\sqrt{x_1^2+y_1^2}} = \frac{\varphi'\Big(\sqrt{x_2^2+y_2^2}\Big)}{\varphi\Big(\sqrt{x_2^2+y_2^2}\Big)\sqrt{x_2^2+y_2^2}} = \frac{f_X'(x_2)}{x_2 f_X(x_2)}.$$

We have shown that for some constant k,

$$\frac{f_X'(x)}{x f_X(x)} = k.$$

Therefore, $\dfrac{f_X'(x)}{f_X(x)} = kx$ and hence $\ln f_X(x) = \dfrac{1}{2}kx^2 + c$, or

$$f_X(x) = e^{(1/2)kx^2 + c} = \alpha e^{(1/2)kx^2},$$

where $\alpha = e^c$. Now since $\displaystyle\int_{-\infty}^{\infty} \alpha e^{(1/2)kx^2}\,dx = 1$, we have that $k < 0$. Let $\sigma = \sqrt{-1/k}$; then $f_X(x) = \alpha e^{-x^2/(2\sigma^2)}$ and $\displaystyle\int_{-\infty}^{\infty} \alpha e^{-x^2/(2\sigma^2)}\,dx = 1$ implies that $\alpha = 1/(\sigma\sqrt{2\pi})$. So $f_X(x) = \dfrac{1}{\sigma\sqrt{2\pi}}e^{-x^2/(2\sigma^2)}$, showing that $X \sim N(0,\sigma^2)$. The fact that $Y \sim N(0,\sigma^2)$ is proved similarly.

8.3 CONDITIONAL DISTRIBUTIONS

1. $p_Y(y) = \displaystyle\sum_{x=1}^{2} p(x,y) = \frac{1}{25}(2y^2+5)$. Thus

$$p_{X|Y}(x|y) = \frac{p(x,y)}{p_Y(y)} = \frac{(1/25)(x^2+y^2)}{(1/25)(2y^2+5)} = \frac{x^2+y^2}{2y^2+5} \quad x = 1, 2, \;\; y = 0, 1, 2,$$

$$P(X = 2 \mid Y = 1) = p_{X|Y}(2|1) = 5/7,$$

$$E(X|Y=1) = \sum_{x=1}^{2} x p_{X|Y}(x|1) = \sum_{x=1}^{2} x\frac{x^2+1}{7} = \frac{12}{7}.$$

2. Since

$$f_Y(y) = \int_0^y 2\,dx = 2y, \quad 0 < y < 1,$$

we have that

$$f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)} = \frac{2}{2y} = \frac{1}{y}, \quad 0 < x < y,\ 0 < y < 1.$$

3. Let X be the number of flips of the coin until the sixth head is obtained. Let Y be the number of flips of the coin until the third head is obtained. Let Z be the number of additional flips of the coin after the third head occurs until the sixth head occurs; Z is a negative binomial random variable with parameters 3 and $1/2$. By the independence of the trials,

$$\begin{aligned} p_{X|Y}(x|5) &= P(Z = x - 5) = \binom{x-6}{2}\Big(\frac{1}{2}\Big)^3\Big(\frac{1}{2}\Big)^{x-8} \\ &= \binom{x-6}{2}\Big(\frac{1}{2}\Big)^{x-5}, \quad x = 8, 9, 10, \dots . \end{aligned}$$

4. Note that

$$f_{X|Y}\Big(x \;\Big|\; \frac{3}{4}\Big) = \frac{3\big[x^2 + (9/16)\big]}{(27/16) + 1} = \frac{1}{43}(48x^2 + 27).$$

Therefore,

$$P\Big(\frac{1}{4} < X < \frac{1}{2} \;\Big|\; Y = \frac{3}{4}\Big) = \int_{1/4}^{1/2} \frac{1}{43}(48x^2 + 27)\,dx = \frac{17}{86}.$$

5. In the discrete case, let $p(x, y)$ be the joint probability mass function of X and Y, and let A be the set of possible values of X. Then

$$E(X \mid Y = y) = \sum_{x\in A} x\frac{p(x,y)}{p_Y(y)} = \sum_{x\in A} \frac{x p_X(x) p_Y(y)}{p_Y(y)} = \sum_{x\in A} x p_X(x) = E(X).$$

In the continuous case, letting $f(x, y)$ be the joint probability density function of X and Y, we get

$$\begin{aligned} E(X \mid Y = y) &= \int_{-\infty}^{\infty} x\frac{f(x,y)}{f_Y(y)}\,dx = \int_{-\infty}^{\infty} \frac{x f_X(x) f_Y(y)}{f_Y(y)}\,dx \\ &= \int_{-\infty}^{\infty} x f_X(x)\,dx = E(X). \end{aligned}$$

6. Since

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx = \int_0^1 (x + y)\,dx = \frac{1}{2} + y,$$

the desired quantity is given by

$$f_{X|Y}(x|y) = \begin{cases} \dfrac{x+y}{(1/2)+y} & 0 \le x \le 1, \quad 0 \le y \le 1 \\ 0 & \text{elsewhere.} \end{cases}$$

7. Clearly,

$$f_Y(y) = \int_0^\infty e^{-x(y+1)}\,dx = \frac{1}{y+1}, \quad 0 \le y \le e-1.$$

Therefore,

$$E(X \mid Y = y) = \int_{-\infty}^{\infty} x f_{X|Y}(x|y)\,dx = \int_0^\infty \frac{xf(x,y)}{f_Y(y)}\,dx$$

$$= \int_0^\infty \frac{xe^{-x(y+1)}}{1/(y+1)}\,dx = \frac{1}{y+1}.$$

Note that, the last integral, $\int_0^\infty x(y+1)e^{-x(y+1)}\,dx$ is $1/(y+1)$ because it is the expected value of an exponential random variable with parameter $y+1$.

8. Let $f(x, y)$ be the joint probability density function of X and Y. Clearly,

$$f(x, y) = f_{X|Y}(x|y)f_Y(y).$$

Thus

$$f_X(x) = \int_{-\infty}^{\infty} f_{X|Y}(x|y)f_Y(y)\,dy.$$

Now

$$f_Y(y) = \begin{cases} 1 & 0 < y < 1 \\ 0 & \text{elsewhere,} \end{cases}$$

and

$$f_{X|Y}(x|y) = \begin{cases} \dfrac{1}{1-y} & 0 < y < 1, \quad y < x < 1 \\ 0 & \text{elsewhere.} \end{cases}$$

Therefore, for $0 < x < 1$,

$$f_X(x) = \int_0^x \frac{1}{1-y}\,dy = -\ln(1-x),$$

and hence

$$f_X(x) = \begin{cases} -\ln(1-x) & 0 < x < 1 \\ 0 & \text{elsewhere.} \end{cases}$$

9. $f(x, y)$, the joint probability density function of X and Y is given by

$$f(x, y) = \begin{cases} \dfrac{1}{\pi} & \text{if } x^2 + y^2 \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

Thus

$$f_Y\Big(\frac{4}{5}\Big) = \int_{-\sqrt{1-(16/25)}}^{\sqrt{1-(16/25)}} \frac{1}{\pi}\,dx = \frac{6}{5\pi}.$$

Now

$$f_{X|Y}\Big(x \;\Big|\; \frac{4}{5}\Big) = \frac{f\Big(x, \dfrac{4}{5}\Big)}{f_Y\Big(\dfrac{4}{5}\Big)} = \frac{5}{6}, \quad -\frac{3}{5} \leq x \leq \frac{3}{5}.$$

Therefore,

$$P\Big(0 \leq X \leq \frac{4}{11} \;\Big|\; y = \frac{4}{5}\Big) = \int_0^{4/11} \frac{5}{6}\,dx = \frac{10}{33}.$$

10. **(a)** $\displaystyle\int_0^{\infty}\int_{-x}^{x} ce^{-x}\,dy\,dx = 1$ implies that $c = 1/2$.

(b) $$f_{X|Y}(x|y) = \frac{f(x, y)}{f_Y(y)} = \frac{(1/2)e^{-x}}{\displaystyle\int_{|y|}^{\infty}(1/2)e^{-x}\,dx} = e^{-x+|y|}, \quad x > |y|,$$

$$f_{Y|X}(y|x) = \frac{(1/2)e^{-x}}{\displaystyle\int_{-x}^{x}(1/2)e^{-x}\,dy} = \frac{1}{2x}, \quad -x < y < x.$$

(c) By part (b), given $X = x$, Y is a uniform random variable over $(-x, x)$. Therefore, $E(Y|X = x) = 0$ and

$$\text{Var}(Y|X = x) = \frac{\big[x - (-x)\big]^2}{12} = \frac{x^2}{3}.$$

11. Let $f(x, y)$ be the joint probability density function of X and Y. Since

$$f_{X|Y}(x|y) = \begin{cases} \dfrac{1}{20 + \big[(2y)/3\big] - 20} = \dfrac{3}{2y} & 20 < x < 20 + \dfrac{2y}{3} \\ 0 & \text{otherwise,} \end{cases}$$

and

$$f_Y(y) = \begin{cases} 1/30 & 0 < y < 30 \\ 0 & \text{elsewhere,} \end{cases}$$

we have that

$$f(x, y) = f_{X|Y}(x|y) f_Y(y) = \begin{cases} \dfrac{1}{20y} & 20 < x < 20 + \dfrac{2y}{3}, \quad 0 < y < 30 \\ 0 & \text{elsewhere.} \end{cases}$$

12. Let X be the first arrival time. Clearly,

$$P\big(X \leq x \mid N(t) = 1\big) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x \geq t. \end{cases}$$

For $0 \leq x < t$,

$$P\big(X \leq x \mid N(t) = 1\big) = \frac{P\big(X \leq x,\ N(t) = 1\big)}{P\big(N(t) = 1\big)} = \frac{P\big(N(x) = 1,\ N(t - x) = 0\big)}{P\big(N(t) = 1\big)}$$

$$= \frac{P\big(N(x) = 1\big)P\big(N(t - x) = 0\big)}{P\big(N(t) = 1\big)} = \frac{\dfrac{e^{-\lambda x}(\lambda x)^1}{1!} \cdot \dfrac{e^{-\lambda(t-x)}\big[\lambda(t - x)\big]^0}{0!}}{\dfrac{e^{-\lambda t}(\lambda t)^1}{1!}} = \frac{x}{t},$$

where the third equality follows from the independence of the random variables $N(x)$ and $N(t-x)$ (recall that Poisson processes possess independent increments). We have shown that

$$P\big(X \leq x \mid N(t) = 1\big) = \begin{cases} 0 & \text{if } x < 0 \\ x/t & \text{if } 0 \leq x < t \\ 1 & \text{if } x \geq t. \end{cases}$$

This shows that the conditional distribution of X given $N(t) = 1$ is uniform on (0, 1).

13. For $x \leq y$, the fact that the conditional distribution of X given $Y = y$ is hypergeometric follows from the following:

$$P(X = x \mid Y = y) = \frac{P(X = x,\ Y = y)}{P(Y = y)} = \frac{P(X = x)P(Y - X = y - x)}{P(Y = y)}$$

$$= \frac{\dbinom{m}{x} p^x (1 - p)^{m-x} \cdot \dbinom{n - m}{y - x} p^{y-x} (1 - p)^{(n-m)-(y-x)}}{\dbinom{n}{y} p^y (1 - p)^{n-y}} = \frac{\dbinom{m}{x}\dbinom{n - m}{y - x}}{\dbinom{n}{y}}.$$

It must be clear that the conditional distribution of Y given that $X = x$ is binomial with parameters $n - m$ and p. That is,

$$P(Y = y \mid X = x) = \binom{n-m}{y-x} p^{y-x}(1-p)^{n-m-y+x}, \quad y = x, x+1, \dots, n-m+x.$$

14. Let $f(x, y)$ be the joint probability density function of X and Y. By the solution to Exercise 25, Section 8.1,

$$f(x, y) = \begin{cases} 1/2 & |x| + |y| \le 1 \\ 0 & \text{elsewhere,} \end{cases}$$

and

$$f_Y(y) = 1 - |y|, \quad -1 \le y \le 1.$$

Hence

$$f_{X|Y}(x|y) = \frac{1/2}{1-|y|} = \frac{1}{2\big(1-|y|\big)}, \quad -1 + |y| \le x \le 1 - |y|, \ -1 \le y \le 1.$$

15. Let λ be the parameter of $\big\{N(t): \ t \ge 0\big\}$. The fact that for $s < t$, the conditional distribution of $N(s)$ given $N(t) = n$ is binomial with parameters n and $p = s/t$, follows from the following relations for $i \le n$.

$$P\big(N(s) = i \mid N(t) = n\big) = \frac{P\big(N(s) = i, \ N(t) = n\big)}{P\big(N(t) = n\big)}$$

$$= \frac{P\big(N(s) = i, \ N(t) - N(s) = n - i\big)}{P\big(N(t) = n\big)} = \frac{P\big(N(s) = i\big)P\big(N(t) - N(s) = n - i\big)}{P\big(N(t) = n\big)}$$

$$= \frac{P\big(N(s) = i\big)P\big(N(t-s) = n - i\big)}{P\big(N(t) = n\big)} = \frac{\dfrac{e^{-\lambda s}(\lambda s)^i}{i!} \cdot \dfrac{e^{-\lambda(t-s)}\big[\lambda(t-s)\big]^{n-i}}{(n-i)!}}{\dfrac{e^{-\lambda t}(\lambda t)^n}{n!}}$$

$$= \binom{n}{i}\Big(\frac{s}{t}\Big)^i\Big(1 - \frac{s}{t}\Big)^{n-i},$$

where the third equality follows since Poisson processes possess independent increments and the fourth equality follows since Poisson processes are stationary.

For $i \geq k$,

$$\begin{aligned}
P\big(N(t) = i \mid N(s) = k\big) &= P\big(N(t) - N(s) = i - k \mid N(s) = k\big) \\
&= P\big(N(t) - N(s) = i - k\big) = P\big(N(t - s) = i - k\big) \\
&= \frac{e^{-\lambda(t-s)}\big[\lambda(t - s)\big]^{i-k}}{(i - k)!}
\end{aligned}$$

shows that the conditional distribution of $N(t)$ given $N(s) = k$ is Poisson with parameter $\lambda(t - s)$.

16. Let $p(x, y)$ be the joint probability mass function of X and Y. Clearly,

$$p_Y(5) = \Big(\frac{12}{13}\Big)^4\Big(\frac{1}{13}\Big),$$

and

$$p(x, 5) = \begin{cases} \Big(\frac{11}{13}\Big)^{x-1}\Big(\frac{1}{13}\Big)\Big(\frac{12}{13}\Big)^{4-x}\Big(\frac{1}{13}\Big) & x < 5 \\ 0 & x = 5 \\ \Big(\frac{11}{13}\Big)^4\Big(\frac{1}{13}\Big)\Big(\frac{12}{13}\Big)^{x-6}\Big(\frac{1}{13}\Big) & x > 5. \end{cases}$$

Using these, we have that

$$\begin{aligned}
E(X \mid Y = 5) &= \sum_{x=1}^{\infty} x p_{X|Y}(x|5) = \sum_{x=1}^{\infty} x \,\frac{p(x, 5)}{p_Y(5)} \\
&= \sum_{x=1}^{4} \frac{1}{11} x\Big(\frac{11}{12}\Big)^x + \sum_{x=6}^{\infty} x\Big(\frac{11}{12}\Big)^4\Big(\frac{1}{13}\Big)\Big(\frac{12}{13}\Big)^{x-6} \\
&= 0.72932 + \Big(\frac{11}{12}\Big)^4\Big(\frac{1}{13}\Big)\sum_{y=0}^{\infty}(y + 6)\Big(\frac{12}{13}\Big)^y \\
&= 0.72932 + \Big(\frac{11}{12}\Big)^4\Big(\frac{1}{13}\Big)\Big[\sum_{y=0}^{\infty} y\Big(\frac{12}{13}\Big)^y + 6\sum_{y=0}^{\infty}\Big(\frac{12}{13}\Big)^y\Big] \\
&= 0.702932 + \Big(\frac{11}{12}\Big)^4\Big(\frac{1}{13}\Big)\Big[\frac{12/13}{(1/13)^2} + 6\frac{1}{1 - (12/13)}\Big] = 13.412.
\end{aligned}$$

Remark: In successive draws of cards from an ordinary deck of 52 cards, one at a time, randomly, and with replacement, the expected value of the number of draws until the first ace is $1/(1/13) = 13$. This exercise shows that knowing the first king occurred on the fifth trial will increase, on the average, the number of trials until the first ace 0.412 draws.

17. Let X be the number of blue chips in the first 9 draws and Y be the number of blue chips drawn altogether. We have that

$$E(X \mid Y = 10) = \sum_{x=0}^{9} x \frac{p(x, 10)}{p_Y(10)}$$

$$= \sum_{x=1}^{9} x \frac{\binom{9}{x}\left(\frac{12}{22}\right)^{x}\left(\frac{10}{22}\right)^{9-x} \cdot \binom{9}{10-x}\left(\frac{12}{22}\right)^{10-x}\left(\frac{10}{22}\right)^{x-1}}{\binom{18}{10}\left(\frac{12}{22}\right)^{10}\left(\frac{10}{22}\right)^{8}}$$

$$= \sum_{x=1}^{9} x \frac{\binom{9}{x}\binom{9}{10-x}}{\binom{18}{10}} = \frac{9 \times 10}{18} = 5,$$

where the last sum is $(9 \times 10)/18$ because it is the expected value of a hypergeometric random variable with $N = 18$, $D = 9$, and $n = 10$.

18. Clearly,

$$f_X(x) = \int_x^1 n(n-1)(y-x)^{n-2}\, dy = n(1-x)^{n-1}.$$

Thus

$$f_{Y|X}(y|x) = \frac{f(x, y)}{f_X(x)} = \frac{n(n-1)(y-x)^{n-2}}{n(1-x)^{n-1}} = \frac{(n-1)(y-x)^{n-2}}{(1-x)^{n-1}}.$$

Therefore,

$$E(Y \mid X = x) = \int_x^1 y \frac{(n-1)(y-x)^{n-2}}{(1-x)^{n-1}}\, dy = \frac{n-1}{(1-x)^{n-1}} \int_x^1 y(y-x)^{n-2}\, dy.$$

But

$$\int_x^1 y(y-x)^{n-2}\, dy = \int_x^1 (y-x+x)(y-x)^{n-2}\, dy$$

$$= \int_x^1 (y-x)^{n-1}\, dy + \int_x^1 x(y-x)^{n-2}\, dy$$

$$= \frac{(1-x)^n}{n} + \frac{x(1-x)^{n-1}}{n-1}.$$

Thus

$$E(Y \mid X = x) = \frac{n-1}{n}(1-x) + x = \frac{n-1}{n} + \frac{1}{n}x.$$

19. **(a)** The area of the triangle is 1/2. So

$$f(x, y) = \begin{cases} 2 & \text{if } x \geq 0,\ y \geq 0,\ x + y \leq 1 \\ 0 & \text{elsewhere.} \end{cases}$$

(b) $f_Y(y) = \int_0^{1-y} 2\,dx = 2(1-y), \quad 0 < y < 1$. Therefore,

$$f_{X|Y}(x|y) = \frac{2}{2(1-y)} = \frac{1}{1-y}, \quad 0 \leq x \leq 1-y,\ 0 \leq y < 1.$$

(c) By part (b), given that $Y = y$, X is a uniform random variable over $(0, 1-y)$. Thus $E(X \mid Y = y) = (1-y)/2, 0 < y < 1$.

20. Clearly,

$$p_X(x) = \sum_{y=0}^{x} \frac{1}{e^2\, y!\,(x-y)!} = \frac{1}{e^2\, x!} \sum_{y=0}^{x} \frac{x!}{y!\,(x-y)!} = \frac{e^{-2}}{x!} \sum_{y=0}^{x} \binom{x}{y} = \frac{e^{-2} \cdot 2^x}{x!},$$

where the last equality follows since $\sum_{y=0}^{x} \binom{x}{y}$ is the number of subsets of a set with x elements and hence is equal to 2^x. Therefore, $p_X(x)$ is Poisson with parameter 2 and so

$$p_{Y|X}(y|x) = \frac{p(x, y)}{p_X(x)} = \binom{x}{y} 2^{-x}.$$

This yields

$$E(Y \mid X = x) = \sum_{y=0}^{x} y \binom{x}{y} 2^{-x} = \sum_{y=0}^{x} y \binom{x}{y} \Big(\frac{1}{2}\Big)^y \Big(\frac{1}{2}\Big)^{x-y} = \frac{x}{2},$$

where the last equality follows because the last sum is the expected value of a binomial random variable with parameters x and 1/2.

21. Let X be the lifetime of the dead battery. We want to calculate $E(X \mid X < s)$. Since X is a continuous random variable, this is the same as $E(X \mid X \leq s)$. To find this quantity, let

$$F_{X|X \leq s}(t) = P(X \leq t \mid X \leq s),$$

and $f_{X|X \leq s}(t) = F'_{X|X \leq s}(t)$. Then

$$E(X \mid X \leq s) = \int_0^{\infty} t f_{X|X \leq s}(t)\, dt.$$

Now

$$F_{X|X\le s}(t) = P(X \le t \mid X \le s) = \frac{P(X \le t,\, X \le s)}{P(X \le s)}$$
$$= \begin{cases} \dfrac{P(X \le t)}{P(X \le s)} & \text{if } t < s \\ 1 & \text{if } t \ge s. \end{cases}$$

Differentiating $F_{X|X\le s}(t)$ with respect to t, we obtain

$$f_{X|X\le s}(t) = \begin{cases} \dfrac{f(t)}{F(s)} & \text{if } t < s \\ 0 & \text{otherwise.} \end{cases}$$

This yields

$$E(X \mid X \le s) = \frac{1}{F(s)} \int_0^s t f(t)\, dt.$$

8.4 TRANSFORMATIONS OF TWO RANDOM VARIABLES

1. Let f be the joint probability density function of X and Y. Clearly,

$$f(x, y) = \begin{cases} 1 & 0 < x < 1,\ 0 < y < 1 \\ 0 & \text{elswhere.} \end{cases}$$

The system of two equations in two unknowns

$$\begin{cases} -2\ln x = u \\ -2\ln y = v \end{cases}$$

defines a one-to-one transformation of

$$R = \big\{(x, y) \colon 0 < x < 1,\ 0 < y < 1\big\}$$

onto the region

$$Q = \big\{(u, v) \colon u > 0,\ v > 0\big\}.$$

It has the unique solution $x = e^{-u/2}$, $y = e^{-v/2}$. Hence

$$\mathbf{J} = \begin{vmatrix} -\dfrac{1}{2}e^{-u/2} & 0 \\ 0 & -\dfrac{1}{2}e^{-v/2} \end{vmatrix} = \frac{1}{4}e^{-(u+v)/2} \ne 0.$$

By Theorem 8.8, $g(u, v)$, the joint probability density function of U and V is

$$g(u, v) = f\big(e^{-u/2}, e^{-v/2}\big)\left|\frac{1}{4}e^{-(u+v)/2}\right| = \frac{1}{4}e^{-(u+v)/2},\ u > 0,\ v > 0.$$

2. Let $f(x, y)$ be the joint probability density function of X and Y. Clearly,

$$f(x, y) = f_1(x)f_2(y), \qquad x > 0, \ y > 0.$$

Let $V = X$ and $g(u, v)$ be the joint probability density functions of U and V. The probability density function of U is $g_U(u)$, its marginal density function. The system of two equations in two unknowns

$$\begin{cases} x/y = u \\ x = v \end{cases}$$

defines a one-to-one transformation of

$$R = \{(x, y) \colon x > 0, \ y > 0\}$$

onto the region

$$Q = \{(u, v) \colon u > 0, \ v > 0\}.$$

It has the unique solution $x = v$, $y = v/u$. Hence

$$\mathbf{J} = \begin{vmatrix} 0 & 1 \\ -\dfrac{v}{u^2} & \dfrac{1}{u} \end{vmatrix} = \frac{v}{u^2} \neq 0.$$

By Theorem 8.8,

$$g(u, v) = f\Big(v, \frac{v}{u}\Big)\Big|\frac{v}{u^2}\Big| = \frac{v}{u^2} f\Big(v, \frac{v}{u}\Big) = \frac{v}{u^2} f_1(v) f_2\Big(\frac{v}{u}\Big) \qquad u > 0, \ v > 0.$$

Therefore,

$$g_U(u) = \int_0^\infty \frac{v}{u^2} f_1(v) f_2\Big(\frac{v}{u}\Big)\, dv, \qquad u > 0.$$

3. Let $g(r, \theta)$ be the joint probability density function of R and Θ. We will show that $g(r, \theta) = g_R(r)g_\Theta(\theta)$. This proves the *surprising* result that R and Θ are independent. Let $f(x, y)$ be the joint probability density function of X and Y. Clearly,

$$f(x, y) = \frac{1}{2\pi} e^{-(x^2+y^2)/2}, \quad -\infty < x < \infty, \ -\infty < y < \infty.$$

Let $\mathbf{R}$ be the entire xy-plane excluded the set of points on the x-axis with $x \geq 0$. This causes no problems since

$$P(Y = 0, \ X \geq 0) = P(Y = 0)P(X \geq 0) = 0.$$

The system of two equations in two unknowns

$$\begin{cases} \sqrt{x^2 + y^2} = r \\ \arctan \dfrac{y}{x} = \theta \end{cases}$$

defines a one-to-one transformation of **R** onto the region

$$Q = \big\{(r, \theta) \colon r > 0,\ 0 < \theta < 2\pi\big\}.$$

It has the unique solution

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta. \end{cases}$$

Hence

$$\mathbf{J} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r \neq 0.$$

By Therorem 8.8, $g(r, \theta)$ is given by

$$g(r, \theta) = f(r\cos\theta, r\sin\theta)|r| = \frac{1}{2\pi} r e^{-r^2/2} \quad 0 < \theta < 2\pi,\ r > 0.$$

Now

$$g_R(r) = \int_0^{2\pi} \frac{1}{2\pi} r e^{-r^2/2}\, d\theta = r e^{-r^2/2}, \quad r > 0,$$

and

$$g_\Theta(\theta) = \int_0^{\infty} \frac{1}{2\pi} r e^{-r^2/2}\, dr = \frac{1}{2\pi}, \quad 0 < \theta < 2\pi.$$

Therefore, $g(r, \theta) = g_R(r) g_\Theta(\theta)$, showing that R and Θ are independent random variables. The formula for $g_\Theta(\theta)$ indicates that Θ is a uniform random variable over the interval $(0, 2\pi)$. The probability density function obtained for R is called *Rayleigh*.

4. ***Method* 1**: By the convolution theorem (Theorem 8.9), g, the probability density function of the sum of X and Y, the two random points selected from (0, 1) is given by

$$g(t) = \int_{-\infty}^{\infty} f_1(x) f_2(t - x)\, dx,$$

where f_1 and f_2 are, respectively, the probability density functions of X and Y. Since

$$f_1(x) = f_2(x) = \begin{cases} 1 & x \in (0, 1) \\ 0 & \text{elsewhere,} \end{cases}$$

the integrand, $f_1(x) f_2(t - x)$ is nonzero if $0 < x < 1$ and $t - 1 < x < t$. This shows that for $t < 0$ and $t \geq 2$, $g(t) = 0$. For $0 \leq t < 1$, $t - 1 < 0$; thus

$$g(t) = \int_0^t dx = t.$$

For $1 \leq t < 2$, $0 < t - 1 < 1$; therefore,

$$g(t) = \int_{t-1}^{1} dx = 1 - (t - 1) = 2 - t.$$

So

$$g(t) = \begin{cases} t & \text{if } 0 \le t < 1 \\ 2 - t & \text{if } 1 \le t < 2 \\ 0 & \text{otherwise.} \end{cases}$$

***Method* 2**: Note that the sample space of the experiment of choosing two random numbers from (0, 1) is

$$S = \left\{(x, y) \in \mathbf{R^2} \colon 0 < x < 1,\ 0 < y < 1\right\}.$$

So, for $0 \le t < 1$, $P(X + Y \le t)$ is the area of the region

$$\left\{(x, y) \in S \colon 0 < x \le t,\ 0 < y \le t,\ x + y \le t\right\}$$

divided by the area of S: $t^2/2$. For $1 \le t < 2$, $P(X + Y \le t)$ is the area of

$$S - \left\{(x, y) \in S \colon t - 1 \le x < 1,\ t - 1 \le y < 1,\ x + y > t\right\}$$

divided by the area of S: $1 - \dfrac{(2-t)^2}{2}$. (Draw figures to verify these regions.) Let G be the probability distribution function of $X + Y$. We have shown that

$$G(t) = \begin{cases} 0 & t < 0 \\ \dfrac{t^2}{2} & 0 \le t < 1 \\ 1 - \dfrac{(2-t)^2}{2} & 1 \le t < 2 \\ 1 & t \ge 2. \end{cases}$$

Therefore,

$$g(t) = G'(t) = \begin{cases} t & 0 \le t < 1 \\ 2 - t & 1 \le t < 2 \\ 0 & \text{otherwise.} \end{cases}$$

5. **(a)** Clearly, $p_X(x) = 1/3$ for $x = -1, 0, 1$ and $p_Y(y) = 1/3$ for $y = -1, 0, 1$. Since

$$P(X + Y = z) = \begin{cases} 1/9 & z = -2, +2 \\ 2/9 & z = -1, +1 \\ 3/9 & z = 0, \end{cases}$$

the relation

$$P(X + Y = z) = \sum_x p_X(x) p_Y(z - x)$$

is easily seen to be true.

(b) $p(x, y) = p_X(x)p_Y(y)$ for all possible values x and y of X and Y if and only if $(1/9)+c = 1/9$ and $(1/9) - c = 1/9$; that is, if and only if $c = 0$.

6. Let $h(x, y)$ be the joint probability density function of X and Y. Then

$$h(x, y) = \begin{cases} \dfrac{1}{x^2y^2} & x \geq 1, \ y \geq 1 \\ \\ 0 & \text{elsewhere.} \end{cases}$$

Consider the system of two equations in two unknowns

$$\begin{cases} x/y = u \\ xy = v. \end{cases} \tag{29}$$

This system has the unique solution

$$\begin{cases} x = \sqrt{uv} \\ y = \sqrt{v/u}. \end{cases} \tag{30}$$

We have that

$$x \geq 1 \iff \sqrt{uv} \geq 1 \iff u \geq \frac{1}{v},$$

$$y \geq 1 \iff \sqrt{v/u} \geq 1 \iff v \geq u.$$

Clearly, $x \geq 1$, $y \geq 1$ imply that $v = xy \geq 1$, so $\dfrac{1}{v} > 0$. Therefore, the system of equations (29) defines a one-to-one transformation of

$$R = \big\{(x, y) \colon x \geq 1, \ y \geq 1\big\}$$

onto the region

$$Q = \Big\{(u, v) \colon 0 < \frac{1}{v} \leq u \leq v\Big\}.$$

By (30),

$$\mathbf{J} = \begin{vmatrix} \dfrac{1}{2}\sqrt{\dfrac{v}{u}} & \dfrac{1}{2}\sqrt{\dfrac{u}{v}} \\ \\ -\dfrac{\sqrt{v}}{2u\sqrt{u}} & \dfrac{1}{2\sqrt{uv}} \end{vmatrix} = \frac{1}{2u} \neq 0.$$

Hence, by Theorem 8.8, $g(u, v)$, the joint probability density function of U and V is given by

$$g(u, v) = h\bigg(\sqrt{uv}, \sqrt{\frac{v}{u}}\bigg)|\mathbf{J}| = \frac{1}{2uv^2}, \quad 0 < \frac{1}{v} \leq u \leq v.$$

7. Let h be the joint probability density function of X and Y. Clearly,

$$h(x,y) = \begin{cases} e^{-(x+y)} & x > 0,\ y > 0 \\ 0 & \text{elsewhere.} \end{cases}$$

Consider the system of two equations in two unknowns

$$\begin{cases} x + y = u \\ e^x = v. \end{cases} \tag{31}$$

This system has the unique solution

$$\begin{cases} x = \ln v \\ y = u - \ln v. \end{cases} \tag{32}$$

We have that

$$x > 0 \iff \ln v > 0 \iff v > 1,$$

$$y > 0 \iff u - \ln v > 0 \iff e^u > v.$$

Therefore, the system of equations (31) defines a one-to-one transformation of

$$R = \big\{(x,y)\colon x > 0,\ y > 0\big\}$$

onto the region

$$Q = \big\{(u,v)\colon u > 0,\ 1 < v < e^u\big\}.$$

By (32),

$$\mathbf{J} = \begin{vmatrix} 0 & \dfrac{1}{v} \\ 1 & -\dfrac{1}{v} \end{vmatrix} = -\frac{1}{v} \neq 0.$$

Hence, by Theorem 8.8, $g(u,v)$, the joint probability density function of U and V is given by

$$g(u,v) = h(\ln v, u - \ln v)|\mathbf{J}| = \frac{1}{v}e^{-u}, \quad u > 0,\ 1 < v < e^u.$$

8. Let $U = X + Y$ and $V = X - Y$. Let $g(u,v)$ be the joint probability density function of U and V. We will show that $g(u,v) = g_U(u)g_V(v)$. To do so, let $f(x,y)$ be the joint probability density function of X and Y. Then

$$f(x,y) = \frac{1}{2\pi}e^{-(x^2+y^2)/2}, \quad -\infty < x < \infty,\ -\infty < y < \infty.$$

The system of two equations in two unkowns

$$\begin{cases} x + y = u \\ x - y = v \end{cases}$$

defines a one-to-one correspondence from the entire xy-plane onto the entire uv-plane. It has the unique solution

$$\begin{cases} x = \dfrac{u+v}{2} \\ y = \dfrac{u-v}{2}. \end{cases}$$

Hence

$$\mathbf{J} = \begin{vmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{vmatrix} = -\frac{1}{2} \neq 0.$$

By Theorem 8.8,

$$\begin{aligned} g(u,v) &= f\Big(\frac{u+v}{2}, \frac{u-v}{2}\Big)|\mathbf{J}| \\ &= \frac{1}{4\pi}\exp\left[-\frac{\Big(\dfrac{u+v}{2}\Big)^2 + \Big(\dfrac{u-v}{2}\Big)^2}{2}\right] = \frac{1}{4\pi}e^{-(u^2+v^2)/4}, \quad -\infty < u,\ v < \infty. \end{aligned}$$

This gives

$$\begin{aligned} g_U(u) &= \frac{1}{4\pi}\int_{-\infty}^{\infty} e^{-(u^2+v^2)/4}\,dv = \frac{1}{4\pi}e^{-u^2/4}\int_{-\infty}^{\infty} e^{-v^2/4}\,dv \\ &= \frac{1}{2\sqrt{\pi}}e^{-u^2/4}\int_{-\infty}^{\infty}\frac{1}{2\sqrt{\pi}}e^{-v^2/4}\,dv = \frac{1}{2\sqrt{\pi}}e^{-u^2/4}, \quad -\infty < u < \infty, \end{aligned}$$

where the last equality follows because $\dfrac{1}{2\sqrt{\pi}}e^{-v^2/4}$ is the probability density function of a normal random variable with mean 0 and variance 2. Thus its integral over the interval $(-\infty, \infty)$ is 1. Similarly,

$$g_V(v) = \frac{1}{2\sqrt{\pi}}e^{-v^2/2}, \quad -\infty < v < \infty.$$

Since $g(u,v) = g_U(u)g_V(v)$, U and V are independent normal random variables each with mean 0 and variance 2.

9. Let f be the joint probability density function of X and Y. Clearly,

$$f(x,y) = \frac{\lambda^{r_1+r_2}x^{r_1-1}y^{r_2-1}e^{-\lambda(x+y)}}{\Gamma(r_1)\Gamma(r_2)}, \quad x > 0,\ y > 0.$$

Consider the system of two equations in two unknowns

$$\begin{cases} x + y = u \\ \dfrac{x}{x+y} = v. \end{cases} \tag{33}$$

Clearly, (33) implies that $u > 0$ and $v > 0$. This system has the unique solution

$$\begin{cases} x = uv \\ y = u - uv. \end{cases} \tag{34}$$

We have that

$$x > 0 \iff uv > 0 \iff u > 0 \text{ and } v > 0,$$

$$y > 0 \iff u - uv > 0 \iff v < 1.$$

Therefore, the system of equations (33) defines a one-to-one transformation of

$$R = \{(x, y) \colon x > 0,\ y > 0\}$$

onto the region

$$Q = \{(u, v) \colon u > 0,\ 0 < v < 1\}.$$

By (34),

$$\mathbf{J} = \begin{vmatrix} v & u \\ 1 - v & -u \end{vmatrix} = -u \neq 0.$$

Hence by Thereom 8.8, the joint probability density function of U and V is given by

$$g(u, v) = f(uv, u - uv)|\mathbf{J}| = \frac{\lambda^{r_1+r_2} u^{r_1+r_2-1} e^{-\lambda u} v^{r_1-1}(1-v)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} \quad u > 0,\ 0 < v < 1.$$

Note that

$$g(u, v) = \frac{\lambda e^{-\lambda u}(\lambda u)^{r_1+r_2-1}}{\Gamma(r_1+r_2)} \cdot \frac{\Gamma(r_1+r_2)}{\Gamma(r_1)\Gamma(r_2)} v^{r_1-1}(1-v)^{r_2-1}$$
$$= \frac{\lambda e^{-\lambda u}(\lambda u)^{r_1+r_2-1}}{\Gamma(r_1+r_2)} \cdot \frac{1}{B(r_1, r_2)} v^{r_1-1}(1-v)^{r_2-1}, \quad u > 0,\ 0 < v < 1.$$

This shows that

$$g(u, v) = g_U(u) g_V(v).$$

That is, U and V are independent. Furthermore, it shows that $g_U(u)$ is the probability density function of a gamma random variable with parameter $r_1 + r_2$ and λ; $g_V(v)$ is the probability density function of a beta random variable with parameters r_1 and r_2.

10. Let f be the joint probability density function of X and Y. Clearly,

$$f(x, y) = \lambda^2 e^{-\lambda(x+y)}, \quad x > 0,\ y > 0.$$

The system of two equations in two unknowns

$$\begin{cases} x + y = u \\ x/y = v \end{cases}$$

defines a one-to-one transformation of

$$R = \{(x, y) \colon x > 0,\ y > 0\}$$

onto the region

$$Q = \{(u, v) \colon u > 0, v > 0\}.$$

It has he unique solution $x = uv/(1+v)$, $y = u/(1+v)$. Hence

$$\mathbf{J} = \begin{vmatrix} \dfrac{v}{1+v} & \dfrac{u}{(1+v)^2} \\ \dfrac{1}{1+v} & -\dfrac{u}{(1+v)^2} \end{vmatrix} = -\frac{u}{(1+v)^2} \neq 0.$$

By Theorem 8.8, $g(u, v)$, the joint probability density function of U and V is

$$g(u, v) = f\Big(\frac{uv}{1+v}, \frac{u}{1+v}\Big)|\mathbf{J}| = \frac{\lambda^2 u}{(1+v)^2} e^{-\lambda u}, \quad u > 0,\ v > 0.$$

This shows that $g(u, v) = g_U(u) g_V(v)$, where

$$g_U(u) = \lambda^2 u e^{-\lambda u}, \quad u > 0,$$

and

$$g_V(v) = \frac{1}{(1+v)^2}, \quad v > 0.$$

Therefore, $U = X + Y$ and $V = X/Y$ are independent random variables.

REVIEW PROBLEMS FOR CHAPTER 8

1. **(a)** We have that

$$\begin{aligned} P(XY \le 6) &= p(1,2) + p(1,4) + p(1,6) + p(2,2) + p(3,2) \\ &= 0.05 + 0.14 + 0.10 + 0.25 + 0.15 = 0.69. \end{aligned}$$

(b) First we calculate $p_X(x)$ and $p_Y(y)$, the marginal probability mass functions of X and Y. They are given by the following table.

	x			
y	1	2	3	$p_Y(y)$
2	0.05	0.25	0.15	0.45
4	0.14	0.10	0.17	0.41
6	0.10	0.02	0.02	0.14
$p_X(x)$	0.29	0.37	0.34	

Therefore,

$$E(X) = 1(0.29) + 2(0.37) + 3(0.34) = 2.05;$$
$$E(Y) = 2(0.45) + 4(0.41) + 6(0.14) = 3.38.$$

2. **(a) and (b)** $p(x, y)$, the joint probability mass function of X and Y, and $p_X(x)$ and $p_Y(y)$, the marginal probability mass functions of X and Y are given by the following table.

	y						
x	1	2	3	4	5	6	$p_X(x)$
2	1/36	0	0	0	0	0	1/36
3	0	2/36	0	0	0	0	2/36
4	0	1/36	2/36	0	0	0	3/36
5	0	0	2/36	2/36	0	0	4/36
6	0	0	1/36	2/36	2/36	0	5/36
7	0	0	0	2/36	2/36	2/36	6/36
8	0	0	0	1/36	2/36	2/36	5/36
9	0	0	0	0	2/36	2/36	4/36
10	0	0	0	0	1/36	2/36	3/36
11	0	0	0	0	0	2/36	2/36
12	0	0	0	0	0	1/36	1/36
$p_Y(y)$	1/36	3/36	5/36	7/36	9/36	11/36	

(c) $E(X) = \sum_{x=2}^{15} x p_X(x) = 7; \quad E(Y) = \sum_{y=1}^{6} y p_Y(y) = 161/36 \approx 4.47.$

3. Let X be the number of spades and Y be the number of hearts in the random bridge hand. The desired probability mass function is

$$p_{X|Y}(x|4) = \frac{p(x, 4)}{p_Y(4)} = \frac{\dfrac{\dbinom{13}{x}\dbinom{13}{4}\dbinom{26}{9-x}}{\dbinom{52}{13}}}{\dfrac{\dbinom{13}{4}\dbinom{39}{9}}{\dbinom{52}{13}}} = \frac{\dbinom{13}{x}\dbinom{26}{9-x}}{\dbinom{39}{9}}, \quad 0 \le x \le 9.$$

4. The set of possible values of X and Y, both, is $\{0, 1, 2, 3\}$. Let $p(x, y)$ be their joint probability mass function; then

$$p(x, y) = \frac{\dbinom{13}{x}\dbinom{13}{y}\dbinom{26}{3-x-y}}{\dbinom{52}{3}}, \quad 0 \le x, \ y, \ x + y \le 3.$$

5. Reducing the sample space, the answer is $\dfrac{\dbinom{13}{x}\dbinom{13}{6-x}}{\dbinom{26}{6}}$, $\quad 0 \le x \le 6$.

6. (a) $\displaystyle\int_0^2 \Big(\int_0^x \frac{c}{x}\,dy\Big)\,dx = 1 \implies c = 1/2.$

(b) $\displaystyle f_X(x) = \int_0^x \frac{1}{2x}\,dy = \frac{1}{2}, \quad 0 < x < \frac{1}{2},$

$$f_Y(y) = \int_y^2 \frac{1}{2x}\,dx = \Big[\frac{1}{2}\ln x\Big]_y^2 = \frac{1}{2}\ln\frac{2}{y}, \quad 0 < y < 2.$$

7. Note that $f(x, y) = \frac{1}{2}y\Big(\frac{3}{2}x^2 + \frac{1}{2}\Big)$, where $\frac{1}{2}y$, $0 < y < 2$ and $\frac{3}{2}x^2 + \frac{1}{2}$, $\quad 0 < x < 1$ are probability density functions. Therefore,

$$f_Y(y) = \frac{1}{2}y, \quad 0 < y < 2,$$

$$f_X(x) = \frac{3}{2}x^2 + \frac{1}{2}, \quad 0 < x < 1.$$

We observe that $f(x, y) = f_X(x) f_Y(y)$. This shows that X and Y are independent random variables and hence $E(XY) = E(X)E(Y)$. This relation can also be verified directly:

$$E(XY) = \int_0^1 \Big[\int_0^2 \Big(\frac{3}{4}x^3y^2 + \frac{1}{4}xy^2\Big)\,dy\Big]\,dx = \frac{5}{6},$$

$$E(X) = \int_0^1 \Big[\int_0^2 \Big(\frac{3}{4}x^3y + \frac{1}{4}xy\Big)\,dy\Big]\,dx = \frac{5}{8},$$

$$E(Y) = \int_0^1 \Big[\int_0^2 \Big(\frac{3}{4}x^2y^2 + \frac{1}{4}y^2\Big)\,dy\Big]\,dx = \frac{4}{3}.$$

Hence

$$E(XY) = \frac{5}{6} = \frac{5}{8}\cdot\frac{4}{3} = E(X)E(Y).$$

8. A distribution function is 0 at $-\infty$ and 1 at ∞, so it cannot be constant everywhere. $F(x, y)$ is not a joint probability distribution function because assuming it is, we get that $F_X(x)$ is constant everywhere:

$$F_X(x) = F(x, \infty) = 1, \quad \forall x.$$

9. The answer is $\dfrac{\pi r_2^2 - \pi r_3^2}{\pi r_1^2} = \dfrac{r_2^2 - r_3^2}{r_1^2}$.

10. Let Y be the total number of heads obtained. Let X be the total number of heads in the first 10 flips. For $2 \le x \le 10$,

$$p_{X|Y}(x \mid 12) = \frac{p(x, 12)}{p_Y(12)} = \frac{\binom{10}{x}\left(\frac{1}{2}\right)^{10} \cdot \binom{10}{12-x}\left(\frac{1}{2}\right)^{10}}{\binom{20}{12}\left(\frac{1}{2}\right)^{20}} = \frac{\binom{10}{x}\binom{10}{12-x}}{\binom{20}{12}}.$$

This is the probability mass function of a hypergeometric random variable with parameters $N = 20$, $D = 10$, and $n = 12$. Its expected value is $\dfrac{nD}{N} = \dfrac{12 \times 10}{20} = 6$, as expected.

11. $f(x, y)$, the joint probability density function of X and Y is given by

$$f(x, y) = \frac{\partial^2}{\partial x\, \partial y} F(x, y) = 4xye^{-x^2}e^{-y^2}, \quad x > 0, \; y > 0.$$

Therefore, by symmetry,

$$P(X > 2Y) + P(Y > 2X) = 2P(X > 2Y) = 2\int_0^\infty \left(\int_{2y}^\infty 4xye^{-x^2}e^{-y^2}\,dx\right)dy = \frac{2}{5}.$$

12. We have that

$$f_X(x) = \int_0^{1-x} 3(x+y)\,dy = -\frac{3}{2}x^2 + \frac{3}{2}, \quad 0 < x < 1,$$

By symmetry,

$$f_Y(y) = -\frac{3}{2}y^2 + \frac{3}{2}, \quad 0 < y < 1.$$

Therefore,

$$P(X + Y > 1/2) = \int_0^{1/2}\left[\int_{(1/2)-x}^{1-x} 3(x+y)\,dy\right]dx + \int_{1/2}^1\left[\int_0^{1-x} 3(x+y)\,dy\right]dx$$
$$= \frac{9}{64} + \frac{5}{16} = \frac{29}{64}.$$

13. Since

$$f_{X|Y}(x|y) = \frac{f(x, y)}{f_Y(y)} = \frac{e^{-y}}{\int_0^1 e^{-y}\,dx} = 1, \quad 0 < x < 1, \; y > 0,$$

we have that

$$E(X^n \mid Y = y) = \int_0^1 x^n \cdot 1\,dx = \frac{1}{n+1}, \quad n \ge 1.$$

14. Let $p(x, y)$ be the joint probability mass function of X and Y. We have that

$$p(x, y) = \binom{10}{x}\left(\frac{1}{4}\right)^x\left(\frac{3}{4}\right)^{10-x} \cdot \binom{15}{y}\left(\frac{1}{4}\right)^y\left(\frac{3}{4}\right)^{15-y}$$

$$= \binom{10}{x}\binom{15}{y}\left(\frac{1}{4}\right)^{x+y}\left(\frac{3}{4}\right)^{25-x-y}, \quad 0 \le x \le 10,\ 0 \le y \le 15.$$

15. $\displaystyle\int_0^1 \left[\int_x^1 cx(1-x)\,dy\right] dx = 1 \implies c = 12.$ Clearly,

$$f_X(x) = \int_x^1 12x(1-x)\,dy = 12x(1-x)^2, \quad 0 < x < 1,$$

$$f_Y(y) = \int_0^y 12x(1-x)\,dx = 6y^2 - 4y^3, \quad 0 < y < 1.$$

Since $f(x, y) \neq f_X(x)f_Y(y)$, X and Y are not independent.

16. The area of the region bounded by $y = x^2 - 1$ and $y = 1 - x^2$ is

$$\int_{-1}^1 \left(\int_{x^2-1}^{1-x^2} dy\right) dx = \frac{8}{3}.$$

Therefore $f(x, y)$, the joint probability density function of X and Y is given by

$$f(x, y) = \begin{cases} 3/8 & x^2 - 1 < y < 1 - x^2, \quad -1 < x < 1 \\ 0 & \text{elsewhere.} \end{cases}$$

Clearly,

$$f_X(x) = \int_{x^2-1}^{1-x^2} \frac{3}{8}\,dy = \frac{3}{4}(1 - x^2), \quad -1 < x < 1.$$

To find $f_Y(y)$, note that for $-1 < y < 0$,

$$f_Y(y) = \int_{-\sqrt{1+y}}^{\sqrt{1+y}} \frac{3}{8}\,dx = \frac{3}{4}\sqrt{1+y}$$

and, for $0 \le y < 1$,

$$f_Y(y) = \int_{-\sqrt{1-y}}^{\sqrt{1-y}} \frac{3}{8}\,dx = \frac{3}{4}\sqrt{1-y}.$$

So

$$f_Y(y) = \begin{cases} \dfrac{3}{4}\sqrt{1+y} & -1 < y < 0 \\ \dfrac{3}{4}\sqrt{1-y} & 0 \le y < 1 \\ 0 & \text{otherwise.} \end{cases}$$

Since $f(x, y) \neq f_X(x) f_Y(y)$, X and Y are not independent.

17. Let $f(x, y)$ be the joint probability density function of X and Y, G be the probability distribution function of X/Y, and g be the probability density function of X/Y. We have that

$$f(x, y) = \begin{cases} 1/2 & 0 < x < 1,\ 0 < y < 2 \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, $P(X/Y \le t) = 0$ if $t < 0$. For $0 \le t < 1/2$,

$$P\Big(\frac{X}{Y} \le t\Big) = \int_0^2 \Big(\int_0^{ty} \frac{1}{2}\,dx\Big)\,dy = t.$$

For $t \ge 1/2$,

$$P\Big(\frac{X}{Y} \ge t\Big) = \int_0^1 \Big(\int_{x/t}^{2} \frac{1}{2}\,dy\Big)\,dx = 1 - \frac{1}{4t}.$$

(Draw appropriate figures to verify the limits of these integrals.) Therefore,

$$G(t) = \begin{cases} 0 & t < 0 \\ t & 0 \le t < \dfrac{1}{2} \\ 1 - \dfrac{1}{4t} & t \ge \dfrac{1}{2}. \end{cases}$$

This gives

$$g(t) = G'(t) = \begin{cases} 0 & t < 0 \\ 1 & 0 \le t < \dfrac{1}{2} \\ \dfrac{1}{4t^2} & t \ge \dfrac{1}{2}. \end{cases}$$

18. No, because $G(\infty, \infty) = F(\infty) + F(\infty) = 2 \neq 1$.

19. The problem is equivalent to the following: Two points X and Y are selected independently and at random from the interval $(0, \ell)$. What is the probability that the length of at least one interval is less than $\ell/20$? The solution to this problem is as follows:

$$P\Big(\min(X,\ Y-X,\ \ell-Y) < \frac{\ell}{20}\ \Big|\ X<Y\Big)P(X<Y)$$

$$+\ P\Big(\min(Y,\ X-Y,\ \ell-X) < \frac{\ell}{20}\ \Big|\ X>Y\Big)P(X>Y)$$

$$= 2P\Big(\min(X,\ Y-X,\ \ell-Y) < \frac{\ell}{20}\ \Big|\ X<Y\Big)P(X<Y)$$

$$= 2P\Big(\min(X,\ Y-X,\ \ell-Y) < \frac{\ell}{20}\ \Big|\ X<Y\Big)\cdot\frac{1}{2}$$

$$= 1-P\Big(\min(X,\ Y-X,\ \ell-Y) \ge \frac{\ell}{20}\ \Big|\ X<Y\Big)$$

$$= 1-P\Big(X\ge\frac{\ell}{20},\ Y-X\ge\frac{\ell}{20},\ \ell-Y\ge\frac{\ell}{20}\ \Big|\ X<Y\Big)$$

$$= 1-P\Big(X\ge\frac{\ell}{20},\ Y-X\ge\frac{\ell}{20},\ Y\le\frac{19\ell}{20}\ \Big|\ X<Y\Big).$$

Now $P\Big(X\ge\frac{\ell}{20},\ Y-X\ge\frac{\ell}{20},\ Y\le\frac{19\ell}{20}\ \Big|\ X<Y\Big)$ is the area of the region

$$\Big\{(x,y)\in\mathbf{R}^2\colon 0<x<\ell,\ 0<y<\ell,\ x\ge\frac{\ell}{20},\ y-x\ge\frac{\ell}{20},\ y\le\frac{19\ell}{20}\Big\}$$

divided by the area of the triangle

$$\big\{(x,y)\in\mathbf{R}^2\colon 0<x<\ell,\ 0<y<\ell,\ y>x\big\};$$

that is,

$$\frac{\frac{17\ell}{20}\times\frac{17\ell}{20}}{2}\div\frac{\ell^2}{2}=0.7225.$$

Therefore, the desired probability is $1-0.7225=0.2775$.

20. Let $p(x,y)$ be the joint probability mass function of X and Y.

$$p(x,y)=P(X=x,\ Y=y)=(0.90)^{x-1}(0.10)(0.90)^{y-1}(0.10)=(0.90)^{x+y-2}(0.10)^2.$$

21. We have that

$$f_X(x)=\int_{-x}^{x}dy=2x,\quad 0<x<1,$$

$$f_Y(y)=\begin{cases}\displaystyle\int_{-y}^{1}dx=1+y & -1<y<0\\[2ex] \displaystyle\int_{y}^{1}dx=1-y & 0<y<1,\end{cases}$$

$$f_{X|Y}(x|y) = \begin{cases} \dfrac{1}{1+y} & -y < x < 1,\ -1 < y < 0 \\ \dfrac{1}{1-y} & y < x < 1,\ 0 < y < 1, \end{cases}$$

and

$$f_{Y|X}(y|x) = \frac{1}{2x},\ -x < y < x.$$

Thus

$$E(Y \mid X = x) = \int_{-x}^{x} \frac{y}{2x}\,dy = 0 = 0 \cdot x + 0,$$

and

$$E(X \mid Y = y) = \begin{cases} \displaystyle\int_{-y}^{1} \frac{x}{1+y}\,dx = \frac{1-y}{2}, & -1 < y < 0 \\ \displaystyle\int_{y}^{1} \frac{x}{1-y}\,dx = \frac{1+y}{2}, & 0 < y < 1. \end{cases}$$

22. We present the solution given by Merryfield, Viet, and Watson, in the August–September 1997 issue of the *American Mathematical Monthly.* Let f be the joint probability density function of X and Y.

$$E(W_A) = \int_a^b \int_a^b W_A(x, y) f(x, y)\,dxdy,$$

$$E(W_B) = \int_a^b \int_a^b W_B(x, y) f(x, y)\,dxdy.$$

Let $U = Y$, $V = X$, $h_1(x, y) = y$ and $h_2(x, y) = x$. Then the system of equations

$$\begin{cases} y = u \\ x = v \end{cases}$$

has the unique solution $x = v$, $y = u$, and

$$\mathbf{J} = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1 \neq 0.$$

Applying the change of variables formula for multiple intergrals, we obtain

$$E(W_A) = \int_a^b \int_a^b W_A(x, y) f(x, y)\,dxdy = \int_a^b \int_a^b W_A(v, u) f(v, u) |\mathbf{J}|\,dudv$$
$$= \int_a^b \int_a^b W_A(v, u) f(v, u)\,dudv.$$

Since the distribution of the money in each player's wallet is the same, the joint distributions of (X, Y) and (Y, X) have the same probability density function f satisfying $f(x, y) = f(y, x)$. Observing that $W_A(Y, X) = W_B(X, Y)$, we have that $W_A(v, u) = W_B(u, v)$. This and $f(v, u) = f(u, v)$ imply that

$$E(W_A) = \int_a^b \int_a^b W_B(u, v) f(u, v)\, du dv = E(W_B).$$

On the other hand, $W_A(X, Y) = -W_B(X, Y)$ implies that $E(W_A) = -E(W_B)$. Thus $E(W_A) = -E(W_A)$, implying that $E(W_A) = E(W_B) = 0$.

Chapter 9

Multivariate Distributions

9.1 JOINT DISTRIBUTIONS OF $n > 2$ RANDOM VARIABLES

1. Let $p(h, d, c, s)$ be the joint probability mass function of the number of hearts, diamonds, clubs, and spades selected. We have

$$p(h, d, c, s) = \frac{\binom{13}{h}\binom{13}{d}\binom{13}{c}\binom{13}{s}}{\binom{52}{13}}, \quad h + d + c + s = 13, \quad 0 \le h, d, c, s \le 13.$$

2. Let $p(a, h, n, w)$ be the joint probability mass function of A, H, N, and W. Clearly,

$$p(a, h, n, w) = \frac{\binom{8}{a}\binom{7}{h}\binom{3}{n}\binom{20}{w}}{\binom{38}{12}},$$

$$a + h + n + w = 12,\ 0 \le a \le 8,\ 0 \le h \le 7,\ 0 \le n \le 3,\ 0 \le w \le 12.$$

The marginal probability mass function of A is given by

$$p_A(a) = \frac{\binom{8}{a}\binom{30}{12 - a}}{\binom{38}{12}}, \quad 0 \le a \le 8.$$

3. **(a)** The desired joint marginal probability mass functions are given by

$$p_{X,Y}(x, y) = \sum_{z=1}^{2} \frac{xyz}{162} = \frac{xy}{54}, \quad x = 4, 5,\ y = 1, 2, 3.$$

$$p_{Y,Z}(y, z) = \sum_{x=4}^{5} \frac{xyz}{162} = \frac{yz}{18}, \quad y = 1, 2, 3,\ z = 1, 2.$$

$$p_{X,Z}(x, z) = \sum_{y=1}^{3} \frac{xyz}{162} = \frac{xz}{27}, \quad x = 4, 5,\ z = 1, 2.$$

(b) $E(YZ) = \sum_{y=1}^{3}\sum_{z=1}^{2} yz p_{Y,Z}(y, z) = \sum_{y=1}^{3}\sum_{z=1}^{2} \frac{(yz)^2}{18} = \frac{35}{9}.$

4. **(a)** The desired marginal joint probability mass functions are given by

$$f_{X,Y}(x, y) = \int_y^\infty 6e^{-x-y-z}\,dz = 6e^{-x-2y}, \quad 0 < x < y < \infty.$$

$$f_{X,Z}(x, z) = \int_x^z 6e^{-x-y-z}\,dy = 6e^{-x-z}(e^{-x} - e^{-z}), \quad 0 < x < z < \infty.$$

$$f_{Y,Z}(y, z) = \int_0^y 6e^{-x-y-z}\,dx = 6e^{-y-z}(1 - e^{-y}), \quad 0 < y < z < \infty.$$

(b) $E(X) = \int_0^\infty \int_x^\infty x f_{X,Y}(x, y)\,dy\,dx = \int_0^\infty \int_x^\infty 6xe^{-x-2y}\,dy\,dx = \int_0^\infty 3xe^{-3x}\,dx = 1/3.$

5. They are not independent because $P(X_1 = 1, X_2 = 1, X_3 = 0) = 1/4$, whereas $P(X_1 = 1)P(X_2 = 1)P(X_3 = 0) = 1/8$.

6. Note that

$$\begin{aligned} f_X(x) &= \int_0^\infty \int_0^\infty x^2 e^{-x(1+y+z)}\,dy\,dz \\ &= x^2 e^{-x} \int_0^\infty e^{-xz}\left(\int_0^\infty e^{-xy}\,dy\right) dz = e^{-x}, \quad x > 0, \end{aligned}$$

$$f_Y(y) = \int_0^\infty \left(\int_0^\infty x^2 e^{-x(1+y+z)}\,dz\right) dx = \frac{1}{(1+y)^2}, \quad y > 0,$$

and similarly,

$$f_Z(z) = \frac{1}{(1+z)^2}, \quad z > 0.$$

Also

$$f_{X,Y}(x, y) = \int_0^\infty x^2 e^{-x(1+y+z)}\,dz = xe^{-x(1+y)}, \quad y > 0.$$

Since

$$f(x, y, z) \neq f_X(x) f_Y(y) f_Z(z),$$

X, Y, and Z are not independent. Since $f_{X,Y}(x, y) \neq f_X(x) f_Y(y)$, X, Y, and Z are not pairwise independent either.

7. **(a)** The marginal probability distribution functions of X, Y, and Z are, respectively, given by

$$F_X(x) = F(x, \infty, \infty) = 1 - e^{-\lambda_1 x}, \quad x > 0,$$
$$F_Y(y) = F(\infty, y, \infty) = 1 - e^{-\lambda_2 y}, \quad y > 0,$$
$$F_Z(z) = F(\infty, \infty, z) = 1 - e^{-\lambda_3 z}, \quad z > 0.$$

Since $F(x, y, z) = F_X(x)F_Y(y)F_Z(z)$, the random variables X, Y, and Z are independent.

(b) From part (a) it is clear that X, Y, and Z are independent exponential random variables with parameters λ_1, λ_2, and λ_3, respectively. Hence their joint probability density functions is given by

$$f(x, y, z) = \lambda_1\lambda_2\lambda_3 e^{-\lambda_1 x - \lambda_2 y - \lambda_3 z}.$$

(c) The desired probability is calculated as follows:

$$\begin{aligned} P(X < Y < Z) &= \int_0^\infty \int_x^\infty \int_y^\infty f(x, y, z)\,dz\,dy\,dx \\ &= \lambda_1\lambda_2\lambda_3 \int_0^\infty e^{-\lambda_1 x}\Big[\int_x^\infty e^{-\lambda_2 y}\Big(\int_y^\infty e^{-\lambda_3 z}\,dz\Big)\,dy\Big]\,dx \\ &= \frac{\lambda_1\lambda_2}{(\lambda_2+\lambda_3)(\lambda_1+\lambda_2+\lambda_3)}. \end{aligned}$$

8. **(a)** Clearly $f(x, y, z) \ge 0$ for the given domain. Since

$$\int_0^1 \Big[\int_0^x \Big(\int_0^y -\frac{\ln x}{xy}\,dz\Big)\,dy\Big]\,dx = 1,$$

f is a joint probability density function.

(b) $$f_{X,Y}(x, y) = \int_0^y -\frac{\ln x}{xy}\,dz = -\frac{\ln x}{x}, \quad 0 \le y \le x \le 1.$$
$$f_Y(y) = \int_y^1 \Big(\int_0^y -\frac{\ln x}{xy}\,dz\Big)\,dx = \frac{1}{2}(\ln y)^2, \quad 0 \le y \le 1.$$

9. For $1 \le i \le n$, let X_i be the distance of the ith point selected at random from the origin. For $r < R$, the desired probability is

$$\begin{aligned} P(X_1 \ge r, X_2 \ge r, \ldots, X_n \ge r) &= P(X_1 \ge r)P(X_2 \ge r)\cdots P(X_n \ge r) \\ &= \Big(\frac{\pi R^2 - \pi r^2}{\pi R^2}\Big)^n = \Big(1 - \frac{r^2}{R^2}\Big)^n. \end{aligned}$$

For $r \ge R$, the desired probability is 0.

10. The sphere inscribed in the cube has radius a and is centered at the origin. Hence the desired probability is $\big[(4/3)\pi a^3\big]/(8a^3) = \pi/6$.

11. Yes, it is because $f \geq 0$ and

$$\int_0^\infty \int_{x_1}^\infty \int_{x_2}^\infty \cdots \int_{x_{n-1}}^\infty e^{-x_n}\,dx_n\,dx_{n-1}\,\cdots\,dx_1$$
$$= \int_0^\infty \int_{x_1}^\infty \int_{x_2}^\infty \cdots \int_{x_{n-2}}^\infty e^{-x_{n-1}}\,dx_{n-1}\,\cdots\,dx_1$$
$$= \cdots = \int_0^\infty \int_{x_1}^\infty e^{-x_2}\,dx_2\,dx_1 = \int_0^\infty e^{-x_1}\,dx_1 = 1.$$

12. Let $f(x_1, x_2, x_3)$ be the joint probability density function of X_1, X_2, and X_3, the lifetimes of the original, the second, and the third transistors, respectively. We have that

$$f(x_1, x_2, x_3) = \frac{1}{5}e^{-x_1/5} \cdot \frac{1}{5}e^{-x_2/5} \cdot \frac{1}{5}e^{-x_3/5} = \frac{1}{125}e^{-(x_1+x_2+x_3)/5}.$$

Now

$$P(X_1 + X_2 + X_3 < 15) = \int_0^{15} \int_0^{15-x_1} \int_0^{15-x_1-x_2} \frac{1}{125}e^{-(x_1+x_2+x_3)/5}\,dx_3\,dx_2\,dx_1$$
$$= \int_0^{15} \int_0^{15-x_1} \left[\frac{1}{25}e^{-(x_1+x_2)/5} - \frac{1}{25}e^{-3}\right] dx_2\,dx_1$$
$$= \int_0^{15} \left(\frac{1}{5}e^{-x_1/5} - \frac{4}{5}e^{-3} + \frac{1}{25}e^{-3}x_1\right) dx_1$$
$$= 1 - \frac{17}{2}e^{-3} = 0.5768.$$

Therefore, the desired probability is $P(X_1 + X_2 + X_3 \geq 15) = 1 - 0.5768 = 0.4232$.

13. Let F be the distribution function of X. We have that

$$\begin{aligned} F(t) &= P(X \leq t) = 1 - P(X > t) = 1 - P(X_1 > t, X_2 > t, \ldots, X_n > t) \\ &= 1 - P(X_1 > t)P(X_2 > t)\cdots P(X_n > t) = 1 - e^{-\lambda_1 t}e^{-\lambda_2 t}\cdots e^{-\lambda_n t} \\ &= 1 - e^{-(\lambda_1+\lambda_2+\cdots+\lambda_n)t}, \quad t > 0. \end{aligned}$$

Thus X is exponential with parameter $\lambda_1 + \lambda_2 + \cdots + \lambda_n$.

14. Let Y be the number of functioning components of the system. The random variable Y is binomial with parameters n and p. The reliability of this system is given by

$$r = P(X = 1) = P(Y \geq k) = \sum_{i=k}^{n} \binom{n}{i} p^i (1-p)^{n-i}.$$

15. Let X_i be the lifetime of the ith part. The time until the item fails is the random variable $\min(X_1, X_2, \ldots, X_n)$ which by the solution to Exercise 13 is exponentially distributed with parameter $n\lambda$. Thus the average life of the item is $1/(n\lambda)$.

16. Let $X_1, X_2, \ldots$ be the lifetimes of the transistors selected at random. Clearly,

$$N = \min\{n\colon X_n > s\}.$$

Note that

$$P(X_N \le t \mid N = n) = P(X_n \le t \mid X_1 \le s,\ X_2 \le s,\ \ldots,\ X_{n-1} \le s,\ X_n > s).$$

This shows that for $s \ge t$, $P(X_N \le t \mid N = n) = 0$. For $s < t$,

$$\begin{aligned}
P(X_N \le t \mid N = n) &= \frac{P(s < X_n \le t,\ X_1 \le s,\ X_2 \le s,\ \ldots,\ X_{n-1} \le s)}{P(X_1 \le s,\ X_2 \le s,\ \ldots,\ X_{n-1} \le s,\ X_n > s)} \\
&= \frac{P(s < X_n \le t)P(X_1 \le s)P(X_2 \le s)\cdots P(X_{n-1} \le s)}{P(X_1 \le s)P(X_2 \le s)\cdots P(X_{n-1} \le s)P(X_n > s)} \\
&= \frac{P(s < X_n \le t)}{P(X_n > s)} = \frac{F(t) - F(s)}{1 - F(s)}.
\end{aligned}$$

This relation shows that the probability distribution function of X_N given $N = n$ does not depend on n. Therefore, X_N and N are independent.

17. Clearly,

$$\begin{aligned}
X &= X_1\Big[1 - (1 - X_2)(1 - X_3)\Big]\Big[1 - (1 - X_4)(1 - X_5X_6)\Big]X_7 \\
&= X_1X_7\big(X_2X_4 + X_3X_4 - X_2X_3X_4 + X_2X_5X_6 + X_3X_5X_6 \\
&\quad - X_2X_3X_5X_6 - X_2X_4X_5X_6 - X_3X_4X_5X_6 + X_2X_3X_4X_5X_6\big).
\end{aligned}$$

The reliability of this system is

$$\begin{aligned}
r &= p_1p_7\big(p_2p_4 + p_3p_4 - p_2p_3p_4 + p_2p_5p_6 + p_3p_5p_6 \\
&\quad - p_2p_3p_5p_6 - p_2p_4p_5p_6 - p_3p_4p_5p_6 + p_2p_3p_4p_5p_6\big).
\end{aligned}$$

18. Let G and F be the distribution functions of $\max_{1\le i\le n} X_i$ and $\min_{1\le i\le n} X_i$, respectively. Let g and f be their probability density functions, respectively. For $0 \le t < 1$,

$$\begin{aligned}
G(t) &= P(X_1 \le t, X_2 \le t, \ldots, X_n \le t) \\
&= P(X_1 \le t)P(X_2 \le t)\cdots P(X_n \le t) = t^n.
\end{aligned}$$

So

$$G(t) = \begin{cases} 0 & t < 0 \\ t^n & 0 \le t < 1 \\ 1 & t \ge 1. \end{cases}$$

Therefore,

$$g(t) = G'(t) = \begin{cases} nt^{n-1} & 0 < t < 1 \\ 0 & \text{elsewhere.} \end{cases}$$

This gives

$$E\big(\max_{1 \le i \le n} X_i\big) = \int_0^1 nt^n\,dt = \frac{n}{n+1}.$$

Similarly, for $0 \le t < 1$,

$$\begin{aligned} F(t) = P\Big(\min_{1 \le i \le n} X_i \le t\Big) &= 1 - P\Big(\min_{1 \le i \le n} X_i > t\Big) \\ &= 1 - P(X_1 > t)P(X_2 > t) \cdots P(X_n > t) \\ &= 1 - (1-t)^n, \quad 0 \le t < 1. \end{aligned}$$

Hence

$$F(t) = \begin{cases} 0 & t < 0 \\ 1 - (1-t)^n & 0 \le t < 1 \\ 1 & t \ge 1, \end{cases}$$

and

$$f(t) = \begin{cases} n(1-t)^{n-1} & 0 < t < 1 \\ 0 & \text{otherwise.} \end{cases}$$

So

$$E\Big(\min_{1 \le i \le n} X_i\Big) = \int_0^1 nt(1-t)^{n-1}\,dt = \frac{1}{n+1}.$$

19. We have that

$$\begin{aligned} P\big(\max(X_1, X_2, \dots, X_n) \le t\big) &= P(X_1 \le t, X_2 \le t, \dots, X_n \le t) \\ &= P(X_1 \le t)P(X_2 \le t) \cdots P(X_n \le t) \\ &= \big[F(t)\big]^n, \end{aligned}$$

and

$$\begin{aligned} P\big(\min(X_1, X_2, \dots, X_n) \le t\big) &= 1 - P\big(\min(X_1, X_2, \dots, X_n) > t\big) \\ &= 1 - P(X_1 > t, X_2 > t, \dots, X_n > t) \\ &= 1 - P(X_1 > t)P(X_2 > t) \cdots P(X_n > t) \\ &= 1 - \big[1 - F(t)\big]^n. \end{aligned}$$

20. We have that

$$\begin{aligned}P(Y_n > x) &= P\Big(\min(X_1, X_2, \dots, X_n) > \frac{x}{n}\Big)\\ &= P\Big(X_1 > \frac{x}{n},\ X_2 > \frac{x}{n},\ \dots,\ X_n > \frac{x}{n}\Big)\\ &= P\Big(X_1 > \frac{x}{n}\Big) P\Big(X_2 > \frac{x}{n}\Big) \cdots P\Big(X_n > \frac{x}{n}\Big)\\ &= \Big(1 - \frac{x}{n}\Big)^n.\end{aligned}$$

Thus

$$\lim_{n\to\infty} P(Y_n > x) = \lim_{n\to\infty}\Big(1 - \frac{x}{n}\Big)^n = e^{-x}, \quad x > 0.$$

21. We have that

$$\begin{aligned}P(X < Y < Z) &= \int_{-\infty}^{\infty}\int_x^{\infty}\int_y^{\infty} h(x)h(y)h(z)\,dz\,dy\,dx\\ &= \int_{-\infty}^{\infty}\int_x^{\infty} h(x)h(y)\big[1 - H(y)\big]\,dy\,dx\\ &= \int_{-\infty}^{\infty} h(x)\Big[-\frac{1}{2}\big[1 - H(y)\big]^2\Big]_x^{\infty}\,dx\\ &= \frac{1}{2}\int_{-\infty}^{\infty} h(x)\big[1 - H(x)\big]^2\,dx\\ &= \frac{1}{2}\Big[-\frac{1}{3}\big[1 - H(x)\big]^3\Big]_{-\infty}^{\infty} = \frac{1}{6}.\end{aligned}$$

22. Noting that $X_i^2 = X_i$, $1 \le i \le 5$, we have

$$\begin{aligned}X &= \max\{X_2X_5, X_2X_3X_4, X_1X_4, X_1X_3X_5\}\\ &= 1 - (1 - X_2X_5)(1 - X_2X_3X_4)(1 - X_1X_4)(1 - X_1X_3X_5)\\ &= X_2X_5 + X_1X_4 + X_1X_3X_5 + X_2X_3X_4 - X_1X_2X_3X_4 - X_1X_2X_3X_5\\ &\quad - X_1X_2X_4X_5 - X_1X_3X_4X_5 - X_2X_3X_4X_5 + 2X_1X_2X_3X_4X_5.\end{aligned}$$

Therefore, whenever the system is turned on for water to flow from A to B, water reaches B with probability r given by,

$$\begin{aligned}r = P(X = 1) = E(X) &= p_2p_5 + p_1p_4 + p_1p_3p_5 + p_2p_3p_4 - p_1p_2p_3p_4\\ &\quad - p_1p_2p_3p_5 - p_1p_2p_4p_5 - p_1p_3p_4p_5 - p_2p_3p_4p_5 + 2p_1p_2p_3p_4p_5.\end{aligned}$$

23. Clearly, $B = (1 \times 1)/2$ and $h = 1$. So the volume of the pyramid is $(1/3)Bh = 1/6$. Therefore, the joint probability density function of X, Y, and Z is

$$f(x, y, z) = \begin{cases} 6 & (x, y, z) \in V\\ 0 & \text{otherwise.}\end{cases}$$

Thus

$$f_X(x) = \int_0^{1-x} \left(\int_0^{1-x-y} 6\,dz \right) dy = 3(1-x)^2, \quad 0 < x < 1.$$

Similarly, $f_Y(y) = 3(1-y)^2,\ \ 0 < y < 1$, and $f_Z(z) = 3(1-z)^2,\ \ 0 < z < 1$. Since

$$f(x, y, z) \neq f_X(x) f_Y(y) f_Z(z),$$

X, Y, and Z are not independent.

24. The probability that $Ax^2+Bx+C = 0$ has real roots is equal to the probability that $B^2-4AC \geq 0$. To calculate this quantity, we will first evaluate the distribution functions of B^2 and $-4AC$ and then use the convolution theorem to find the distribution function of $B^2 - 4AC$.

$$F_{B^2}(t) = P(B^2 \leq t) = \begin{cases} 0 & \text{if } t < 0 \\ \sqrt{t} & \text{if } 0 \leq t < 1 \\ 1 & \text{if } t \geq 1, \end{cases}$$

$$f_{B^2}(t) = F'_{B^2}(t) = \begin{cases} \dfrac{1}{2\sqrt{t}} & \text{if } 0 < t < 1 \\ 0 & \text{otherwise,} \end{cases}$$

and

$$F_{-4AC}(t) = P(-4AC \leq t) = \begin{cases} 0 & \text{if } t < -4 \\ P\Big(AC \geq -\dfrac{t}{4}\Big) & \text{if } -4 \leq t < 0 \\ 1 & \text{if } t \geq 0. \end{cases}$$

Now A and C are random numbers from $(0, 1)$; hence (A, C) is a random point from the square $(0, 1) \times (0, 1)$ in the ac-plane. Therefore, $P(AC \geq -t/4) = P\big(C \geq -t/(4A)\big)$ is the area of the shaded region $\Big($bounded by $a = 1$, $c = 1$, $c = -\dfrac{t}{4a}\Big)$ of Figure 1.

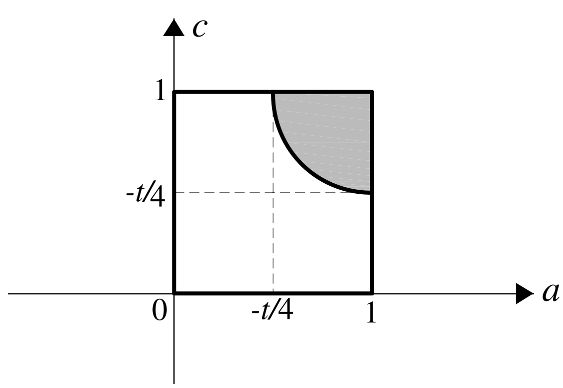

Figure 1 The shaded region of Exercise 24.

Thus, for $-4 \le t < 0$,

$$F_{-4AC}(t) = \int_{-t/4}^{1} \Big(\int_{-t/(4a)}^{1} dc \Big) da = 1 + \frac{t}{4} - \frac{t}{4} \ln\Big(-\frac{t}{4}\Big).$$

Therefore,

$$F_{-4AC}(t) = P(-4AC \le t) = \begin{cases} 0 & \text{if } t < -4 \\ 1 + \dfrac{t}{4} - \dfrac{t}{4} \ln\Big(-\dfrac{t}{4}\Big) & \text{if } -4 \le t < 0 \\ 1 & \text{if } t > 0. \end{cases}$$

Applying convolution theorem, we obtain

$$\begin{aligned} P\big(B^2 - 4AC \ge 0\big) &= 1 - P\big(B^2 - 4AC < 0\big) \\ &= 1 - \int_{-\infty}^{\infty} F_{-4AC}(0 - x) f_{B^2}(x) dx \\ &= 1 - \int_0^1 \Big(1 - \frac{x}{4} + \frac{x}{4} \ln \frac{x}{4}\Big) \frac{1}{2\sqrt{x}} dx. \end{aligned}$$

Letting $y = \sqrt{x}/2$, we get $dy = \dfrac{1}{4\sqrt{x}} dx$. So

$$\begin{aligned} P\big(B^2 - 4AC \ge 0\big) &= 1 - \int_0^{1/2} (1 - y^2 + y^2 \ln y^2) 2dy \\ &= 1 - \int_0^{1/2} 2dy + 2\int_0^{1/2} (y^2 - y^2 \ln y^2) dy \\ &= 2\int_0^{1/2} (y^2 - y^2 \ln y^2) dy. \end{aligned}$$

Now by integration by parts ($u = \ln y^2$, $dv = y^2 dy$),

$$\int y^2 \ln y^2 dy = \frac{1}{3} y^3 \ln y^2 - \frac{2}{9} y^3.$$

Thus

$$P\big(B^2 - 4AC \ge 0\big) = \Big[\frac{10}{9} y^3 - \frac{2}{3} y^3 \ln y^2\Big]_0^{1/2} = \frac{5}{36} + \frac{1}{6} \ln 2 \approx 0.25.$$

25. The following solution by Scott Harrington, Duke University, Durham, NC, was given in *The College Mathematics Journal*, September 1993.

> Let V be the set of points $(A, B, C) \in [0, 1]^3$ such that $f(x) = x^3 + Ax^2 + Bx + C = 0$ has all real roots. The probability that all of the roots are real is the volume of V.

The function is cubic, so it either has one real root and two complex roots or three real roots. Since the coefficient of x^3 is positive, $\lim_{x\to-\infty} f(x) = -\infty$ and $\lim_{x\to+\infty} f(x) = +\infty$. The number of real roots of the graph of $f(x)$ depends on the nature of the critical points of the function f.

$$f'(x) = 3x^2 + 2Ax + B = 0,$$

with roots

$$x = -\frac{1}{3}A \pm \frac{1}{3}\sqrt{A^2 - 3B}.$$

Let $D = \sqrt{A^2 - 3B}$, $x_1 = -\frac{1}{3}(A + D)$, and $x_2 = -\frac{1}{3}(A - D)$. If $A^2 < 3B$ then the critical points are imaginary, so the graph of $f(x)$ is strictly increasing and there must be exactly one real root. Thus we may assume $A^2 \geq 3B$.

In order for there to be three real roots, counting multiplicities, the local maximum $\big(x_1, f(x_1)\big)$ and local minimum $\big(x_2, f(x_2)\big)$ must satisfy $f(x_1) \geq 0$ and $f(x_2) \leq 0$; that is,

$$\begin{aligned} f(x_1) &= -\frac{1}{27}(A^3 + 3A^2D + 3AD^2 + D^3) \\ &\quad + \frac{1}{9}A(A^2 + 2AD + D^2) - \frac{1}{3}B(A + D) + C \geq 0, \\ f(x_2) &= -\frac{1}{27}(A^3 - 3A^2D + 3AD^2 - D^3) \\ &\quad + \frac{1}{9}A(A^2 - 2AD + D^2) - \frac{1}{3}B(A - D) + C \leq 0. \end{aligned}$$

Simplifying produces two half-spaces:

$$C \geq \frac{1}{27}\Big(-2A^3 + 9AB - 2(A^2 - 3B)^{3/2}\Big), \quad \text{(constraint surface 1);}$$

$$C \leq \frac{1}{27}\Big(-2A^3 + 9AB + 2(A^2 - 3B)^{3/2}\Big), \quad \text{(constraint surface 2).}$$

These two surfaces intersect at the curve given parametrically by $A = t$, $B = \frac{1}{3}t^2$ and $C = \frac{1}{27}t^3$. Note that all points in the intersection of these two half-spaces satisfy $B \leq \frac{1}{3}A^2$. Surface 2 intersects the plane $C = 0$ at the A-axis, but surface 1 intersects the plane $C = 0$ at the curve $B = \frac{1}{4}A^2$, which is a quadratic curve in the plane $C = 0$ located between the A-axis and the upper limit $B = \frac{1}{3}A^2$. Therefore, V is the region above the plane $C = 0$ and constraint surface 1, and below constraint surface 2. The volume of V is the volume V_2 under surface 2 minus the volume V_1 under surface 1. Now

$$V_1 = \int_{a=0}^{1}\int_{b=(1/4)a^2}^{(1/3)a^2} \frac{1}{27}\Big(-2a^3 + 9ab - 2(a^2 - 3b)^{3/2}\Big)\,db\,da$$

$$= \int_0^1 \frac{1}{27}\Big[-2a^3b + \frac{9}{2}ab^2 + \frac{4}{15}(a^2-3b)^{5/2}\Big]_{b=(1/4)a^2}^{(1/3)a^2} da$$

$$= \int_0^1 \frac{1}{27}\cdot\frac{7}{160}a^5\,da = \frac{7}{25,920}, \quad \text{and}$$

$$V_2 = \int_{a=0}^1 \int_{b=0}^{(1/3)a^2} \frac{1}{27}\Big(-2a^3 + 9ab + 2(a^2-3b)^{3/2}\Big)\,db\,da$$

$$= \int_0^1 \frac{1}{27}\Big[-2a^3b + \frac{9}{2}ab^2 - \frac{4}{15}(a^2-3b)^{5/2}\Big]_{b=0}^{(1/3)a^2} da = \int_0^1 \frac{1}{270}a^5\,da = \frac{1}{1620}.$$

Thus

$$V = V_2 - V_1 = \frac{1}{1,620} - \frac{7}{25,920} = \frac{1}{2,880}.$$

9.2 ORDER STATISTICS

1. By Theorem 9.5, we have that

$$f_3(x) = \frac{4!}{2!\,1!} f(x)\big[F(x)\big]^2\big[1-F(x)\big],$$

where

$$f(x) = \begin{cases} 1 & 0 < x < 1 \\ 0 & \text{otherwise,} \end{cases}$$

and

$$F(x) = \begin{cases} 0 & x < 0 \\ x & 0 \le x < 1 \\ 1 & x \ge 1. \end{cases}$$

Therefore,

$$f_3(x) = 12x^2(1-x), \quad 0 < x < 1.$$

Hence the desired probability is

$$\int_{1/4}^{1/2} 12x^2(1-x)\,dx = \frac{67}{256} = 0.26172.$$

2. Let X_1 and X_2 be the points selected at random. By Theorem 9.6, the joint probability density function of $X_{(1)}$ and $X_{(2)}$ is given by

$$f_{12}(x,y) = \frac{2!}{(1-1)!\,(2-1-1)!\,(2-2)!}\,x^{1-1}(y-x)^{2-1-1}, \quad 0 < x < y < 1.$$

So

$$f_{12}(x, y) = 2, \quad 0 < x < y < 1.$$

We have that, the desired probability is given by

$$P\big(X_{(2)} \geq 3X_{(1)}\big) = \int_0^1 \int_0^{y/3} 2\,dx\,dy = \frac{1}{3}.$$

3. By Theorem 9.5, $f_4(x)$, the probability density function of $X_{(4)}$ is given by

$$f_4(x) = \frac{4!}{3!\,0!}\lambda e^{-\lambda x}\big(1 - e^{-\lambda x}\big)^3\big(e^{-\lambda x}\big)^{4-4} = 4\lambda e^{-\lambda x}\big(1 - e^{-\lambda x}\big)^3.$$

The desired probability is

$$\int_{3\lambda}^{\infty} 4\lambda e^{-\lambda x}\big(1 - e^{-\lambda x}\big)^3\,dx = 1 - \big(1 - e^{-3\lambda^2}\big)^4.$$

4. By Remark 6.4,

$$E\big[X_{(n)}\big] = \int_0^{\infty} P\big(X_{(n)} > x\big)\,dx.$$

Now

$$\begin{aligned} P\big(X_{(n)} > x\big) &= 1 - P\big(X_{(n)} \leq x\big) \\ &= 1 - P(X_1 \leq x, X_2 \leq x, \dots, X_n \leq x) = 1 - \big[F(x)\big]^n. \end{aligned}$$

So

$$E\big[X_{(n)}\big] = \int_0^{\infty} \Big(1 - \big[F(x)\big]^n\Big)\,dx.$$

5. To find $P\big(X_{(i)} = k\big)$, $0 \leq k \leq n$, note that

$$P\big(X_{(i)} = k\big) = 1 - P\big(X_{(i)} < k\big) - P\big(X_{(i)} > k\big).$$

Let N be the number of X_j's that are less than k. Then N is a binomial random variable with parameters m and

$$p_1 = \sum_{l=0}^{k-1} \binom{n}{l} p^l (1-p)^{n-l}. \tag{35}$$

Let L be the number of X_j's that are greater than k. Then L is a binomial random variable with parameters m and

$$p_2 = \sum_{l=k+1}^{n} \binom{n}{l} p^l (1-p)^{n-l}. \tag{36}$$

Clearly,

$$P\big(X_{(i)} < k\big) = P(N \geq i) = \sum_{j=i}^{m} \binom{m}{j} p_1^j (1-p_1)^{m-j},$$

and

$$P\big(X_{(i)} > k\big) = P(L \geq m-i+1) = \sum_{j=m-i+1}^{m} \binom{m}{j} p_2^j (1-p_2)^{m-j}.$$

Thus, for $0 \leq k \leq n$,

$$P\big(X_{(i)} = k\big) = 1 - \sum_{j=i}^{m} \binom{m}{j} p_1^j (1-p_1)^{m-j} - \sum_{j=m-i+1}^{m} \binom{m}{j} p_2^j (1-p_2)^{m-j},$$

where p_1 and p_2 are given by (35) and (36).

6. By Theorem 9.6, the joint probability density function of $X_{(1)}$ and $X_{(n)}$ is given by

$$f_{1n}(x, y) = n(n-1)f(x)f(y)\big[F(y) - F(x)\big]^{n-2}, \quad x < y.$$

Therefore,

$$\begin{aligned} G(t) &= P\bigg(\frac{X_{(1)} + X_{(n)}}{2} \leq t\bigg) = P\big(X_{(1)} + X_{(n)} \leq 2t\big) \\ &= \int_{-\infty}^{t} \int_{x}^{2t-x} n(n-1)f(x)f(y)\big[F(y) - F(x)\big]^{n-2}\,dy\,dx \\ &= n \int_{-\infty}^{t} \big[F(2t-x) - F(x)\big]^{n-1} f(x)\,dx. \end{aligned}$$

7. By Theorem 9.5, $f_1(x)$, the probability density function of $X_{(1)}$ is given by

$$f_1(x) = \frac{2!}{(1-1)!\,(2-1)!} \lambda e^{-\lambda x} \big(1 - e^{-\lambda x}\big)^{1-1} \big(e^{-\lambda x}\big)^{2-1} = 2\lambda e^{-2\lambda x}, \quad x \geq 0.$$

By Theorem 9.6, $f_{12}(x, y)$, the joint probability density function of $X_{(1)}$ and $X_{(2)}$ is given by

$$\begin{aligned} f_{12}(x, y) &= \frac{2!}{(1-1)!\,(2-1-1)!\,(2-2)!} \lambda e^{-\lambda x} \lambda e^{-\lambda y} \\ &= \big(1 - e^{-\lambda x}\big)^{1-1} \big(e^{-\lambda x} - e^{-\lambda y}\big)^{2-1-1} = 2\lambda^2 e^{-\lambda(x+y)}, \quad 0 \leq x < y < \infty. \end{aligned}$$

Let $U = X_{(1)}$ and $V = X_{(2)} - X_{(1)}$. We will show that $g(u, v)$, the joint probability density function of U and V satisfy $g(u, v) = g_U(u) g_V(v)$. This proves that U and V are independent. To find $g(u, v)$, note that the system of two equations in two unknowns

$$\begin{cases} x = u \\ y - x = v \end{cases}$$

defines a one-to-one transformation of

$$R = \{(x, y) : 0 \le x < y < \infty\}$$

onto the region

$$Q = \{(u, v) : u \ge 0,\ v > 0\}.$$

It has the unique solution $x = u$, $y = u + v$. Hence

$$\mathbf{J} = \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = 1 \neq 0.$$

By Thereom 8.8,

$$g(u, v) = f_{12}(u, u + v)|\mathbf{J}| = 2\lambda^2 e^{-\lambda(u+2v)}, \quad u \ge 0,\ v > 0.$$

Since

$$g(u, v) = g_U(u)g_V(v),$$

where

$$g_U(u) = 2\lambda e^{-2\lambda u}, \quad u \ge 0,$$

and

$$g_V(v) = \lambda e^{-\lambda v}, \quad v > 0,$$

we have that U and V are independent. Furthermore, U is exponential with parameter 2λ and V is exponential with parameter λ.

8. Let $f_{12}(x, y)$ be the joint probability density function of $X_{(1)}$ and $X(2)$. By Theorem 9.6,

$$\begin{aligned} f_{12}(x, y) &= 2!\, f(x) f(y) = 2 \cdot \frac{1}{\sigma\sqrt{2\pi}} e^{-x^2/2\sigma^2} \cdot \frac{1}{\sigma\sqrt{2\pi}} e^{-y^2/2\sigma^2} \\ &= \frac{1}{\sigma^2\pi} e^{-x^2/2\sigma^2} \cdot e^{-y^2/2\sigma^2}, \quad -\infty < x < y < \infty. \end{aligned}$$

Therefore,

$$\begin{aligned} E\big[X_{(1)}\big] &= \int_{-\infty}^{\infty} \int_{-\infty}^{y} x \cdot \frac{1}{\sigma^2\pi} e^{-x^2/2\sigma^2} \cdot e^{-y^2/2\sigma^2}\, dx\, dy \\ &= \frac{1}{\sigma^2\pi} \int_{-\infty}^{\infty} e^{-y^2/2\sigma^2} \left(\int_{-\infty}^{y} x e^{-x^2/2\sigma^2}\, dx \right) dy \\ &= \frac{1}{\sigma^2\pi} \int_{-\infty}^{\infty} e^{-y^2/2\sigma^2} \cdot (-\sigma^2) e^{-y^2/2\sigma^2}\, dy \\ &= -\frac{1}{\pi} \int_{-\infty}^{\infty} e^{-y^2/\sigma^2}\, dy \end{aligned}$$

$$= -\frac{1}{\pi} \cdot \sigma\sqrt{\pi} \cdot \frac{1}{\dfrac{\sigma}{\sqrt{2}} \cdot \sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\dfrac{y^2}{2(\sigma/\sqrt{2})^2}}\, dy$$

$$= -\frac{1}{\pi} \cdot \sigma\sqrt{\pi} \cdot 1 = -\frac{\sigma}{\sqrt{\pi}}.$$

9. (a) By Theorem 9.6, the joint probability density function of $X_{(1)}$ and $X_{(n)}$ is given by

$$f_{1n}(x, y) = \begin{cases} n(n-1)f(x)f(y)\big[F(y) - F(x)\big]^{n-2} & x < y \\ 0 & \text{elsewhere.} \end{cases}$$

We will use this to find $g(r, v)$, the joint probability density function of $R = X_{(n)} - X_{(1)}$ and $V = X_{(n)}$. The probability density function of the sample range, R, is then the marginal probability density function of R. That is,

$$g_R(r) = \int_{-\infty}^{\infty} g(r, v)dv.$$

To find $g(r, v)$, we will use Theorem 8.8. The system of two equations in two unknowns

$$\begin{cases} y - x = r \\ y = v \end{cases}$$

defines a one-to-one transformation of

$$\big\{(x, y) :\ -\infty < x < y < \infty\big\}$$

onto the region

$$\big\{(r, v) :\ -\infty < v < \infty,\ r > 0\big\}.$$

It has the unique solution $x = v - r$, $y = v$. Hence

$$\mathbf{J} = \begin{vmatrix} -1 & 1 \\ 0 & 1 \end{vmatrix} = -1 \neq 0.$$

By Theorem 8.8, $g(u, v)$ is given by

$$\begin{aligned} g(r, v) &= f_{1n}(v - r, v)|\mathbf{J}| \\ &= n(n-1)f(v-r)f(v)\big[F(v) - F(v-r)\big]^{n-2}, \quad -\infty < v < \infty,\ r > 0. \end{aligned}$$

This implies

$$g_R(r) = \int_{-\infty}^{\infty} n(n-1)f(v-r)f(v)\big[F(v) - F(v-r)\big]^{n-2}\, dv, \quad r > 0. \tag{37}$$

(b) The probability density function of n random numbers from (0, 1) is obtained by letting

$$f(v) = \begin{cases} 1 & 0 < v < 1 \\ 0 & \text{otherwise,} \end{cases}$$

and $F(v) - F(v - r) = v - (v - r) = r$ in (37). Note that the integrand of the integral in (37) is nonzero if $0 < v < 1$ and if $0 < v - r < 1$; that is, if $0 < r < v < 1$. Therefore,

$$g_R(r) = \int_r^1 n(n-1)r^{n-2}\,dv = n(n-1)r^{n-2}(1-r), \quad 0 < r < 1.$$

10. Let f and F be the probability density and distribution functions of X_i, $1 \le i \le n$, respectively. We have that

$$f(x) = \begin{cases} 1/\theta & 0 < x < \theta \\ 0 & \text{elsewhere} \end{cases}$$

and

$$F(x) = \begin{cases} 0 & x < 0 \\ x/\theta & 0 \le x < \theta \\ 1 & x \ge \theta. \end{cases}$$

Let $g(r)$ be the probability density function of $R = X_{(n)} - X_{(1)}$. By part (a) of Exercise 9,

$$g(r) = \int_r^\theta n(n-1)\Big(\frac{v}{\theta} - \frac{v-r}{\theta}\Big)^{n-2}\,dv = \frac{n(n-1)r^{n-2}}{\theta^n}(\theta - r) \quad 0 < r < \theta.$$

(Note that $0 < v < \theta$ and $0 < v - r < \theta$ imply that $r < v < \theta$.) Therefore,

$$E(R) = \int_0^\infty r\frac{n(n-1)r^{n-2}}{\theta^n}(\theta - r)\,dr = \frac{n-1}{n+1}\theta.$$

9.3 MULTINOMIAL DISTRIBUTIONS

1. The desired probability is

$$\frac{8!}{3!\,2!\,3!}\left(\frac{150}{800}\right)^3\left(\frac{400}{800}\right)^2\left(\frac{250}{800}\right)^3 = 0.028.$$

2. We have that

$$P(B = i,\ R = j,\ G = 20 - i - j)$$
$$= \frac{20!}{i!\,j!\,(20-i-j)!}(0.2)^i(0.3)^j(0.5)^{20-i}, \quad 0 \le i,\ j \le 20, \quad i + j \le 20.$$

3. Let U, D, and S be the number of days among the next six days that the stock market moves up, moves down, and remains the same, respectively. The desired probability is

$$\begin{aligned}
&P(U=0,\ D=0,\ S=6)+P(U=1,\ D=1,\ S=4)\\
&\quad+P(U=2,\ D=2,\ S=2)+P(U=3,\ D=3,\ S=0)\\
&\quad=\frac{6!}{0!\,0!\,6!}\Big(\frac{1}{4}\Big)^0\Big(\frac{5}{12}\Big)^0\Big(\frac{1}{3}\Big)^6+\frac{6!}{1!\,1!\,4!}\Big(\frac{1}{4}\Big)^1\Big(\frac{5}{12}\Big)^1\Big(\frac{1}{3}\Big)^4\\
&\quad+\frac{6!}{2!\,2!\,2!}\Big(\frac{1}{4}\Big)^2\Big(\frac{5}{12}\Big)^2\Big(\frac{1}{3}\Big)^2+\frac{6!}{3!\,3!\,0!}\Big(\frac{1}{4}\Big)^3\Big(\frac{5}{12}\Big)^3\Big(\frac{1}{3}\Big)^0=0.171.
\end{aligned}$$

4. Let A, B, C, D, and F be the number of students who get A, B, C, D, and F, respectively. The desired probability is given by

$$\begin{aligned}
&P(A=2,\ B=5,\ C=5,\ D=2,\ F=1)+P(A=3,\ B=5,\ C=5,\ D=2,\ F=0)\\
&\quad=\frac{15!}{2!\,5!\,5!\,2!\,1!}(0.16)^2(0.34)^5(0.34)^5(0.14)^2(0.02)^1\\
&\qquad+\frac{15!}{3!\,5!\,5!\,2!\,0!}(0.16)^3(0.34)^5(0.34)^5(0.14)^2(0.02)^0\\
&\quad=0.0172.
\end{aligned}$$

5. Let L, M, and S be the number of large, medium, and small watermelons among the five watermelons Joanna buys, respectively.

(a) We have that

$$\begin{aligned}
P(L\geq 2)&=1-P(L=0)-P(L=1)\\
&=1-\binom{5}{0}(0.50)^0(0.50)^5-\binom{5}{1}(0.50)^1(0.50)^4=0.8125.
\end{aligned}$$

(b) $P(L=2,\ M=2,\ S=1)=\dfrac{5!}{2!\,2!\,1!}(0.5)^2(0.3)^2(0.2)^1=0.135.$

(c) Using parts (a) and (b) and

$$P(L=3,\ M=2,\ S=0)=\frac{5!}{3!\,2!\,0!}(0.5)^3(0.3)^2(0.2)^0=0.1125,$$

we have that

$$\begin{aligned}
P(M=2\mid L\geq 2)&=\frac{P(M=2,\ L\geq 2)}{P(L\geq 2)}\\
&=\frac{P(L=2,\ M=2,\ S=1)+P(L=3,\ M=2,\ S=0)}{P(L\geq 2)}\\
&=\frac{0.135+0.1125}{0.8125}=0.3046.
\end{aligned}$$

6. Let X be the number of faculty members who are below 40 and Y be the number of those who are above 50 in the committee. The desired probability mass function is

$$p_{X|Y}(x|2) = \frac{\dfrac{10!}{x!\,2!\,(8-x)!}(0.5)^x(0.3)^2(0.2)^{8-x}}{\dbinom{10}{2}(0.3)^2(0.7)^8} = \binom{8}{x}\left(\frac{5}{7}\right)^x\left(\frac{2}{7}\right)^{8-x}, \quad 0 \le x \le 8.$$

7. The probability is 1/4 that the blood type of a child of this man and woman is AB. The probability is 1/4 that it is A, and the probability is 1/2 that it is B. The desired probability is equal to

$$\frac{6!}{3!\,2!\,1!}\left(\frac{1}{2}\right)^3\left(\frac{1}{4}\right)^2\left(\frac{1}{4}\right)^1 = \frac{15}{128} = 0.117.$$

8. The probability of two AA's, two Aa's, and two aa's is

$$g(p) = \frac{6!}{2!\,2!\,2!}(p^2)^2\big[2p(1-p)\big]^2\big[(1-p)^2\big]^2 = 360p^6(1-p)^6.$$

To find the maximum of this function, set $g'(p) = 0$ to obtain $p = 1/2$.

9. Let $N(t)$ be the number of customers who arrive at the store by time t. We are given that $\big\{N(t)\colon t \ge 0\big\}$ is a Poisson process with $\lambda = 3$. Let X, Y, and Z be the number of customers who use charge cards, write personal checks, and pay cash in five operating minutes, respectively. Then

$$\begin{aligned}
&P(X=5,\ Y=2,\ Z=3)\\
&\quad = \sum_{n=10}^{\infty} P\big(X=5,\ Y=2,\ Z=3 \mid N(5)=n\big)P\big(N(5)=n\big)\\
&\quad = \sum_{n=10}^{\infty} \frac{n!}{5!\,2!\,3!\,(n-10)!}(0.40)^5(0.10)^2(0.20)^3(0.30)^{n-10}\cdot\frac{e^{-15}15^n}{n!}\\
&\quad = \frac{(0.40)^5(0.10)^2(0.20)^3e^{-15}15^{10}}{3!\,5!\,2!}\sum_{n=10}^{\infty}\frac{(0.30)^{n-10}(15)^{n-10}}{(n-10)!}\\
&\quad = (0.00010035)\sum_{n=10}^{\infty}\frac{(4.5)^{n-10}}{(n-10)!} = (0.00010035)e^{4.5} = 0.009033.
\end{aligned}$$

REVIEW PROBLEMS FOR CHAPTER 9

1. Let $p(b, r, g)$ be the joint probability mass function of B, R, and G. Then

$$p(b, r, g) = \frac{\binom{20}{b}\binom{30}{r}\binom{50}{g}}{\binom{100}{20}}, \quad b + r + g = 20,\ 0 \le b,\ r,\ g \le 20.$$

2. Let F be the distribution function of X. Let $X_1, X_2, \dots, X_n$ be the outcomes of the first, second, ..., and the nth rolls, respectively. Then $X = \min(X_1, X_2, \dots, X_n)$. Therefore,

$$F(t) = P(X \le t) = 1 - P(X > t) = 1 - P(X_1 > t,\ X_2 > t,\ \dots,\ X_n > t)$$

$$= 1 - \big[P(X_1 > t)\big]^n = \begin{cases} 0 & t < 1 \\ 1 - \left(\frac{5}{6}\right)^n & 1 \le t < 2 \\ 1 - \left(\frac{4}{6}\right)^n & 2 \le t < 3 \\ 1 - \left(\frac{3}{6}\right)^n & 3 \le t < 4 \\ 1 - \left(\frac{2}{6}\right)^n & 4 \le t < 5 \\ 1 - \left(\frac{1}{6}\right)^n & 5 \le t < 6 \\ 1 & t \ge 6. \end{cases}$$

The probability mass function of X is

$$p(x) = P(X = x) = \left(\frac{7 - x}{6}\right)^n - \left(\frac{6 - x}{6}\right)^n, \quad x = 1, 2, 3, 4, 5, 6.$$

3. Let $D_1, D_2, \dots, D_n$ be the distances of the points selected from the origin. Let $D = \min(D_1, D_2, \dots, D_n)$. The desired probability is

$$P(D \ge r) = P(D_1 \ge r,\ D_2 \ge r,\ \dots,\ D_n \ge r) = \big[P(D_1 \ge r)\big]^n = \big[1 - P(D_1 < r)\big]^n$$

$$= \left[1 - \frac{(4/3)\pi r^3}{8a^3}\right]^n = \left[1 - \frac{\pi}{6}\left(\frac{r}{a}\right)^3\right]^n.$$

4. **(a)** $c\displaystyle\int_0^1\int_0^1\int_0^1 (x + y + 2z)\,dz\,dy\,dx = 1 \implies c = 1/2.$

(b) We have that

$$P\Big(X < \frac{1}{3} \,\Big|\, Y < \frac{1}{2},\ Z < \frac{1}{4}\Big) = \frac{P\Big(X < \frac{1}{3},\ Y < \frac{1}{2},\ Z < \frac{1}{4}\Big)}{P\Big(Y < \frac{1}{2},\ Z < \frac{1}{4}\Big)}$$

$$= \frac{\displaystyle\int_0^{1/3}\int_0^{1/2}\int_0^{1/4} \frac{1}{2}(x + y + 2z)\,dz\,dy\,dx}{\displaystyle\int_0^{1}\int_0^{1/2}\int_0^{1/4} \frac{1}{2}(x + y + 2z)\,dz\,dy\,dx} = \frac{1/36}{1/8} = \frac{2}{9}.$$

5. The joint probability mass function of the number of times each face appears is multinomial. Hence the desired probability is $\dfrac{18!}{(3!)^6}\left(\dfrac{1}{6}\right)^{18} = 0.00135.$

6. Using the multinomial distribution, the answer is

$$\frac{7!}{3!\,2!\,2!}(0.4)^3(0.35)^2(0.25)^2 = 0.1029.$$

7. For $1 \le i \le n$, let X_i be the lifetime of the ith component. Then $\min(X_1, X_2, \dots, X_n)$ is the lifetime of the system. Let $\bar{F}(t)$ be the survival function of the system. By the independence of the lifetimes of the components, for all $t > 0$,

$$\begin{aligned}\bar{F}(t) &= P\big(\min(X_1, X_2, \dots, X_n) > t\big) = P(X_1 > t, X_2 > t, \dots, X_n > t)\\ &= P(X_1 > t)P(X_2 > t)\cdots P(X_n > t) = \bar{F}_1(t)\bar{F}_2(t)\cdots\bar{F}_n(t).\end{aligned}$$

8. For $1 \le i \le n$, let X_i be the lifetime of the ith component. Then $\max(X_1, X_2, \dots, X_n)$ is the lifetime of the system. Let $\bar{F}(t)$ be the survival function of the system. By the independence of the lifetimes of the components, for all $t > 0$,

$$\begin{aligned}\bar{F}(t) &= P\big(\max(X_1, X_2, \dots, X_n) > t\big)\\ &= 1 - P\big(\max(X_1, X_2, \dots, X_n) \le t\big)\\ &= 1 - P(X_1 \le t, X_2 \le t, \dots, X_n \le t)\\ &= 1 - P(X_1 \le t)P(X_2 \le t)\cdots P(X_n \le t)\\ &= 1 - F_1(t)F_2(t)\cdots F_n(t).\end{aligned}$$

9. The problem is equivalent to the following: Two points X and Y are selected independently and at random from the interval $(0, \ell)$. What is the probability that the length of at least one interval is less than $\ell/20$? The solution to this problem is as follows:

$$P\Big(\min(X,\ Y-X,\ \ell-Y)<\frac{\ell}{20}\ \Big|\ X<Y\Big)P(X<Y)$$

$$+\,P\Big(\min(Y,\ X-Y,\ \ell-X)<\frac{\ell}{20}\ \Big|\ X>Y\Big)P(X>Y)$$

$$=2P\Big(\min(X,\ Y-X,\ \ell-Y)<\frac{\ell}{20}\ \Big|\ X<Y\Big)P(X<Y)$$

$$=2P\Big(\min(X,\ Y-X,\ \ell-Y)<\frac{\ell}{20}\ \Big|\ X<Y\Big)\cdot\frac{1}{2}$$

$$=1-P\Big(\min(X,\ Y-X,\ \ell-Y)\geq\frac{\ell}{20}\ \Big|\ X<Y\Big)$$

$$=1-P\Big(X\geq\frac{\ell}{20},\ Y-X\geq\frac{\ell}{20},\ \ell-Y\geq\frac{\ell}{20}\ \Big|\ X<Y\Big)$$

$$=1-P\Big(X\geq\frac{\ell}{20},\ Y-X\geq\frac{\ell}{20},\ Y\leq\frac{19\ell}{20}\ \Big|\ X<Y\Big).$$

Now $P\Big(X\geq\frac{\ell}{20},\ Y-X\geq\frac{\ell}{20},\ Y\leq\frac{19\ell}{20}\ \Big|\ X<Y\Big)$ is the area of the region

$$\Big\{(x,y)\in\mathbf{R}^2\colon 0<x<\ell,\ 0<y<\ell,\ x\geq\frac{\ell}{20},\ y-x\geq\frac{\ell}{20},\ y\leq\frac{19\ell}{20}\Big\}$$

divided by the area of the triangle

$$\big\{(x,y)\in\mathbf{R}^2\colon 0<x<\ell,\ 0<y<\ell,\ y>x\big\};$$

that is,

$$\frac{\dfrac{17\ell}{20}\times\dfrac{17\ell}{20}}{2}\div\frac{\ell^2}{2}=0.7225.$$

Therefore, the desired probability is $1-0.7225=0.2775$.

10. Let $f_{13}(x,y)$ be the joint probability density function of $X_{(1)}$ and $X_{(3)}$. By Theorem 9.6,

$$f_{13}(x,y)=6(y-x),\quad 0<x<y<1.$$

Let $U=\dfrac{X_{(1)}+X_{(3)}}{2}$ and $V=X_{(1)}$. Using Theorem 8.8, we will find $g(u,v)$, the joint probability density function of U and V. The probability density function of the midrange of these three random variables is $g_U(u)$. The system of two equations in two unknowns

$$\begin{cases}\dfrac{x+y}{2}=u\\ x=v\end{cases}$$

defines a one-to-one transformation of

$$R = \{(x, y) : 0 < x < y < 1\}$$

onto the region

$$Q = \Big\{(u, v) : 0 < v < u < \frac{v+1}{2} < 1\Big\}$$

that has the unique solution

$$\begin{cases} x = v \\ y = 2u - v. \end{cases}$$

Hence

$$\mathbf{J} = \begin{vmatrix} 0 & 1 \\ 2 & -1 \end{vmatrix} = -2 \neq 0;$$

therefore,

$$g(u, v) = f_{13}(v, 2u - v)|\mathbf{J}| = 24(u - v), \quad 0 < v < u < \frac{v+1}{2} < 1.$$

To find $g_U(u)$, draw the region Q to see that

$$g_U(u) = \begin{cases} \displaystyle\int_0^u 24(u - v)\,dv & 0 < u < 1/2 \\ \displaystyle\int_{2u-1}^u 24(u - v)\,dv & 1/2 \leq u < 1. \end{cases}$$

Therefore,

$$g_U(u) = \begin{cases} 12u^2 & 0 < u < 1/2 \\ 12(u - 1)^2 & 1/2 \leq u < 1. \end{cases}$$

The expected value of U is given by

$$E(U) = \int_0^{1/2} 12u^3\,du + \int_{1/2}^1 12u(u - 1)^2\,du = \frac{3}{16} + \frac{5}{16} = \frac{1}{2}.$$

Chapter 10

More Expectations and Variances

10.1 EXPECTED VALUES OF SUMS OF RANDOM VARIABLES

1. Since

$$E(X) = \int_0^1 x(1-x)\,dx + \int_1^2 x(x-1)\,dx = \frac{2}{3},$$

and

$$E(X^2) = \int_0^1 x^2(1-x)\,dx + \int_1^2 x^2(x-1)\,dx = \frac{3}{2},$$

we have that

$$E(X^2 + X) = \frac{3}{2} + \frac{2}{3} = \frac{13}{6}.$$

2. By Example 10.7, the answer is $\dfrac{5}{2/5} = 12.5$.

3. We have that $E(X^2) = \text{Var}(X) + \big[E(X)\big]^2 = 1$. Similarly, $E(Y^2) = E(Z^2) = 1$. Thus

$$\begin{aligned} E\big[X^2(Y+5Z)^2\big] &= E(X^2)E\big[(Y+5Z)^2\big] = E(Y^2 + 25Z^2 + 10YZ) \\ &= E(Y^2) + 25E(Z^2) + 10E(Y)E(Z) = 26. \end{aligned}$$

4. Since $f(x, y) = e^{-x} \cdot 2e^{-2y}$, X and Y are independent exponential random variables with parameters 1 and 2, respectively. Thus $E(X) = 1$, $E(Y) = 1/2$,

$$E(X^2) = \text{Var}(X) + \big[E(X)\big]^2 = 1 + 1 = 2,$$

and

$$E(Y^2) = \text{Var}(Y) + \big[E(Y)\big]^2 = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}.$$

Therefore, $E(X^2 + Y^2) = 2 + \dfrac{1}{2} = \dfrac{5}{2}$.

5. let X_1, X_2, X_3, X_4, and X_5 be geometric random variables with parameters 1, 4/5, 3/5, 2/5, and 1/5, respectively. The desired quantity is

$$\begin{aligned} E(X_1+X_2+X_3+X_4+X_5) &= E(X_1)+E(X_2)+E(X_3)+E(X_4)+E(X_5) \\ &= 1+\frac{5}{4}+\frac{5}{3}+\frac{5}{2}+5 = 11.42. \end{aligned}$$

6. Clearly,

$$E(X_i) = 1\cdot\frac{1}{n}+0\cdot\Big(1-\frac{1}{n}\Big)=\frac{1}{n}.$$

Thus $E(X_1+X_2+\cdots+X_n)=n\cdot\dfrac{1}{n}=1$ is the desired quantity.

7. Let X_1, X_2, X_3, and X_4 be the cost of a band to play music, the amount the caterer will charge, the rent of a hall to give the party, and other expenses, respectively. Let N be the number of people who participate. We have that $E(X_1)=1550$, $E(X_2)=1900$, $E(X_3)=1000$, $E(X_4)=550$, and

$$E(N)=\sum_{i=151}^{200} i\cdot\frac{1}{50}=\frac{1}{50}\Big(\sum_{i=1}^{200} i-\sum_{i=1}^{150} i\Big)=\frac{1}{50}\Big(\frac{200\times 201}{2}-\frac{150\times 151}{2}\Big)=175.50.$$

To have no loss on average, let x be the amount (in dollars) that the society should charge each participant. We must have

$$E(X_1+X_2+X_3+X_4)\le E(xN)=xE(N).$$

This gives

$$x\ge\frac{E(X_1)+E(X_2)+E(X_3)+E(X_4)}{175.50}=\frac{1550+1900+1000+550}{175.50}=28.49.$$

So to have no loss on the average, the society should charge each participant \$28.49.

8. **(a)** $E(\to 007)=E(007\to 007)=1,000.$
(b)

$$\begin{aligned} E(\to 156156) &= E(\to 156)+E(156\to 156156) \\ &= E(156\to 156)+E(156156\to 156156) \\ &= 1,000+1,000,000=1,001,000. \end{aligned}$$

(c)

$$\begin{aligned} E(\to 575757) &= E(\to 57)+E(57\to 5757)+E(5757\to 575757) \\ &= E(57\to 57)+E(5757\to 5757)+E(575757\to 575757) \\ &= 100+10,000+1,000,000=1,010,100. \end{aligned}$$

9. Let X be the number of students standing at the front of the room after k, $1 \leq k < n$ names have been called. The k students whose names have been called are not standing. Let A_1, A_2, $\dots$, A_{n-k} be the students whose names have not been called. Let

$$X_i = \begin{cases} 1 & \text{if } A_i \text{ is standing} \\ 0 & \text{otherwise.} \end{cases}$$

Clearly,

$$X = X_1 + X_2 + \cdots + X_{n-k}.$$

For i, $1 \leq i \leq n-k$,

$$E(X_i) = P\big(A_i \text{ is standing}\big) = \frac{k}{n}.$$

This is because A_i is standing if and only if his or her original seat was among the first k. Hence

$$E(X) = E(X_1) + E(X_2) + \cdots + E(X_{n-k}) = (n-k)\cdot\frac{k}{n} = \frac{(n-k)k}{n}.$$

10. By Theorem 10.2,

$$\begin{aligned} E\big[\min(X_1, X_2, \dots, X_n)\big] &= \sum_{k=1}^{\infty} P\big(\min(X_1, X_2, \dots, X_n) \geq k\big) \\ &= \sum_{k=1}^{\infty} P(X_1 \geq k,\ X_2 \geq k, \dots X_n \geq k) \\ &= \sum_{k=1}^{\infty} P(X_1 \geq k)P(X_2 \geq k)\cdots P(X_n \geq k) \\ &= \sum_{k=1}^{\infty}\big[P(X_1 \geq k)\big]^n = \sum_{k=1}^{\infty}\bigg[\Big(\sum_{i=k}^{\infty} p_i\Big)^n\bigg] = \sum_{k=1}^{\infty} h_k^n. \end{aligned}$$

11. Let E_1 be the event that the first three outcomes are heads and the fourth outcome is tails. For $2 \leq i \leq n-3$, let E_i be as defined in the hint. Let E_{n-2} be the event that the outcome $(n-3)$ is tails and the last three outcomes are heads. The expected number of exactly three consecutive heads is

$$\begin{aligned} E\Big(X_1 + \sum_{i=2}^{n-3} X_i + X_{n-2}\Big) &= E(X_1) + \sum_{i=2}^{n-3} E(X_i) + E(X_{n-2}) \\ &= P(E_1) + \sum_{i=2}^{n-3} P(E_i) + P(E_{n-2}) \\ &= \Big(\frac{1}{2}\Big)^4 + \sum_{i=2}^{n-3}\Big(\frac{1}{2}\Big)^5 + \Big(\frac{1}{2}\Big)^4 \\ &= \Big(\frac{1}{2}\Big)^3 + (n-4)\Big(\frac{1}{2}\Big)^5 = \frac{n}{32}. \end{aligned}$$

12. Let

$$X_i = \begin{cases} 1 & \text{if the } i\text{th box is empty} \\ 0 & \text{otherwise;} \end{cases}$$

The expected number of the empty boxes is

$$E(X_1 + X_2 + \cdots + X_{40}) = 40E(X_i) = 40P(X_i = 1) = 40\Big(\frac{39}{40}\Big)^{80} \approx 5.28.$$

13. The expected number of birthdays that belong to one student is

$$E(X_1 + X_2 + \cdots + X_{25}) = 25E(X_i) = 25P(X_i = 1) = 25\Big(\frac{364}{365}\Big)^{24} = 23.41.$$

14. Let $X_i = 1$, if the birthdays of at least two students are on the ith day of the year, and $X_i = 0$, otherwise. The desired quantity is

$$\begin{aligned} E\Big(\sum_{i=1}^{365} X_i\Big) &= 365E(X_i) = 365P(X_i = 1) \\ &= 365\bigg[1 - \Big(\frac{364}{365}\Big)^{25} - \binom{25}{1}\Big(\frac{1}{365}\Big)\Big(\frac{364}{365}\Big)^{24}\bigg] = 0.788. \end{aligned}$$

15. Let $u_1, u_2, \ldots, u_{39}$ be an enumeration of the nonheart cards. Let

$$X_i = \begin{cases} 1 & \text{if no heart is drawn before } u_i \text{ is drawn} \\ 0 & \text{otherwise.} \end{cases}$$

Let N be the number of cards drawn until a heart is drawn. Clearly, $N = 1 + \sum_{i=1}^{39} X_i$. By the result of Exercise 9, Section 3.2,

$$\begin{aligned} E(N) &= 1 + \sum_{i=1}^{39} E(X_i) = 1 + \sum_{i=1}^{39} P(X_i = 1) \\ &= 1 + \sum_{i=1}^{39} \frac{1}{14} = 1 + 39 \cdot \frac{1}{14} = 3.786. \end{aligned}$$

Note that if the experiment was performed *with* replacement, then $E(N) = 4$.

16. We have that

$$\begin{aligned} E(\to \text{THTHTTHTHT}) &= E(\to \text{T}) + E(\text{T} \to \text{THT}) + E(\text{THT} \to \text{THTHT}) \\ &\quad + E(\text{THTHT} \to \text{THTHTTHTHT}) \\ &= E(\to \text{T}) + E(\text{THT} \to \text{THT}) + E(\text{THTHT} \to \text{THTHT}) \\ &\quad + E(\text{THTHTTHTHT} \to \text{THTHTTHTHT}) \\ &= 2 + 8 + 32 + 1,024 = 1,066. \end{aligned}$$

17. **(a)** $\int_0^{\infty}\int_0^{\infty} I(x, y)\,dx\,dy$ is the area of the rectangle

$$\big\{(x, y) \in \mathbf{R^2}\colon 0 \le x < X,\ 0 \le y < Y\big\};$$

therefore it is equal to XY.

(b) Part (a) implies that

$$E(XY) = \int_0^{\infty}\int_0^{\infty} E\big[I(x, y)\big]\,dx\,dy = \int_0^{\infty}\int_0^{\infty} P(X > x, Y > y)\,dx\,dy.$$

18. Clearly $N > i$ if and only if

$$X_1 \ge X_2 \ge X_3 \ge \cdots \ge X_i.$$

Hence for $i \ge 2$,

$$P(N > i) = P\big(X_1 \ge X_2 \ge X_3 \ge \cdots \ge X_{i-1} \ge X_i\big) = \frac{1}{i!}$$

because X_i's are independent and identically distributed. So, by Theorem 10.2,

$$E(N) = \sum_{i=1}^{\infty} P(N \ge i) = \sum_{i=0}^{\infty} P(N > i) = P(N > 0) + P(N > 1) + \sum_{i=2}^{\infty}\frac{1}{i!}$$
$$= 1 + 1 + \sum_{i=2}^{\infty}\frac{1}{i!} = \sum_{i=0}^{\infty}\frac{1}{i!} = e.$$

19. If the first red chip is drawn on or before the 10th draw, let N be the number of chips before the first red chip. Otherwise, let $N = 10$. Clearly,

$$P(N = i) = \Big(\frac{1}{2}\Big)^i\Big(\frac{1}{2}\Big) = \Big(\frac{1}{2}\Big)^{i+1}, \quad 0 \le i \le 9; \quad P(N = 10) = \Big(\frac{1}{2}\Big)^{10}.$$

The desired quantity is

$$E(10 - N) = \sum_{i=0}^{9}(10 - i)\Big(\frac{1}{2}\Big)^{i+1} + (10 - 10)\cdot\Big(\frac{1}{2}\Big)^{10} \approx 9.001.$$

20. Clearly, if for some $\lambda \in \mathbf{R}$, $X = \lambda Y$, Cauchy-Schwarz's inequality becomes equality. We show that the converse of this is also true. Suppose that for random variables X and Y,

$$E(XY) = \sqrt{E(X^2)E(Y^2)}.$$

Then

$$4\big[E(XY)\big]^2 - 4E(X^2)E(Y^2) = 0.$$

Now the left side of this equation is the discriminant of the quadratic equation

$$E(Y^2)\lambda^2 - 2\big[E(XY)\big]\lambda + E(X^2) = 0.$$

Hence this quadratic equation has exactly one root. On the other hand,

$$E(Y^2)\lambda^2 - 2\big[E(XY)\big]\lambda + E(X^2) = E\big[(X - \lambda Y)^2\big].$$

So the equation

$$E\big[(X - \lambda Y)^2\big] = 0$$

has a unique solution. That is, there exists a unique number $\lambda_1 \in \mathbf{R}$ such that

$$E\big[(X - \lambda_1 Y)^2\big] = 0.$$

Since the expected value of a positive random variable is positive, this implies that with probability 1, $X - \lambda_1 Y = 0$ or $X = \lambda_1 Y$.

10.2 COVARIANCE

1. Since X and Y are independent random variables, $\text{Cov}(X, Y) = 0$.

2. $$E(X) = \sum_{x=1}^{3}\sum_{y=3}^{4}\frac{1}{70}x^2(x+y) = \frac{17}{7};$$

$E(Y) = \sum_{x=1}^{3}\sum_{y=3}^{4}\frac{1}{70}xy(x+y) = \frac{124}{35};$

$$E(XY) = \sum_{x=1}^{3}\sum_{y=3}^{4}\frac{1}{70}x^2y(x+y) = \frac{43}{5}.$$

Therefore,

$$\text{Cov}(X, Y) = E(XY) - E(X)E(Y) = \frac{43}{5} - \frac{17}{7}\cdot\frac{124}{35} = -\frac{1}{245}.$$

3. Intuitively, $E(X)$ is the average of 1, 2, ..., 6 which is 7/2; $E(Y)$ is $(7/2)(1/2) = 7/4$. To show these, note that

$$E(X) = \sum_{x=1}^{6} x p_X(x) = \sum_{x=1}^{6} x(1/6) = 7/2.$$

By the table constructed for $p(x, y)$ in Example 8.2,

$$E(Y) = 0 \cdot \frac{63}{384} + 1 \cdot \frac{120}{384} + 2 \cdot \frac{99}{384} + 3 \cdot \frac{64}{384} + 4 \cdot \frac{29}{384} + 5 \cdot \frac{8}{384} + 6 \cdot \frac{1}{384} = \frac{7}{4}.$$

By the same table,

$$E(XY) = \sum_{x=1}^{6} \sum_{y=0}^{6} xyp(x, y) = 91/12.$$

Therefore,

$$\text{Cov}(X, Y) = E(XY) - E(X)E(Y) = \frac{91}{12} - \frac{7}{2} \cdot \frac{7}{4} = \frac{35}{24} > 0.$$

This shows that X and Y are positively correlated. The higher the outcome from rolling the die, the higher the number of tails obtained—a fact consistent with our intuition.

4. Let X be the number of sheep stolen; let Y be the number of goats stolen. Let $p(x, y)$ be the joint probability mass function of X and Y. Then, for $0 \le x \le 4$, $0 \le y \le 4$, $0 \le x + y \le 4$,

$$p(x, y) = \frac{\binom{7}{x}\binom{8}{y}\binom{5}{4-x-y}}{\binom{20}{4}};$$

$p(x, y) = 0$, for other values of x and y. Clearly, X is a hypergeometric random variable with parameters $n = 4$, $D = 7$, and $N = 20$. Therefore,

$$E(X) = \frac{nD}{N} = \frac{28}{20} = \frac{7}{5}.$$

Y is a hypergeometric random variable with parameters $n = 4$, $D = 8$, and $N = 20$. Therefore,

$$E(Y) = \frac{nD}{N} = \frac{32}{20} = \frac{8}{5}.$$

Since

$$E(XY) = \sum_{x=0}^{4} \sum_{y=0}^{4-x} xyp(x, y) = \frac{168}{95},$$

we have

$$\text{Cov}(X, Y) = E(XY) - E(X)E(Y) = \frac{168}{95} - \frac{7}{5} \cdot \frac{8}{5} = -\frac{224}{475} < 0.$$

Therefore, X and Y are negatively correlated as expected.

5. Since $Y = n - X$,

$$E(XY) = E(nX - X^2) = nE(X) - E(X^2) = nE(X) - \big[\text{Var}(X) + E(X)^2\big]$$
$$= n \cdot np - \big[np(1-p) + n^2p^2\big] = n(n-1)p(1-p),$$

and

$$\text{Cov}(X, Y) = E(XY) - E(X)E(Y) = n(n-1)p(1-p) - np \cdot n(1-p) = -np(1-p).$$

This confirms the (obvious) fact that X and Y are negatively correlated.

6. Both (a) and (b) are straightforward results of relation (10.6).

7. Since $\text{Cov}(X, Y) = 0$, we have

$$\text{Cov}(X, Y + Z) = \text{Cov}(X, Y) + \text{Cov}(X, Z) = \text{Cov}(X, Z).$$

8. By relation (10.6),

$$\text{Cov}(X + Y, X - Y) = E(X^2 - Y^2) - E(X + Y)E(X - Y)$$
$$= E(X^2) - E(Y^2) - \big[E(X)\big]^2 + \big[E(Y)\big]^2 = \text{Var}(X) - \text{Var}(Y).$$

9. In Theorem 10.4, let $a = 1$ and $b = -1$.

10. **(a)** This is an immediate result of Exercise 8 above.

(b) By relation (10.6),

$$\text{Cov}(X, XY) = E(X^2Y) - E(X)E(XY)$$
$$= E(X^2)E(Y) - \big[E(X)\big]^2 E(Y) = E(Y)\text{Var}(X).$$

11. The probability density function of Θ is given by

$$f(\theta) = \begin{cases} \dfrac{1}{2\pi} & \text{if } \theta \in [0, 2\pi] \\ 0 & \text{otherwise.} \end{cases}$$

Therefore,

$$E(XY) = \int_0^{2\pi} \sin\theta \cos\theta \, \frac{1}{2\pi} \, d\theta = 0, \qquad E(X) = \int_0^{2\pi} \sin\theta \, \frac{1}{2\pi} \, d\theta = 0,$$

$$E(Y) = \int_0^{2\pi} \cos\theta \, \frac{1}{2\pi} \, d\theta = 0.$$

Thus $\text{Cov}(X, Y) = E(XY) - E(X)E(Y) = 0$.

12. The joint probability density function of X and Y is given by

$$f(x, y) = \begin{cases} \dfrac{1}{\pi} & x^2 + y^2 \leq 1 \\ 0 & \text{elsewhere.} \end{cases}$$

X and Y are dependent because, for example,

$$P\Big(0 < X < \frac{1}{2} \;\Big|\; Y = 0\Big) = \frac{1}{4}$$

while,

$$P\Big(0 < X < \frac{1}{2}\Big) = 2\int_0^{1/2} \int_0^{\sqrt{1-x^2}} \frac{1}{\pi}\, dy\, dx = \frac{2}{\pi} \int_0^{1/2} \sqrt{1 - x^2}\, dx$$

$$= \frac{1}{6} + \frac{\sqrt{3}}{4\pi} \neq P\Big(0 < X < \frac{1}{2} \;\Big|\; Y = 0\Big).$$

X and Y are uncorrelated because

$$E(X) = \iint_{x^2+y^2 \leq 1} x \frac{1}{\pi}\, dx\, dy = \frac{1}{\pi} \int_0^1 \int_0^{2\pi} r^2 \cos\theta\, d\theta\, dr = 0,$$

$$E(Y) = \iint_{x^2+y^2 \leq 1} y \frac{1}{\pi}\, dx\, dy = \frac{1}{\pi} \int_0^1 \int_0^{2\pi} r^2 \sin\theta\, d\theta\, dr = 0,$$

and

$$E(XY) = \iint_{x^2+y^2 \leq 1} xy \frac{1}{\pi}\, dx\, dy = \frac{1}{\pi} \int_0^1 \int_0^{2\pi} r^3 \cos\theta \sin\theta\, d\theta\, dr = 0,$$

implying that $\text{Cov}(X, Y) = E(XY) - E(X)E(Y) = 0$.

13. We have that

$$E(X) = \int_{1/2}^{2} \frac{8}{15} x^2\, dx = 1.4, \qquad E\big(X^2\big) = \int_{1/2}^{2} \frac{8}{15} x^3\, dx = 2.125,$$

$$E(Y) = \int_{1/4}^{9/4} \frac{6}{13} y^{3/2}\, dy = 1.396, \qquad E\big(Y^2\big) = \int_{1/4}^{9/4} \frac{6}{13} y^{5/2}\, dy = 2.252.$$

These give $\text{Var}(X) = 2.125 - 1.4^2 = 0.165$, and $\text{Var}(Y) = 2.252 - 1.396^2 = 0.303$. Hence $E(X + Y) = 1.4 + 1.396 = 2.796$, and by independence of X and Y,

$$\text{Var}(X + Y) = \text{Var}(X) + \text{Var}(Y) = 0.165 + 0.303 = 0.468.$$

Therefore, the expected value and variance of the total raise Mr. Jones will get next year are \$2796 and \$468, respectively.

14. We have that

$$\begin{aligned}\text{Var}(XY) &= E(X^2Y^2) - \big[E(X)E(Y)\big]^2 = E(X^2)E(Y^2) - \mu_1^2\mu_2^2 \\ &= (\mu_1^2 + \sigma_1^2)(\mu_2^2 + \sigma_2^2) - \mu_1^2\mu_2^2 = \sigma_1^2\sigma_2^2 + \mu_1^2\sigma_2^2 + \mu_2^2\sigma_1^2.\end{aligned}$$

15. **(a)** Let U_1 and U_2 be the measurements obtained using the voltmeter for V_1 and V_2, respectively. Then $V_1 = U_1 + X_1$ and $V_2 = U_2 + X_2$, where X_1 and X_2, the measurement errors, are independent random variables with mean 0 and variance σ^2. So the error variance in the estimation of V_1 and V_2 using the first method is σ^2.

(b) Let U_3 and U_4 be the measurements obtained,using the voltmeter, for V and W, respectively. Then $V = U_3 + X_3$ and $W = U_4 + X_4$, where X_3 and X_4, the measurement errors, are independent random variables with mean 0 and variance σ^2. Since $(U_3 + U_4)/2$ is used to estimate V_1, and $(U_3 - U_4)/2$ is used to estimate V_2,

$$V_1 = \frac{V + W}{2} = \frac{U_3 + U_4}{2} + \frac{X_3 + X_4}{2},$$

and

$$V_2 = \frac{V - W}{2} = \frac{U_3 - U_4}{2} + \frac{X_3 - X_4}{2},$$

we have that, for part (b), $(X_3 + X_4)/2$ and $(X_3 - X_4)/2$ are the measurement errors in measuring V_1 and V_2, respectively. The independence of X_3 and X_4 yields

$$\text{Var}\Big(\frac{X_3 + X_4}{2}\Big) = \frac{1}{4}\big[\text{Var}(X_3) + \text{Var}(X_4)\big] = \frac{1}{4}(\sigma^2 + \sigma^2) = \frac{\sigma^2}{2},$$

and

$$\text{Var}\Big(\frac{X_3 - X_4}{2}\Big) = \frac{1}{4}\big[\text{Var}(X_3) + \text{Var}(X_4)\big] = \frac{1}{4}(\sigma^2 + \sigma^2) = \frac{\sigma^2}{2}.$$

Therefore, the error variances in the estimation of V_1 and V_2, using the second method, is $\sigma^2/2$, showing that the second method is preferable.

16. Let r be the annual rate of return for Mr. Ingham's total investment. We have

$$\begin{aligned}\text{Var(r)} &= \text{Var}(0.18r_1 + 0.40r_2 + 0.42r_3) \\ &= (0.18)^2\,\text{Var}(r_1) + (0.40)^2\,\text{Var}(r_2) + (0.42)^2\,\text{Var}(r_3) \\ &\quad + 2(0.18)(0.40)\text{Cov}(r_1, r_2) + 2(0.18)(0.42)\text{Cov}(r_1, r_3) \\ &\quad + 2(0.40)(0.42)\text{Cov}(r_2, r_3) \\ &= (0.18)^2(0.064) + (0.40)^2(0.0144) + (0.42)^2(0.01) \\ &\quad + 2(0.18)(0.40)(0.03) + 2(0.18)(0.42)(0.015) + 2(0.40)(0.42)(0.021) \\ &= 0.01979.\end{aligned}$$

Hence the standard deviation of the annual rate of return for Mr. Ingham's total investment is $\sqrt{0.01979} = 0.14$.

17. Let r_1, r_2, and r_3 be the annual rates of return for Mr. Kowalski's investments in financial assets 1, 2, and 3, respectively. Let r be the annual rate of return for his total investment. Then, by Example 4.25,

$$r = 0.25r_1 + 0.40r_2 + 0.35r_3.$$

Since the assets are uncorrelated, we have

$$\begin{aligned} E(r) &= (0.25)(0.12) + (0.40)(0.15) + (0.35)(0.18) = 0.153, \\ \text{Var}(r) &= (0.25)^2(0.08)^2 + (0.40)^2(0.12)^2 + (0.35)^2(0.15)^2 = 0.00546, \\ \sigma_r &= \sqrt{\text{Var}(r)} = 0.074. \end{aligned}$$

Hence $r \sim N(0.153, 0.00546)$. Let X be the total investment of Mr. Kowalski. We are given that $X = 50,000$. Let Y be the total return of Mr. Kowalski's investment next year. The desired probability is

$$\begin{aligned} P(Y - X \geq 10,000) &= P\Big(\frac{Y - X}{X} \geq \frac{10,000}{50,000}\Big) \\ &= P(r \geq 0.2) = P\Big(Z \geq \frac{0.2 - 0.153}{0.074}\Big) \\ &= P(Z \geq 0.64) = 1 - \Phi(0.64) = 1 - 0.7389 = 0.2611. \end{aligned}$$

18. **(a)** We have that

$$E(X) = \int_0^1 \int_x^1 8x^2y\,dy\,dx = \frac{8}{15}, \qquad E(Y) = \int_0^1 \int_x^1 8xy^2\,dy\,dx = \frac{4}{5},$$

$$E(X^2) = \int_0^1 \int_x^1 8x^3y\,dy\,dx = \frac{1}{3}, \qquad E(Y^2) = \int_0^1 \int_x^1 8xy^3\,dy\,dx = \frac{2}{3},$$

$$E(XY) = \int_0^1 \int_x^1 8x^2y^2\,dy\,dx = \frac{4}{9},$$

$$\text{Cov}(X, Y) = E(XY) - E(X)E(Y) = \frac{4}{9} - \frac{8}{15} \cdot \frac{4}{5} = \frac{4}{225},$$

$$\text{Var}(X) = \frac{1}{3} - \Big(\frac{8}{15}\Big)^2 = \frac{11}{225}, \qquad \text{Var}(Y) = \frac{2}{3} - \Big(\frac{4}{5}\Big)^2 = \frac{2}{75}.$$

Therefore,

$$\text{Var}(X + Y) = \frac{11}{225} + \frac{2}{75} + 2 \cdot \frac{4}{225} = \frac{1}{9}.$$

(b) Since $\text{Cov}(X, Y) \neq 0$, X and Y are not independent. This does not contradict Exercise 23 of Section 8.2 because although $f(x, y)$ is the product of a function of x and a function of y, its domain is not of the form

$$\big\{(x, y) : a \leq x \leq b,\ c \leq y \leq d\big\}.$$

In the domain of f, x and y are related by $x \leq y$.

19. For $1 \leq i \leq n$, let X_i be the ith random number selected; we have

$$\text{Var}\Big(\sum_{i=1}^{n} X_i\Big) = \sum_{i=1}^{n} \text{Var}(X_i) = \sum_{i=1}^{n} \frac{(1-0)^2}{12} = \frac{n}{12}.$$

20. By the hint,

$$E(X) = \int_0^{\infty}\int_0^{\infty} \frac{1}{2}x^4 e^{-(y+1)x}\,dx\,dy = \int_0^{\infty} \frac{1}{2}\Big[\frac{4!}{(y+1)^5}\Big]\,dy = 3,$$

$$E(Y) = \int_0^{\infty}\int_0^{\infty} \frac{1}{2}x^3 y e^{-x(y+1)}\,dx\,dy = \int_0^{\infty} \frac{1}{2}y\Big[\frac{3!}{(y+1)^4}\Big]\,dy = \frac{1}{2},$$

and

$$E(XY) = \int_0^{\infty}\int_0^{\infty} \frac{1}{2}x^4 y e^{-(y+1)x}\,dx\,dy = \int_0^{\infty} \frac{1}{3}y\Big[\frac{4!}{(y+1)^5}\Big]\,dy = 1.$$

Since $\text{Cov}(X, Y) = 1 - \dfrac{3}{2} = -\dfrac{1}{2} < 0$,, X and Y are negatively correlated.

21. Note that

$$\begin{aligned} E\big[(X-t)^2\big] &= E\big[(X-\mu+\mu-t)^2\big] \\ &= E\big[(X-\mu)^2\big] + 2(\mu-t)E(X-\mu) + (\mu-t)^2 \\ &= E\big[(X-\mu)^2\big] + (\mu-t)^2. \end{aligned}$$

This relation shows that $E\big[(X-t)^2\big]$ is minimum if $(\mu-t)^2 = 0$; that is, if $t = \mu$. For this value, $E\big[(X-t)^2\big] = \text{Var}(X)$.

22. Clearly,

$$\text{Cov}(I_A, I_B) = E(I_A I_B) - E(I_A)E(I_B) = P(AB) - P(A)P(B).$$

This shows that $\text{Cov}(I_A, I_B) > 0 \Longleftrightarrow P(AB) > P(A)P(B) \Longleftrightarrow \dfrac{P(AB)}{P(B)} > P(A), \Longleftrightarrow P(A \mid B) > P(A)$. The proof that I_A and I_B are positively correlated if and only if $P(B|A) > P(B)$ follows by symmetry.

23. By Exercise 6,

$$\begin{aligned}\text{Cov}(aX+bY, cZ+dW) &= a\,\text{Cov}(X, cZ+dW) + b\,\text{Cov}(Y, cZ+dW)\\ &= ac\,\text{Cov}(X,Z) + ad\,\text{Cov}(X,W) + bc\,\text{Cov}(Y,Z) + bd\,\text{Cov}(Y,W).\end{aligned}$$

24. By Exercise 6 and an induction on n,

$$\text{Cov}\Big(\sum_{i=1}^n a_i X_i,\ \sum_{j=1}^m b_j Y_j\Big) = \sum_{i=1}^n a_i \text{Cov}\Big(X_i,\ \sum_{j=1}^m b_j Y_j\Big).$$

By Exercise 6 and an induction on m,

$$\text{Cov}\Big(X_i,\ \sum_{j=1}^m b_j Y_j\Big) = \sum_{j=1}^m b_j \text{Cov}(X_i,\ Y_j).$$

The desired identity follows from these two identities.

25. For $1 \le i \le n$, let $X_i = 1$ if the outcome of the ith throw is 1; let $X_i = 0$, otherwise. For $1 \le j \le n$, let $Y_j = 1$ if the outcome of the jth throw is 6; let $Y_j = 0$, otherwise. Clearly, $\text{Cov}(X_i, Y_j) = 0$ if $i \ne j$. By Exercise 24,

$$\begin{aligned}\text{Cov}\Big(\sum_{i}^n X_i,\ \sum_{j=1}^n Y_j\Big) &= \sum_{j=1}^n\sum_{i=1}^n \text{Cov}(X_i,\ Y_j) = \sum_{i=1}^n \text{Cov}(X_i,\ Y_i)\\ &= \sum_{i=1}^n \big[E(X_iY_i) - E(X_i)E(Y_i)\big] = \sum_{i=1}^n \Big(0 - \frac{1}{6}\cdot\frac{1}{6}\Big) = -\frac{n}{36}.\end{aligned}$$

As expected, in n throws of a fair die, the number of ones and the number of sixes are negatively correlated.

26. Let $S_n = \sum_{i=1}^n a_i X_i$, $\mu_i = E(X_i)$; then

$$E(S_n) = \sum_{i=1}^n a_i\mu_i, \qquad S_n - E(S_n) = \sum_{i=1}^n a_i(X_i - \mu_i).$$

Thus

$$\begin{aligned}\text{Var}(S_n) &= E\Big(\Big[\sum_{i=1}^n a_i(X_i-\mu_i)\Big]^2\Big)\\ &= \sum_{i=1}^n a_i^2 E\big[(X_i-\mu_i)^2\big] + 2\sum_{i<j}\sum a_i a_j E\big[(X_i-\mu_i)(X_j-\mu_j)\big]\\ &= \sum_{i=1}^n a_i^2 \text{Var}(X_i) + 2\sum_{i<j}\sum a_i a_j \text{Cov}(X_i, X_j).\end{aligned}$$

27. To find $\text{Var}(X)$, we use the following identity:

$$\text{Var}\Big(\sum_{i=1}^{n} X_i\Big) = \sum_{i=1}^{n} \text{Var}(X_i) + 2\sum_{i<j}\sum \text{Cov}(X_i,\ X_j). \tag{38}$$

Now for $1 \le i \le n$,

$$E(X_i) = P(A_i) = \frac{D}{N}, \qquad E(X_i^2) = P(A_i) = \frac{D}{N}.$$

Thus

$$\text{Var}(X_i) = E(X_i^2) - \big[E(X_i)\big]^2 = \frac{D}{N} - \Big(\frac{D}{N}\Big)^2 = \frac{D(N-D)}{N^2}.$$

Also for $i < j$,

$$X_i X_j = \begin{cases} 1 & \text{if } A_i A_j \text{ occurs} \\ 0 & \text{otherwise.} \end{cases}$$

Therefore,

$$E(X_i X_j) = P(A_i A_j) = P(A_j \mid A_i)P(A_i) = \frac{D-1}{N-1} \cdot \frac{D}{N} = \frac{(D-1)D}{(N-1)N},$$

and

$$\begin{aligned} \text{Cov}(X_i,\ X_j) &= E(X_i X_j) - E(X_i)E(X_j) \\ &= \frac{(D-1)D}{(N-1)N} - \frac{D}{N} \cdot \frac{D}{N} = \frac{-D(N-D)}{(N-1)N^2}. \end{aligned}$$

Substituting the values of $\text{Var}(X_i)$'s and $\text{Cov}(X_i,\ X_j)$ back into (38), we get

$$\begin{aligned} \text{Var}(X) &= n\Big[\frac{D(N-D)}{N^2}\Big] + 2\binom{n}{2}\Big[\frac{-D(N-D)}{(N-1)N^2}\Big] \\ &= \frac{nD(N-D)}{N^2}\Big(1 - \frac{n-1}{N-1}\Big). \end{aligned}$$

This follows since in (38), $\sum$ and $\sum_{i<j}\sum$ have n and $\binom{n}{2} = \dfrac{n(n-1)}{2}$ equal terms, respectively.

28. Let $X_i = 1$, if the ith couple is left intact; 0, otherwise. We are interested in $\text{Var}(\sum_{i=1}^{n} X_i)$, where

$$\text{Var}\Big(\sum_{i=1}^{n} X_i\Big) = \sum_{i=1}^{n} \text{Var}(X_i) + 2\sum_{i<j}\sum \text{Cov}(X_i,\ X_j).$$

To find $\text{Var}(X_i)$, note that since $X_i^2 = X_i$,

$$\text{Var}(X_i) = E\big(X_i^2\big) - \big[E(X_i)\big]^2 = E(X_i) - \big[E(X_i)\big]^2.$$

By Example 10.3,

$$E(X_i) = \frac{(2n-m)(2n-m-1)}{2n(2n-1)}.$$

So

$$\text{Var}(X_i) = \frac{(2n-m)(2n-m-1)}{2n(2n-1)}\Big[1 - \frac{(2n-m)(2n-m-1)}{2n(2n-1)}\Big].$$

To find $\text{Cov}(X_i,\ X_j)$, note that $X_iX_j = 1$ if the ith and jth couples are left intact; 0, otherwise. Now

$$\begin{aligned}\text{Cov}(X_i,\ X_j) &= E(X_iX_j) - E(X_i)E(X_j) = P(X_iX_j = 1) - E(X_i)E(X_j)\\ &= \frac{\dbinom{2n-4}{m}}{\dbinom{2n}{m}} - \Big[\frac{(2n-m)(2n-m-1)}{2n(2n-1)}\Big]^2.\end{aligned}$$

Therefore,

$$\text{Cov}(X_i,\ X_j) = \frac{(2n-m)(2n-m-1)(2n-m-2)(2n-m-3)}{2n(2n-1)(2n-2)(2n-3)} - \Big[\frac{(2n-m)(2n-m-1)}{2n(2n-1)}\Big]^2.$$

So

$$\begin{aligned}\text{Var}\Big(\sum_{i=1}^{n} X_i\Big) &= n\frac{(2n-m)(2n-m-1)}{2n(2n-1)}\Big[1 - \frac{(2n-m)(2n-m-1)}{2n(2n-1)}\Big]\\ &\quad + 2\frac{n(n-1)}{2}\Big[\frac{(2n-m)(2n-m-1)(2n-m-2)(2n-m-3)}{2n(2n-1)(2n-2)(2n-3)}\\ &\qquad - \frac{(2n-m)^2(2n-m-1)^2}{4n^2(2n-1)^2}\Big]\\ &= \frac{(2n-m)(2n-m-1)}{2(2n-1)}\Big[1 - \frac{(2n-m)(2n-m-1)}{2n(2n-1)}\Big]\\ &\quad + (n-1)\Big[\frac{(2n-m)(2n-m-1)(2n-m-2)(2n-m-3)}{2(2n-1)(2n-2)(2n-3)}\\ &\qquad - \frac{(2n-m)^2(2n-m-1)^2}{4n(2n-1)^2}\Big]\end{aligned}$$

$$= \frac{(2n-m)(2n-m-1)}{2(2n-1)}\Big[1 - \frac{(2n-m)(2n-m-1)}{2n(2n-1)}$$
$$+ \frac{(n-1)(2n-m-2)(2n-m-3)}{(2n-2)(2n-3)}$$
$$- \frac{(n-1)(2n-m)(2n-m-1)}{2n(2n-1)}\Big]$$
$$= \frac{(2n-m)(2n-m-1)}{2(2n-1)}\Big[1 + \frac{(n-1)(2n-m-2)(2n-m-3)}{(2n-2)(2n-3)}$$
$$- \frac{(2n-m)(2n-m-1)}{2(2n-1)}\Big].$$

10.3 CORRELATION

1. We have that $\text{Cov}(X, Y) = \rho(X, Y)\sigma_X\sigma_Y = 3$; thus

$$\begin{aligned}\text{Var}(2X - 4Y + 3) &= \text{Var}(2X - 4Y) = 4\text{Var}(X) + 16\text{Var}(Y) - 16\text{Cov}(X, Y)\\ &= 4(4) + 16(9) - 16(3) = 112.\end{aligned}$$

2. By Exercise 23 of Section 8.2, X and Y are independent random variables. [This can also be shown directly by verifying the relation $f(x, y) = f_X(x)f_Y(y)$.] Hence $\text{Cov}(X, Y) = 0$, and therefore $\rho(X, Y) = 0$.

3. Let X and Y be the lengths of the pieces obtained. Since $Y = 1 - X$, by Theorem 10.5, $\rho(X, Y) = -1$. Since X and Y are uniform over $(0, 1)$, $\sigma_X = 1/\sqrt{12}$ and $\sigma_Y = 1/\sqrt{12}$. Therefore,

$$\text{Cov}(X, Y) = \rho(X, Y)\sigma_X\sigma_Y = (-1)\Big(\frac{1}{\sqrt{12}}\Big)\Big(\frac{1}{\sqrt{12}}\Big) = -\frac{1}{12}.$$

4. If $\alpha_1\beta_1 = 0$, both sides of the relation are 0 and the equality holds. If $\alpha_1\beta_1 \neq 0$, then

$$\rho(\alpha_1 X + \alpha_2,\ \beta_1 Y + \beta_2) = \frac{\text{Cov}(\alpha_1 X + \alpha_2,\ \beta_1 Y + \beta_2)}{\sigma_{\alpha_1 X+\alpha_2} \cdot \sigma_{\beta_1 Y+\beta_2}}$$

$$= \frac{\text{Cov}(\alpha_1 X,\ \beta_1 Y)}{|\alpha_1|\sigma_X \cdot |\beta_1|\sigma_Y} = \frac{\alpha_1\beta_1\text{Cov}(X, Y)}{|\alpha_1||\beta_1|\sigma_X\sigma_Y} = \text{sgn}(\alpha_1\beta_1)\rho(X, Y).$$

5. No, because for all random variables X and Y, $-1 \leq \rho(X, Y) \leq 1$.

6. By Exercise 6 of Section 10.2,

$$\text{Cov}(X+Y, X-Y) = \text{Var}(X) - \text{Var}(Y).$$

Since $\text{Cov}(X, Y) = 0$,

$$\begin{aligned}\sigma_{X+Y} \cdot \sigma_{X-Y} &= \sqrt{\text{Var}(X+Y) \cdot \text{Var}(X-Y)}\\ &= \sqrt{\big[\text{Var}(X) + \text{Var}(Y)\big]\big[\text{Var}(X) + \text{Var}(Y)\big]}\\ &= \text{Var}(X) + \text{Var}(Y).\end{aligned}$$

Therefore,

$$\rho(X+Y, X-Y) = \frac{\text{Cov}(X+Y, X-Y)}{\sigma_{X+Y} \cdot \sigma_{X-Y}} = \frac{\text{Var}(X) - \text{Var}(Y)}{\text{Var}(X) + \text{Var}(Y)}.$$

7. Using integration by parts, we obtain

$$\begin{aligned}E(X) &= \frac{1}{2}\int_0^{\pi/2}\int_0^{\pi/2} x \sin(x+y)\,dx\,dy = \frac{\pi}{4},\\ E(X^2) &= \frac{1}{2}\int_0^{\pi/2}\int_0^{\pi/2} x^2 \sin(x+y)\,dx\,dy = \frac{\pi^2}{8} + \frac{\pi}{2} - 2.\end{aligned}$$

Hence

$$\text{Var}(X) = \Big(\frac{\pi^2}{8} + \frac{\pi}{2} - 2\Big) - \frac{\pi^2}{16} = \frac{\pi}{2} - 2 + \frac{\pi^2}{16}.$$

By symmetry, $E(Y) = \dfrac{\pi}{4}$ and $\text{Var}(Y) = \dfrac{\pi}{2} - 2 + \dfrac{\pi^2}{16}$. Since

$$E(XY) = \frac{1}{2}\int_0^{\pi/2}\int_0^{\pi/2} xy \sin(x+y)\,dx\,dy = \frac{\pi}{2} - 1,$$

$\text{Cov}(X, Y) = \Big(\dfrac{\pi}{2} - 1\Big) - \dfrac{\pi^2}{16}$. Therefore,

$$\rho(X, Y) = \frac{\text{Cov}(X, Y)}{\sqrt{\text{Var}(X)} \cdot \sqrt{\text{Var}(Y)}} = \frac{(\pi/2) - 1 - (\pi^2/16)}{(\pi/2) - 2 + (\pi^2/16)} = -0.245.$$

Since $\rho(X, Y) \neq \pm 1$, there is no linear relation between X and Y.

10.4 CONDITIONING ON RANDOM VARIABLES

1. Let N be the number of tosses required; then

$$E(N) = E\big[E(N|X)\big] = E(N \mid X=0)P(X=0) + E(N \mid X=1)P(X=1)$$
$$= \big[1+E(N)\big]\frac{1}{2} + \Big(\frac{1}{2}\cdot 1 + \frac{1}{2}\big[2+E(N)\big]\Big)\frac{1}{2}.$$

Solving this equation for $E(N)$, we obtain $E(N)=5$.

2. We have that

$$E\big[Y(t)\big] = E\Big[E\big[Y(t)|X\big]\Big] = E\big[Y(t) \mid X<t\big]P(X<t) + E\big[Y(t) \mid X \geq t\big]P(X \geq t)$$
$$= E\Big[aX - \frac{a}{3}(t-X)\Big]P(X<t) + E(at)P(X \geq t)$$
$$= E\Big(\frac{4a}{3}X - \frac{at}{3}\Big)P(X<t) + atP(X \geq t)$$
$$= \Big[\frac{4a}{3}\Big(\frac{11}{2}\Big) - \frac{at}{3}\Big]\Big(\frac{t-4}{7-4}\Big) + at\Big(\frac{7-t}{3}\Big)$$
$$= \frac{1}{9}a(22-t)(t-4) + \frac{1}{3}at(7-t).$$

To find the value of t that maximizes $E\big[Y(t)\big]$, we solve $\frac{d}{dt}E\big[Y(t)\big] = \frac{1}{9}a(-8t+47) = 0$ for t. We get $t = 47/8 = 5.875$.

3. **(a)** Clearly,

$$E(X_n \mid X_{n-1}=x) = x\cdot\frac{x}{b} + (x+1)\cdot\frac{b-x}{b} = 1 + \Big(1-\frac{1}{b}\Big)x.$$

This implies that

$$E(X_n|X_{n-1}) = 1 + \Big(1-\frac{1}{b}\Big)X_{n-1}.$$

Therefore,

$$E(X_n) = E\big[E(X_n|X_{n-1})\big] = 1 + \Big(1-\frac{1}{b}\Big)E(X_{n-1}). \tag{39}$$

Now we use induction to prove that

$$E(X_n) = b - d\Big(1-\frac{1}{b}\Big)^n. \tag{40}$$

For $n=1$, (40) holds since

$$E(X_1) = (b-d)\frac{b-d}{b} + (b-d+1)\frac{d}{b} = b - d\Big(1-\frac{1}{b}\Big).$$

Suppose that (40) is valid for n, we show that it is valid for $n+1$ as well. By (39),

$$\begin{aligned} E(X_{n+1}) &= 1 + \Big(1 - \frac{1}{b}\Big)E(X_n) = 1 + \Big(1 - \frac{1}{b}\Big)\Big[b - d\Big(1 - \frac{1}{b}\Big)^n\Big] \\ &= 1 + b\Big(1 - \frac{1}{b}\Big) - d\Big(1 - \frac{1}{b}\Big)^{n+1} = b - d\Big(1 - \frac{1}{b}\Big)^{n+1}. \end{aligned}$$

This shows that (40) holds for $n+1$, and hence for all n.

(b) We have that

$$\begin{aligned} P(E_n) &= \sum_{x=b-d}^{b} P(E_n \mid X_{n-1} = x)P(X_{n-1} = x) = \sum_{x=b-d}^{b} \frac{x}{b} P(X_{n-1} = x) \\ &= \frac{1}{b} \sum_{x=b-d}^{b} xP(X_{n-1} = x) = \frac{1}{b}E(X_{n-1}) = 1 - \frac{d}{b}\Big(1 - \frac{1}{b}\Big)^{n-1}. \end{aligned}$$

4. Let V be a random variable defined by

$$V = \begin{cases} 1 & \text{with probability } p \\ 0 & \text{with probability } 1 - p. \end{cases}$$

Then

$$X = \begin{cases} Y & \text{if } V = 1 \\ Z & \text{if } V = 0. \end{cases}$$

Therefore,

$$\begin{aligned} E(X) &= E\big[E(X \mid V)\big] = E(X \mid V = 1)P(V = 1) + E(X \mid V = 0)P(V = 0) \\ &= E(Y)p + E(Z)(1 - p). \end{aligned}$$

5. The probability that a page should be retyped is

$$p = 1 - \frac{e^{-3/2}(3/2)^0}{0!} - \frac{e^{-3/2}(3/2)^1}{1!} - \frac{e^{-3/2}(3/2)^2}{2!} = 0.1912.$$

Thus $E(X_1) = 200(0.1912)$ and

$$\begin{aligned} E(X_2) &= E\big[E(X_2|X_1)\big] = \sum_{x=0}^{200} E(X_2 \mid X_1 = x)P(X_1 = x) \\ &= \sum_{x=0}^{200} (0.1912)xP(X_1 = x) = (0.1912)E(X_1) = (0.1912)^2(200). \end{aligned}$$

Similarly,

$$E(X_3) = E\big[E(X_3|X_2)\big] = (0.1912)^3(200)$$

and, in general,

$$E(X_n) = (0.1912)^n(200).$$

Therefore, by (10.2),

$$E\Big(\sum_{i=1}^{\infty} X_i\Big) = \sum_{i=1}^{\infty} E(X_i) = \sum_{i=1}^{\infty}(0.1912)^i(200) = 200\Big(\frac{0.1912}{1-0.1912}\Big) = 47.28.$$

6. For $i \geq 1$, let X_i be the length of the ith character of the message. Since the total number of the bits of the message is $\sum_{i=1}^{K} X_i$, and since it will take (1/1000)th of a second to emit a bit, we have that $T = (1/1000)\sum_{i=1}^{K} X_i$. By Wald's equation and Theorem 10.8,

$$E(T) = \frac{1}{1000}E(K)E(X_1) = \frac{1}{1000}\mu \cdot \frac{1}{p} = \frac{\mu}{1000p}$$

$$\text{Var}(T) = \Big(\frac{1}{1000}\Big)^2\Big[E(K)\text{Var}(X_1) + \big[E(X_1)\big]^2\text{Var}(K)\Big]$$

$$= \Big(\frac{1}{1000}\Big)^2\Big[\mu \cdot \frac{1-p}{p^2} + \frac{1}{p^2}\sigma^2\Big] = \frac{\mu(1-p)+\sigma^2}{1,000,000p^2}.$$

7. We have that

$$E(X_n) = E\big[E(X_n|Y)\big] = E(X_n \mid Y=1)P(Y=1) + E(X_n \mid Y=0)P(Y=0)$$

$$= 0 \cdot P(Y=1) + \big[1 + E(X_{n+1})\big]\frac{39-n}{52-n}.$$

This recursive relation and $E(X_{39}) = 0$ imply that $E(X_{38}) = 1/14$, $E(X_{37}) = 2/14$, $E(X_{36}) = 3/14$, and, in general, $E(X_i) = (39-i)/14$. The answer is

$$1 + E(X_0) = 1 + \frac{39}{14} = \frac{53}{14} = 3.786.$$

8. Let F be the distribution function of X. We have

$$P(X < Y) = \int_{-\infty}^{\infty} P(X < Y \mid Y = y)g(y)\,dy$$

$$= \int_{-\infty}^{\infty} P(X < y)\,g(y)\,dy = \int_{-\infty}^{\infty} F(y)\,g(y)\,dy.$$

9. Let f be the probability density function of λ; then

$$\begin{aligned} P(N = i) &= \int_0^\infty P(N = i \mid \lambda = x) f(x)\, dx \\ &= \int_0^\infty \frac{e^{-x} x^i}{i!} e^{-x}\, dx = \int_0^\infty \frac{e^{-2x} x^i}{i!}\, dx \\ &= \frac{1}{i!}\Big(\frac{1}{2}\Big)^i \int_0^\infty e^{-2x} (2x)^i\, dx \\ &= \frac{1}{i!}\Big(\frac{1}{2}\Big)^{i+1} \int_0^\infty e^{-u} u^i\, du = \Big(\frac{1}{2}\Big)^{i+1}. \end{aligned}$$

In these calculations, we have used the substitution $u = 2x$ and the relation

$$\int_0^\infty e^{-u} u^i\, du = i!.$$

10. Suppose that player A carries x dollars in his wallet. Then player A wins if and only if player B carries y dollars, $y \in (x, 1]$ in his wallet. Thus player A wins y dollars with probability $1 - x$. In such a case, the expected amount player A wins is $(1+x)/2$. Player A loses x dollars with probability x. Therefore,

$$E(W_A \mid X = x) = \frac{1+x}{2} \cdot (1 - x) + (-x) \cdot x = \frac{1}{2} - \frac{3}{2}x^2.$$

Let f_X be the probability density function of X, then

$$f_X(x) = \begin{cases} 1 & \text{if } 0 \le x \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

Therefore,

$$\begin{aligned} E(W_A) &= E\big[E(W_A \mid X)\big] = \int_0^1 E(W_A \mid X = x) f_X(x)\, dx \\ &= \int_0^1 \Big(\frac{1}{2} - \frac{3}{2}x^2\Big)\, dx = \Big[\frac{1}{2}x - \frac{1}{2}x^3\Big]_0^1 = 0. \end{aligned}$$

The solution above was presented by Kent G. Merryfield, Ngo Viet, and Saleem Watson in their joint paper "The Wallet Paradox" published in the August-September 1977 issue of the *American Mathematical Monthly.* Note the following observations by the authors.

> It is interesting to consider special cases of this formula for the conditional expectation. Since $E(W_A \mid X = 1) = -1$ and $E(W_A \mid X = 0) = 1/2$, we see that a player carrying one dollar in his wallet should expect to lose it, whereas a player carrying nothing in his wallet should expect to gain half a dollar (the mean). Interestingly, if a player is carrying half a dollar (the mean) in his wallet, then $E(W_A \mid X = 1/2) = 1/8$; that is, his expectation of winning is positive.

11. **(a)** To derive the relation

$$E(K_n \mid K_{n-1} = i) = (i+1)\frac{1}{2} + \big[i + 1 + E(K_n)\big]\frac{1}{2}$$
$$= (i+1) + \frac{1}{2}E(K_n),$$

we noted the following. It took i tosses of the coin to obtain $n-1$ consecutive heads. If the result of the next toss is heads, we have the desired n consecutive heads. This occurs with probability $1/2$. However, if the result of the next toss is tails, then, on the average, we need an additional $E(K_n)$ tosses [a total of $i + 1 + E(K_n)$ tosses] to obtain n consecutive heads. This also happens with probability 1/2.

(b) From (a) it should be clear that

$$E(K_n \mid K_{n-1}) = (K_{n-1} + 1) + \frac{1}{2}E(K_n).$$

(c) Finding the expected values of both sides of (b) yields

$$E(K_n) = E(K_{n-1}) + 1 + \frac{1}{2}E(K_n).$$

Solving this for $E(K_n)$, we obtain

$$E(K_n) = 2 + 2E(K_{n-1}).$$

(d) Note that K_1 is a geometric random variable with parameter 1/2. Thus $E(K_1) = 2$. Solving $E(K_n) = 2 + 2E(K_{n-1})$ recursively, we get

$$E(K_n) = 2 + 2^2 + 2^3 + \cdots + 2^n = 2(1 + 2 + \cdots + 2^{n-1})$$
$$= 2 \cdot \frac{2^n - 1}{2 - 1} = 2(2^n - 1).$$

12. Suppose that the last tour left at time 0. Let X be the time from 0 until the next guided tour begins. Let S_{10} be the time from 0 until 10 new tourists arrive. The random variable S_{10} is gamma with parameters $\lambda = 1/5$ and $n = 10$. Let F and f be the probability distribution and density functions of S_{10}. Then, for $t \geq 0$,

$$f(t) = \frac{1}{5}e^{-t/5}\frac{(t/5)^9}{9!}.$$

To find $E(X)$, note that

$$E(X) = E(X \mid S_{10} < 60)P(S_{10} < 60) + E(X \mid S_{10} \geq 60)P(S_{10} \geq 60)$$
$$= E(S_{10} \mid S_{10} < 60)P(S_{10} < 60) + 60P(S_{10} \geq 60).$$

Now

$$P(S_{10} < 60) = \int_0^{60} \frac{1}{5}\, e^{-t/5} \frac{(t/5)^9}{9!}\, dt = 0.7576,$$

and, by Remark 8.1,

$$\begin{aligned} E(S_{10} \mid S_{10} < 60) &= \frac{1}{F(60)} \int_0^{60} t f(t)\, dt \\ &= \frac{1}{0.7576} \int_0^{60} \frac{1}{5} t e^{-t/5}\, \frac{(t/5)^9}{9!}\, dt = 43.0815. \end{aligned}$$

Therefore,

$$E(X) = (43.0815)(0.7576) + 60(1 - 0.7576) = 47.18.$$

This shows that the expected length of time between two consecutive tours is approximately 47 minutes and 10 seconds.

13. Let X_1 be the time until the first application arrives. Let X_2 be the time between the first and second applications, and so forth. Then X_i's are independent exponential random variables with mean $1/\lambda = 1/5$ of a day. Let N be the first integer for which

$$X_1 \le 2,\ X_2 \le 2,\ \ldots,\ X_N \le 2,\ X_{N+1} > 2.$$

The time that the admissions office has to wait before doubling its student recruitment efforts is $S_{N+1} = X_1 + X_2 + \cdots + X_{N+1}$. Therefore,

$$E(S_{N+1}) = E\big[E(S_{N+1} \mid N)\big] = \sum_{i=0}^{\infty} E(S_{N+1} \mid N = i) P(N = i).$$

Now, for $i \ge 0$,

$$\begin{aligned} E(S_{N+1} \mid N = i) &= E(X_1 + X_2 + \cdots + X_{i+1} \mid N = i) = \sum_{j=1}^{i+1} E(X_j \mid N = i) \\ &= \Big[\sum_{j=1}^{i} E(X_j \mid X_j \le 2)\Big] + E(X_{i+1} \mid X_{i+1} > 2), \end{aligned}$$

where by Remark 8.1,

$$E(X_j \mid X_j \le 2) = \frac{1}{F(2)} \int_0^2 t f(t)\, dt,$$

$$E(X_{i+1} \mid X_{i+1} > 2) = \frac{1}{1 - F(2)} \int_2^{\infty} t f(t)\, dt,$$

F and f being the probability distribution and density functions of X_i's, respectively. That is, for $t \geq 0$, $F(t) = 1 - e^{-5t}$, $f(t) = 5e^{-5t}$. Thus, for $1 \leq j \leq i$,

$$E(X_j \mid X_j \leq 2) = \frac{1}{1 - e^{-10}} \int_0^2 5t\, e^{-5t}\, dt = (1.0000454)\Big[\Big(-t - \frac{1}{5}\Big)e^{-5t}\Big]_0^2$$
$$= (1.0000454)(0.19999) = 0.1999092$$

and, for $j = i + 1$,

$$E(X_{i+1} \mid X_{i+1} > 2) = \frac{1}{e^{-10}} \int_2^\infty 5t\, e^{-5t}\, dt = e^{10}\Big[\Big(-t - \frac{1}{5}\Big)e^{-5t}\Big]_2^\infty = 2.2.$$

Thus, for $i \geq 0$,

$$E(S_{N+1} \mid N = i) = (0.1999092)i + 2.2.$$

To find $P(N = i)$, note that for $i \geq 0$,

$$P(N = i) = P(X_1 \leq 2,\ X_2 \leq 2,\ \ldots,\ X_i \leq 2,\ X_{i+1} > 2)$$
$$= \big[F(2)\big]^i\big[1 - F(2)\big] = (0.9999546)^i(0.0000454).$$

Putting all these together, we obtain

$$E(S_{N+1}) = \sum_{i=0}^{\infty} E(S_{N+1} \mid N = i)P(N = i)$$
$$= \sum_{i=0}^{\infty} \big[(0.1999092)i + 2.2\big](0.9999546)^i(0.0000454)$$
$$= (0.00000908)\sum_{i=0}^{\infty} i(0.9999546)^i + (0.00009988)\sum_{i=0}^{\infty}(0.9999546)^i$$
$$= (0.00000908) \cdot \frac{0.9999546}{(1 - 0.9999546)^2} + (0.00009988) \cdot \frac{1}{1 - 0.9999546}$$
$$= 4407.286,$$

where the next to last equality follows from $\sum_{i=1}^{\infty} ir^i = r/(1-r)^2$, and $\sum_{i=0}^{\infty} r^i = 1/(1-r)$, $|r| < 1$. Since an academic year is 9 months long, and contains approximately 180 business days, the admission officers should not be concerned about this rule at all. It will take 4,407.286 business days, on average, until there is a lapse of two days between two consecutive applications.

14. Let X_i be the number of calls until Steven has not missed Adam in exactly i consecutive calls. We have that

$$E\big(X_i \mid X_{i-1}\big) = \begin{cases} X_{i-1} + 1 & \text{with probability } p \\ X_{i-1} + 1 + E(X_i) & \text{with probability } 1 - p. \end{cases}$$

Therefore,

$$E(X_i) = E\big[E(X_i \mid X_{i-1})\big] = \big[E(X_{i-1}) + 1\big]p + \big[E(X_{i-1}) + 1 + E(X_i)\big](1-p).$$

Solving this equation for $E(X_i)$, we obtain

$$E(X_i) = \frac{1}{p}\big[1 + E(X_{i-1})\big].$$

Now X_1 is a geometric random variable with parameter p. So $E(X_1) = 1/p$. Thus

$$E(X_2) = \frac{1}{p}\big[1 + E(X_1)\big] = \frac{1}{p}\Big(1 + \frac{1}{p}\Big),$$

$$E(X_3) = \frac{1}{p}\big[1 + E(X_2)\big] = \frac{1}{p}\Big(1 + \frac{1}{p} + \frac{1}{p^2}\Big),$$

$$\vdots$$

$$E(X_k) = \frac{1}{p}\Big(1 + \frac{1}{p} + \frac{1}{p^2} + \cdots + \frac{1}{p^{k-1}}\Big) = \frac{1}{p}\cdot\frac{(1/p^k) - 1}{(1/p) - 1} = \frac{1 - p^k}{p^k(1-p)}.$$

15. Let N be the number of games to be played until Emily wins two of the most recent three games. Let X be the number of games to be played until Emily wins a game for the first time. The random variable X is geometric with parameter 0.35. Hence $E(X) = 1/0.35$. First, we find the random variable $E(N \mid X)$ in terms of X. Then we obtain $E(N)$ by calculating the expected value of $E(N \mid X)$. Let W be the event that Emily wins the $(X+1)$st game as well. Let LW be the event that Emily loses the $(X+1)$st game but wins the $(X+2)$nd game. Let LL be the event that Emily loses both the $(X+1)$st and the $(X+2)$nd games. Given $X = x$, we have

$$E(N \mid X = x) = (x+1)P(W) + (x+2)P(LW) + \big[(x+2) + E(N)\big]P(LL).$$

So

$$E(N \mid X = x) = (x+1)(0.35) + (x+2)(0.65)(0.35) + \big[(x+2) + E(N)\big](0.65)^2.$$

This gives

$$E(N \mid X = x) = x + (0.4225)E(N) + 1.65.$$

Therefore,

$$E(N \mid X) = X + (0.4225)E(N) + 1.65.$$

Hence

$$E(N) = E\big[E(N \mid X)\big] = E(X) + (0.4225)E(N) + 1.65 = \frac{1}{0.35} + (0.4225)E(N) + 1.65.$$

Solving this for $E(N)$ gives $E(N) = 7.805$.

16. Since hemophilia is a sex-linked disease, and John is phenotypically normal, John is H. Therefore, no matter what Kim's genotype is, none of the daughters has hemophilia. Whether a boy has hemophilia or not depends solely on the genotype of Kim. Let X be the number of the boys who have hemophilia. To find, $E(X)$, the expected number of the boys who have hemophilia, let

$$Z = \begin{cases} 0 & \text{if Kim is } hh \\ 1 & \text{if Kim is } Hh \\ 2 & \text{if Kim is } HH. \end{cases}$$

Then

$$\begin{aligned} E(X) &= E\big[E(X \mid Z)\big] \\ &= E(X \mid Z = 0)P(Z = 0) + E(X \mid Z = 1)P(Z = 1) + E(X \mid Z = 2)P(Z = 2) \\ &= 4(0.02)(0.02) + 4(1/2)\big[2(0.98)(0.02)\big] + 0\big[0.98)(0.98)\big] = 0.08. \end{aligned}$$

Therefore, on average, 0.08 of the boys and hence 0.08 of the children are expected to have hemophilia.

17. Let X be the number of bags inspected until an unacceptable bag is found. Let K_n be the number of consequent bags inspected until n consecutive acceptable bags are found. The number of bags inspected in one inspection cycle is $X + K_m$. We are interested in $E(X + K_m) = E(X) + E(K_m)$. Clearly, X is a geometric random variable with parameter $\alpha(1 - p)$. So $E(X) = 1/\big[\alpha(1 - p)\big]$. To find $E(K_m)$, note that $\forall n$,

$$E(K_n) = E\big[E(K_n \mid K_{n-1})\big].$$

Now

$$\begin{aligned} E(K_n \mid K_{n-1} = i) &= (i + 1)p + \big[i + 1 + E(K_n)\big](1 - p) \\ &= (i + 1) + (1 - p)E(K_n). \end{aligned} \tag{41}$$

To derive this relation, we noted the following. It took i inspections to find $n - 1$ consecutive acceptable bags. If the next bag inspected is also acceptable, we have the n consecutive acceptable bags required in $i + 1$ inspections. This occurs with probability p. However, if the next bag inspected is unacceptable, then, on the average, we need an additional $E(K_n)$ inspections $\big[$a total of $i + 1 + E(K_n)$ inspections$\big]$ until we get n consecutive acceptable bags of cinnamon. This happens with probability $1 - p$.

From (41), we have

$$E(K_n \mid K_{n-1}) = (K_{n-1} + 1) + (1 - p)E(K_n).$$

Finding the expected values of both sides of this relation gives

$$E(K_n) = E(K_{n-1}) + 1 + (1 - p)E(K_n).$$

Solving for $E(K_n)$, we obtain

$$E(K_n) = \frac{1}{p} + \frac{E(K_{n-1})}{p}.$$

Noting that $E(K_1) = 1/p$ and solving recursively, we find that

$$E(K_n) = \frac{1}{p} + \frac{1}{p^2} + \cdots + \frac{1}{p^n}.$$

Therefore, the desired quantity is

$$\begin{aligned}
E(X + K_m) &= E(X) + E(K_m) \\
&= \frac{1}{\alpha(1-p)} + \frac{1}{p}\Big(1 + \frac{1}{p} + \cdots + \frac{1}{p^{m-1}}\Big) \\
&= \frac{1}{\alpha(1-p)} + \frac{1}{p} \cdot \frac{\Big(\dfrac{1}{p}\Big)^m - 1}{\dfrac{1}{p} - 1} = \frac{(1-\alpha)p^m + \alpha}{\alpha p^m (1-p)}.
\end{aligned}$$

18. For $0 < t \leq 1$, let $N(t)$ be the number of batteries changed by time t. Let X be the lifetime of the initial battery used; X is a uniform random variable over the interval (0, 1). Therefore, f_X, the probability density function of X, is given by

$$f_X(x) = \begin{cases} 1 & \text{if } 0 < x < 1 \\ 0 & \text{otherwise.} \end{cases}$$

We are interested in $K(t) = E\big[N(t)\big]$. Clearly,

$$\begin{aligned}
E\big[N(t)\big] &= E\Big[E\big[N(t) \mid X\big]\Big] = \int_0^\infty E\big[N(t) \mid X = x\big] f_X(x)\,dx \\
&= \int_0^t \Big[1 + E\big[N(t-x)\big]\Big]\,dx = t + \int_0^t E\big[N(t-x)\big]\,dx \\
&= t + \int_0^t K(u)\,du,
\end{aligned}$$

where the last equality follows from the substitution $u = t - x$. Differentiating both sides of $K(t) = t + \int_0^t K(u)\,du$ with respect to t, we obtain $K'(t) = 1 + K(t)$ which is equivalent to

$$\frac{K'(t)}{1 + K(t)} = 1.$$

Thus, for some constant c,

$$\ln\big[1 + K(t)\big] = t + c,$$

or,

$$1 + K(t) = e^{t+c}.$$

The initial condition $K(0) = E\big[N(0)\big] = 0$ yields $e^c = 1$; so

$$K(t) = e^t - 1.$$

On average, after 950 hours of operation, $K(0.95) = 1.586$ batteries are used.

19. Since $E(X|Y)$ is a function of Y, by Example 10.23,

$$E(XZ) = E\big[E(XZ|Y)\big] = E\Big[E\big[XE(X|Y)|Y\big]\Big]$$
$$= E\big[E(X|Y)E(X|Y)\big] = E(Z^2).$$

Therefore,

$$\begin{aligned} E\Big[\big(X - E(X|Y)\big)^2\Big] &= E\big[(X - Z)^2\big] \\ &= E(X^2 - 2ZX + Z^2) = E(X^2) - 2E(Z^2) + E(Z^2) \\ &= E(X^2) - E(Z^2) = E(X^2) - E\big[E(X|Y)^2\big]. \end{aligned}$$

20. Let $Z = E(X|Y)$; then

$$\begin{aligned} \mathrm{Var}(X|Y) &= E\big[(X - Z)^2|Y\big] \\ &= E(X^2 - 2XZ + Z^2|Y) \\ &= E(X^2|Y) - 2E(XZ|Y) + E(Z^2|Y). \end{aligned}$$

Since $E(X|Y)$ is a function of Y, by Example 10.23,

$$E(XZ|Y) = E\big[XE(X|Y)|Y\big] = E(X|Y)E(X|Y) = Z^2.$$

Also

$$E(Z^2|Y) = E\big[E(X|Y)^2|Y\big] = E(X|Y)^2 = Z^2$$

since, in general, $E\big[f(Y)|Y\big] = f(Y)$: if $Y = y$, then $E\big[f(Y)|Y\big]$ is defined to be

$$E\big[f(Y)|Y = y\big] = E\big[f(y)|Y = y\big] = f(y).$$

Therefore,

$$\mathrm{Var}(X|Y) = E(X^2|Y) - 2Z^2 + Z^2 = E(X^2|Y) - E(X|Y)^2.$$

21. By the definition of variance,

$$\mathrm{Var}\Big(\sum_{i=1}^{N} X_i\Big) = E\Big[\Big(\sum_{i=1}^{N} X_i\Big)^2\Big] - \Big[E\Big(\sum_{i=1}^{N} X_i\Big)\Big]^2, \qquad (42)$$

where by Wald's equation,

$$\Big[E\Big(\sum_{i=1}^{N} X_i\Big)\Big]^2 = \big[E(X)E(N)\big]^2 = \big[E(N)\big]^2 \cdot \big[E(X)\big]^2. \tag{43}$$

Now since N is independent of $\{X_1, X_2, \dots\}$,

$$\begin{aligned}
E\Big[\Big(\sum_{i=1}^{N} X_i\Big)^2\Big] &= E\Big[E\Big(\sum_{i=1}^{N} X_i\Big)^2 \;\Big|\; N\Big] \\
&= \sum_{n=1}^{\infty} E\Big[\Big(\sum_{i=1}^{N} X_i\Big)^2 \;\Big|\; N = n\Big] P(N = n) \\
&= \sum_{n=1}^{\infty} E\Big[\Big(\sum_{i=1}^{n} X_i\Big)^2 \;\Big|\; N = n\Big] P(N = n) \\
&= \sum_{n=1}^{\infty} E\Big(\sum_{i=1}^{n} X_i\Big)^2 P(N = n).
\end{aligned}$$

Thus

$$\begin{aligned}
E\Big[\Big(\sum_{i=1}^{N} X_i\Big)^2\Big] &= \sum_{n=1}^{\infty} E\Big(\sum_{i=1}^{n} X_i^2 + 2\sum\sum_{i<j} X_i X_j\Big) P(N = n) \\
&= \sum_{n=1}^{\infty} \Big[nE(X^2) + 2\sum\sum_{i<j} E(X_i)E(X_j)\Big] P(N = n) \\
&= E(X^2)\sum_{n=1}^{\infty} nP(N = n) + \sum_{n=1}^{\infty} 2\binom{n}{2} E(X)E(X)P(N = n) = \\
&= E(X^2)E(N) + \big[E(X)\big]^2 \sum_{n=1}^{\infty} n(n-1)P(N = n) \\
&= E(X^2)E(N) + \big[E(X)\big]^2 E\big[N(N-1)\big] \\
&= E(X^2)E(N) + \big[E(X)\big]^2 E(N^2) - \big[E(X)\big]^2 E(N).
\end{aligned}$$

Putting this and (43) in (42), we obtain

$$\begin{aligned}
\text{Var}\Big(\sum_{i=1}^{N} X_i\Big) &= E(X^2)E(N) + \big[E(X)\big]^2 E(N^2) - \big[E(X)\big]^2 E(N) - \big[E(N)\big]^2\big[E(X)\big]^2 \\
&= E(N)\Big(E(X^2) - \big[E(X)\big]^2\Big) + \big[E(X)\big]^2\Big(E(N^2) - \big[E(N)\big]^2\Big).
\end{aligned}$$

Therefore,

$$\text{Var}\Big(\sum_{i=1}^{N} X_i\Big) = E(N)\text{Var}(X) + [E(X)]^2\text{Var}(N).$$

10.5 BIVARIATE NORMAL DISTRIBUTION

1. The conditional probability density function of Y, given that $X = 70$ is normal with mean

$$E(Y \mid X = x) = \mu_Y + \rho\frac{\sigma_Y}{\sigma_X}(x - \mu_X) = 60 + (0.45)\Big(\frac{2.7}{3}\Big)(70 - 71) = 59.595,$$

and standard deviation

$$\sigma^2_{Y|X=x} = \sqrt{(1-\rho^2)\sigma_Y^2} = 2.7\sqrt{1-(0.45)^2} = 2.411.$$

Therefore, the desired probability is

$$\begin{aligned} P(Y \geq 59 \mid X = 70) &= P\Big(\frac{Y - 59.595}{2.411} \geq \frac{59 - 59.595}{2.411} \,\Big|\, X = 70\Big) \\ &= 1 - \Phi(-0.25) = \Phi(0.25) = 0.5987. \end{aligned}$$

2. By (10.24),

$$f(x, y) = \frac{1}{162\pi}\exp\Big[\frac{1}{162}(x^2 + y^2)\Big].$$

(a) Since $\rho = 0$, X and Y are independent normal random variables with mean 0 and standard deviation 9. Therefore,

$$\begin{aligned} P\big(X \leq 6,\ Y \leq 12\big) = P(X \leq 6)P(Y \leq 12) &= P\Big(\frac{X-0}{9} \leq \frac{6}{9}\Big)P\Big(\frac{Y-0}{12} \leq \frac{12}{9}\Big) \\ &= \Phi(0.67)\Phi(1.23) = (0.7486)(0.9082) = 0.68. \end{aligned}$$

(b) To find $P\big(X^2 + Y^2 \leq 36\big)$, we use polar coordinates.

$$\begin{aligned} P\big(X^2 + Y^2 \leq 36\big) &= \frac{1}{162\pi}\iint_{x^2+y^2\leq 36} \exp\Big[-\frac{1}{162}(x^2+y^2)\Big]\,dy\,dx \\ &= \frac{1}{2\pi}\int_0^{2\pi}\int_0^6 \exp\Big(-\frac{1}{162}r^2\Big)\cdot\frac{2r}{162}\,dr\,d\theta. \end{aligned}$$

Now let $u = r^2/162$; $du = (2r/162)dr$ and we get

$$P\big(X^2 + Y^2 \leq 36\big) = \frac{1}{2\pi}\int_0^{2\pi}\Big(\int_0^{2/9} e^{-u}\,du\Big)\,d\theta = 1 - e^{-2/9} = 0.8.$$

3. Note that

$$\text{Var}(\alpha X + Y) = \alpha^2\sigma_X^2 + \sigma_Y^2 + 2\alpha\rho(X, Y)\sigma_X\sigma_Y.$$

Setting $\dfrac{d}{d\alpha}\text{Var}(\alpha X + Y) = 0$, we get $\alpha = -\rho(X, Y)\dfrac{\sigma_Y}{\sigma_X}$.

4. By (10.24), $f(x, y)$ is maximum if and only if $Q(x, y)$ is minimum. Let $z_1 = \dfrac{x - \mu_X}{\sigma_X}$ and $z_2 = \dfrac{y - \mu_Y}{\sigma_Y}$. Then $|\rho| \le 1$ implies that

$$\begin{aligned} Q(x, y) &= z_1^2 - 2\rho z_1 z_2 + z_2^2 \ge z_1^2 - 2|\rho z_1 z_2| + z_2^2 \\ &\ge z_1^2 - 2|z_1 z_2| + z_2^2 = \big(|z_1| - |z_2|\big)^2 \ge 0. \end{aligned}$$

This inequality shows that Q is minimum if $Q(x, y) = 0$. This happens at $x = \mu_X$ and $y = \mu_Y$. Therefore, (μ_X, μ_Y) is the point at which the maximum of f is obtained.

5. We have that

$$f_X(x) = \int_0^x 2\,dy = 2x, \quad 0 < x < 1,$$

$$f_Y(y) = \int_y^1 2\,dx = 2(1 - y), \quad 0 < y < 1,$$

$$f_{X|Y}(x|y) = \frac{2}{2(1 - y)} = \frac{1}{1 - y}, \quad y < x < 1.$$

$$f_{Y|X}(y|x) = \frac{2}{2x} = \frac{1}{x}, \quad 0 < y < x.$$

Therefore,

$$E(X \mid Y = y) = \int_y^1 x f_{X|Y}(x|y)\,dx = \int_y^1 x\frac{1}{1 - y}\,dx = \frac{1 + y}{2}, \quad 0 < y < 1,$$

$$E(Y \mid X = x) = \int_0^x y f_{Y|X}(y|x)\,dy = \int_0^x y\frac{1}{x}\,dy = \frac{x}{2}, \quad 0 < x < 1.$$

Now since $E(Y \mid X = x)$ is a linear function of x and $E(X \mid Y = y)$ is a linear function of y, by Lemma 10.3,

$$\mu_Y + \rho\frac{\sigma_Y}{\sigma_X}(x - \mu_X) = \frac{x}{2}$$

and

$$\mu_X + \rho\frac{\sigma_X}{\sigma_Y}(y - \mu_Y) = \frac{1+y}{2}.$$

These relations imply that

$$\rho\frac{\sigma_Y}{\sigma_X} = \frac{1}{2} \quad \text{and} \quad \rho\frac{\sigma_X}{\sigma_Y} = \frac{1}{2}.$$

Hence $\rho > 0$ and $\rho^2 = \rho\frac{\sigma_Y}{\sigma_X}\cdot\rho\frac{\sigma_X}{\sigma_Y} = \frac{1}{4}$. Therefore $\rho = 1/2$.

6. We use Theorem 8.8 to find the joint probability density function of X and Y. The joint probability density function of Z and W is given by

$$f(z, w) = \frac{1}{2\pi}\exp\Big[-\frac{1}{2}\big(z^2 + w^2\big)\Big].$$

Let $h_1(z, w) = \sigma_1 z + \mu_1$ and $h_2(z, w) = \sigma_2\big(\rho z + \sqrt{1-\rho^2}\, w\big) + \mu_2$. The system of equations

$$\begin{cases} \sigma_1 z + \mu_1 = x \\ \sigma_2\big(\rho z + \sqrt{1-\rho^2}\, w\big) + \mu_2 = y \end{cases}$$

defines a one-to-one transformation of $\mathbf{R}^2$ in the zw-plane onto $\mathbf{R}^2$ in the xy-plane. It has a unique solution

$$z = \frac{x - \mu_1}{\sigma_1},$$

$$w = \frac{1}{\sqrt{1-\rho^2}}\Big[\frac{y-\mu_2}{\sigma_2} - \frac{\rho(x-\mu_1)}{\sigma_1}\Big]$$

for z and w in terms of x and y. Moreover,

$$\mathbf{J} = \begin{vmatrix} \dfrac{1}{\sigma_1} & 0 \\ -\dfrac{\rho}{\sigma_1\sqrt{1-\rho^2}} & \dfrac{1}{\sigma_2\sqrt{1-\rho^2}} \end{vmatrix} = \frac{1}{\sigma_1\sigma_2\sqrt{1-\rho^2}} \neq 0.$$

Hence, by Theorem 8.8, the joint probability density function of X and Y is given by

$$\frac{1}{\sigma_1\sigma_2\sqrt{1-\rho^2}}\, f\Big(\frac{x-\mu_1}{\sigma_1},\ \frac{1}{\sqrt{1-\rho^2}}\Big[\frac{y-\mu_2}{\sigma_2} - \rho\frac{x-\mu_1}{\sigma_1}\Big]\Big).$$

Noting that $f(z, w) = \frac{1}{2\pi}\exp\Big[-\frac{1}{2}\big(z^2 + w^2\big)\Big]$. Straightforward calculations will result in (10.24), showing that the joint probability density function of X and Y is bivariate normal.

7. Using Theorem 8.8, it is straightforward to show that the joint probability density function of $X + Y$ and $X - Y$ is bivariate normal. Since

$$\rho(X+Y, X-Y) = \frac{\text{Cov}(X+Y, X-Y)}{\sigma_{X+Y} \cdot \sigma_{X-Y}} = \frac{\text{Var}(X) - \text{Var}(Y)}{\sigma_{X+Y} \cdot \sigma_{X-Y}} = 0,$$

$X + Y$ and $X - Y$ are uncorrelated. But for bivariate normal, uncorrelated and independence are equivalent. So $X + Y$ and $X - Y$ are independent.

REVIEW PROBLEMS FOR CHAPTER 10

1. Number the last 10 graduates who will walk on the stage 1 through 10. Let $X_i = 1$ if the ith graduate receives his or her own diploma; 0, otherwise. The number of graduates who will receive their own diploma is $X = X_1 + X_2 + \cdots + X_n$. Since

$$E(X_i) = 1 \cdot \frac{1}{n} + 0 \cdot \Big(1 - \frac{1}{n}\Big) = \frac{1}{n},$$

we have

$$E(X) = E(X_1) + E(X_2) + \cdots + E(X_n) = n \cdot \frac{1}{n} = 1.$$

2. Since

$$E(X) = \int_1^2 (2x^2 - 2x)\,dx = \frac{5}{3},$$

and

$$E(X^3) = \int_1^2 (2x^4 - 2x^3)\,dx = \frac{49}{10},$$

we have that

$$E(X^3 + 2X - 7) = \frac{49}{10} + \frac{10}{3} - 7 = \frac{37}{30}.$$

3. Since

$$E(X^2) = \frac{1}{3}\int_0^1 \int_0^2 (3x^5 + x^3 y)\,dy\,dx = \frac{1}{2},$$

and

$$E(XY) = \frac{1}{3}\int_0^1 \int_0^2 (3x^4 y + x^2 y^2)\,dy\,dx = \frac{94}{135},$$

we have that $E(X^2 + 2XY) = \dfrac{1}{2} + \dfrac{188}{135} = \dfrac{511}{270}$.

4. Let $X_1, X_2, \ldots, X_n$ be geometric random variables with parameters $1, (n-1)/n, (n-2)/n, \ldots, 1/n$, respectively. The desired quantity is

$$\begin{aligned} E(X_1 + X_2 + \cdots + X_n) &= 1 + \frac{n}{n-1} + \frac{n}{n-2} + \cdots + n \\ &= 1 + n\Big(\frac{1}{n-1} + \frac{1}{n-2} + \cdots + \frac{1}{2} + 1\Big) = 1 + na_{n-1}. \end{aligned}$$

5. Let X be the number of tosses until 4 consecutive sixes. Let Y be the number of tosses until the first non-six outcome is obtained. We have

$$\begin{aligned} E(X) = E\big[E(X|Y)\big] &= \sum_{i=1}^{\infty} E(X \mid Y = i)P(Y = i) \\ &= \sum_{i=1}^{4} E(X \mid Y = i)P(Y = i) + \sum_{i=5}^{\infty} E(X \mid Y = i)P(Y = i) \\ &= \sum_{i=1}^{4} \big[i + E(X)\big]\Big(\frac{1}{6}\Big)^{i-1}\Big(\frac{5}{6}\Big) + \sum_{i=5}^{\infty} 4\Big(\frac{1}{6}\Big)^{i-1}\Big(\frac{5}{6}\Big). \end{aligned}$$

This equation reduces to

$$\begin{aligned} E(X) = \big[1 + E(X)\big]\frac{5}{6} + \big[2 + E(X)\big]\frac{1}{6}\cdot\frac{5}{6} + \big[3 + E(X)\big]\Big(\frac{1}{6}\Big)^2\Big(\frac{5}{6}\Big) \\ + \big[4 + E(X)\big]\Big(\frac{1}{6}\Big)^3\Big(\frac{5}{6}\Big) + 4\Big(\frac{5}{6}\Big)\frac{(1/6)^4}{1 - (1/6)}. \end{aligned}$$

Solving this equation for $E(X)$, we obtain $E(X) = 1554$.

6. $f(x, y, z) = (2x)(2y)(2z)$, $0 < x < 1$, $0 < y < 1$, $0 < z < 1$. Since $2x$, $0 < x < 1$ is a probability density function, $2y$, $0 < y < 1$ is a probability density function, and $2z$, $0 < z < 1$ is also a probability density function, these three functions are $f_X(x)$, $f_Y(y)$, and $f_Z(z)$, respectively. Therefore, $f(x, y, z) = f_X(x)f_Y(y)f_Z(z)$ showing that X, Y, and Z are independent. Thus

$$\rho(X, Y) = \rho(Y, Z) = \rho(X, Z) = 0.$$

7. Since $\text{Cov}(X, Y) = \sigma_X\sigma_Y\rho(X, Y) = 2$,

$$\begin{aligned} \text{Var}(3X - 5Y + 7) = \text{Var}(3X - 5Y) &= 9\text{Var}(X) + 25\text{Var}(Y) - 15\text{Cov}(X, Y) \\ &= 9 + 225 - 30 = 204. \end{aligned}$$

8. Clearly,

$$p_X(1) = p(1, 1) + p(1, 3) = 12/25, \qquad p_X(2) = p(2, 3) = 13/25;$$
$$p_Y(1) = p(1, 1) = 2/25, \qquad p_Y(3) = p(1, 3) + p(2, 3) = 23/25.$$

Therefore,

$$p_X(x) = \begin{cases} 12/25 & \text{if } x = 1 \\ 13/25 & \text{if } x = 2, \end{cases} \qquad p_Y(y) = \begin{cases} 2/25 & \text{if } y = 1 \\ 23/25 & \text{if } y = 3. \end{cases}$$

These yield

$$E(X) = 1 \cdot \frac{12}{25} + 2 \cdot \frac{13}{25} = \frac{38}{25};$$
$$E(Y) = 1 \cdot \frac{2}{25} + 3 \cdot \frac{23}{25} = \frac{71}{25};$$
$$E(XY) = (1)(1)\frac{1}{25}(1^2 + 1^2) + (1)(3)\frac{1}{25}(1^2 + 3^2) + (2)(3)\frac{1}{25}(2^2 + 3^2) = \frac{22}{5}.$$

Thus

$$\text{Cov}(X, Y) = E(XY) - E(X)E(Y) = \frac{22}{5} - \frac{38}{25} \cdot \frac{71}{25} = \frac{52}{625}.$$

9. In Exercise 6, Section 8.1, we calculated $p(x, y)$, $p_X(x)$, and $p_Y(y)$. The results of that exercise yield

$$E(X) = \sum_{x=2}^{12} x p_X(x) = 7;$$

$$E(Y) = \sum_{y=0}^{5} y p_Y(y) = 35/18;$$

$$E(XY) = \sum_{x=2}^{12} \sum_{y=0}^{5} xyp(x, y) = 245/18.$$

Therefore,

$$\text{Cov}(X, Y) = E(XY) - E(X)E(Y) = (245/18) - 7(35/18) = 0.$$

This shows that X and Y are uncorrelated. Note that X and Y are not independent as the following shows.

$$1/36 = p(2, 0) \neq p_X(2)p_Y(0) = (1/36)(6/36) = 1/216.$$

10. Let p be the probability mass function of $|X - Y|$, q be the probability mass function of $X + Y$, and r be the probability mass function of $|X^2 - Y^2|$. We have

x	0	1	2
$p(x)$	726/1296	520/1296	50/1296,

x	0	1	2	3	4
$q(x)$	625/1296	500/1296	150/1296	20/1296	1/1296,

x	0	1	3	4
$r(x)$	726/1296	500/1296	20/1296	50/1296.

Using these we obtain

$$E\big(|X^2-Y^2|\big)=\frac{760}{1296},\qquad E\big(|X-Y|\big)=\frac{620}{1296},\qquad E(X+Y)=\frac{864}{1296},$$
$$E\big(|X-Y|^2\big)=\frac{720}{1296},\qquad E\big[(X+Y)^2\big]=1,\qquad \sigma_{|X-Y|}=0.572,$$
$$\sigma_{X+Y}=0.831.$$

Therefore,

$$\begin{aligned}\rho\big(|X-Y|,\ |X+Y|\big)&=\frac{\mathrm{Cov}\big(|X-Y|,\ X+Y\big)}{\sigma_{|X-Y|}\cdot\sigma_{X+Y}}\\&=\frac{E\big(|X^2-Y^2|\big)-E\big(|X-Y|\big)E(X+Y)}{\sigma_{|X-Y|}\cdot\sigma_{X+Y}}\\&=\frac{(760/1296)-(620/1296)(864/1296)}{(0.831)(0.572)}=0.563.\end{aligned}$$

11. One way to solve this problem is to note that the desired probability is the area of the region under the curve $y=\sin x$ from $x=0$ to $x=\pi/2$ divided by the area of the rectangle $[0,\pi/2]\times[0,1]$. Hence it is

$$\frac{\displaystyle\int_0^{\pi/2}\sin x\,dx}{\pi/2}=\frac{2}{\pi}.$$

A second way to find this probability is to note that (X,Y) lies below the curve $y=\sin x$ if and only if $Y<\sin X$. Noting that f, the probability density function of X is given by

$$f(x)=\begin{cases}\dfrac{2}{\pi} & \text{if } 0<x<\dfrac{\pi}{2}\\[2mm] 0 & \text{otherwise,}\end{cases}$$

and conditioning on X, we obtain

$$\begin{aligned}P(Y<\sin X)&=\int_0^{\pi/2}P(Y<\sin X\mid X=x)f(x)dx=\int_0^{\pi/2}\frac{\sin x-0}{1-0}\cdot\frac{2}{\pi}\,dx\\&=-\frac{2}{\pi}\cos x\Big|_0^{\pi/2}=\frac{2}{\pi}.\end{aligned}$$

12. **(a)** Clearly,

$$f_X(x) = \int_0^x e^{-x}\,dy = xe^{-x}, \quad 0 < x < \infty,$$

$$f_Y(y) = \int_y^\infty e^{-x}\,dx = e^{-y}, \quad 0 < y < \infty.$$

(b) we have that

$$E(X) = \int_0^\infty x^2 e^{-x} = 2, \qquad E(Y) = \int_0^\infty ye^{-y} = 1,$$

$$E\big(X^2\big) = \int_0^\infty x^3 e^{-x}\,dx = 6, \qquad E\big(Y^2\big) = \int_0^\infty y^2 e^{-y} = 2.$$

Therefore, $\text{Var}(X) = 2$ and $\text{Var}(Y) = 1$. Also

$$E(XY) = \int_0^\infty \int_y^\infty e^{-x}\,dx\,dy = 3.$$

Thus

$$\rho(X, Y) = \frac{E(XY) - E(X)E(Y)}{\sigma_X \sigma_Y} = \frac{3-2}{\sqrt{2}\cdot 1} = \frac{1}{\sqrt{2}}.$$

13. Let $h(\alpha, \beta) = E\big[(Y - \alpha - \beta X)^2\big]$. Then

$$h(\alpha, \beta) = E\big(Y^2\big) + \alpha^2 + \beta^2 E\big(X^2\big) - 2\alpha E(Y) - 2\beta E(XY) + 2\alpha\beta E(X).$$

Setting $\dfrac{\partial h}{\partial \alpha} = 0$ and $\dfrac{\partial h}{\partial \beta} = 0$, we obtain

$$\begin{cases} \alpha + E(X)\beta = E(Y) \\ E(X)\alpha + E\big(X^2\big)\beta = E(XY). \end{cases}$$

Solving this system of two equations in two unknowns, we obtain

$$\beta = \frac{\text{Cov}(X, Y)}{\sigma_X^2} = \frac{\rho\sigma_X\sigma_Y}{\sigma_X^2} = \rho\frac{\sigma_Y}{\sigma_X},$$

$$\alpha = \mu_Y - \rho\frac{\sigma_Y}{\sigma_X}\mu_X.$$

Therefore, $Y = \mu_Y + \rho\dfrac{\sigma_Y}{\sigma_X}(X - \mu_X)$.

14. We have that

$$E(X) = \int_0^\infty \int_0^\infty xye^{-y(1+x)}\,dy\,dx = \int_0^\infty \frac{x}{1+x}\left(\int_0^\infty (1+x)ye^{-y(1+x)}\,dy\right)dx.$$

Now $\int_0^\infty (1+x)ye^{-y(1+x)}\,dy$ is the expected value of an exponential random variable with parameter $1+x$, so it is $1/(1+x)$. Letting $u = 1+x$, we have

$$E(X) = \int_0^\infty \frac{x}{(1+x)^2}\,dx = \int_1^\infty \frac{u-1}{u^2}\,du$$
$$= \int_1^\infty \frac{1}{u}\,du - \int_1^\infty \frac{1}{u^2}\,du = \ln u \Big|_1^\infty - 1 = \infty.$$

(b) To find $E(X|Y)$, note that

$$E(X \mid Y = y) = \int_0^\infty x f_{X|Y}(x|y)\,dx = \int_0^\infty x \frac{f(x,y)}{f_Y(y)}\,dx,$$

where

$$f_Y(y) = \int_0^\infty ye^{-y(1+x)}\,dx = e^{-y}\int_0^\infty ye^{-yx}\,dx = e^{-y}.$$

Note that $\int_0^\infty ye^{-yx}\,dx = 1$ because ye^{-yx} is the probability density function of an exponential random variable with parameter 1. So

$$E(X \mid Y = y) = \int_0^\infty x \frac{ye^{-y}e^{-yx}}{e^{-y}}\,dx = \int_0^\infty xye^{-xy}\,dx = \frac{1}{y},$$

where the last equality holds because the last integral is the expected value of an exponential random variable with parameter y. Since $\forall y > 0$, $E(X \mid Y = y) = 1/y$, $E(X|Y) = 1/Y$.

15. Let X and Y denote the number of minutes past 10:00 A.M. that bus A and bus B arrive at the station, respectively. X is uniformly distributed over $(0, 30)$. Given that $X = x$, Y is uniformly distributed over $(0, x)$. Let $f(x, y)$ be the joint probability density function of X and Y. We calculate $E(Y)$ by conditioning on X:

$$E(Y) = E\big[E(Y|X)\big] = \int_{-\infty}^{\infty} E(Y \mid X = x) f_X(x)\,dx = \int_0^{30} \frac{x}{2}\cdot\frac{1}{30}\,dx = \frac{30}{4}.$$

Thus the expected arrival time of bus B is 7.5 minutes past 10:00 A.M.

16. To find the distribution function of $\sum_{i=1}^N X_i$, note that

$$P\Big(\sum_{i=1}^N X_i \le t\Big) = \sum_{n=1}^\infty P\Big(\sum_{i=1}^N X_i \le t \;\Big|\; N = n\Big)P(N = n)$$
$$= \sum_{n=1}^\infty P\Big(\sum_{i=1}^n X_i \le t \;\Big|\; N = n\Big)P(N = n)$$
$$= \sum_{n=1}^\infty P\Big(\sum_{i=1}^n X_i \le t\Big)P(N = n),$$

where the last inequality follows since N is independent of $\{X_1, X_2, X_3, \dots\}$. Now $\sum_{i=1}^{n} X_i$ is a gamma random variable with parameters n and λ. Thus

$$\begin{aligned}
P\Big(\sum_{i=1}^{N} X_i \le t\Big) &= \sum_{n=1}^{\infty}\Big[\int_0^t \lambda e^{-\lambda x}\frac{(\lambda x)^{n-1}}{(n-1)!}\,dx\Big](1-p)^{n-1}p \\
&= \sum_{n=1}^{\infty}\int_0^t \lambda p e^{-\lambda x}\frac{\big[\lambda(1-p)x\big]^{n-1}}{(n-1)!}\,dx \\
&= \int_0^t \lambda p e^{-\lambda x}\sum_{n=1}^{\infty}\frac{\big[\lambda(1-p)x\big]^{n-1}}{(n-1)!}\,dx \\
&= \int_0^t \lambda p e^{-\lambda x}e^{\lambda(1-p)x}\,dx \\
&= \int_0^t \lambda p e^{-\lambda p x}\,dx = 1 - e^{-\lambda p t}.
\end{aligned}$$

This shows that $\sum_{i=1}^{N} X_i$ is exponential with parameter λp.

17. Let $X_1, X_2, \dots, X_i, \dots, X_{20}$ be geometric random variables with parameters 1, 19/20, ..., $\big[20-(i-1)\big]/20$, ..., 1/20. The desired quantity is

$$E\Big(\sum_{i=1}^{20} X_i\Big) = \sum_{i=1}^{20} E(X_i) = \sum_{i=1}^{20}\frac{20}{20-(i-1)} = 71.9548.$$

Chapter 11

Sums of Independent Random Variables and Limit Theorems

11.1 MOMENT-GENERATING FUNCTIONS

1. $M_X(t) = E\big(e^{tX}\big) = \sum_{x=1}^{5} e^{tx} p(x) = \frac{1}{5}\big(e^t + e^{2t} + e^{3t} + e^{4t} + e^{5t}\big).$

2. **(a)** For $t \neq 0$,

$$M_X(t) = E\big(e^{tX}\big) = \int_{-1}^{3} \frac{1}{4} e^{tx}\,dx = \frac{1}{4}\Big(\frac{e^{3t} - e^{-t}}{t}\Big),$$

whereas for $t = 0$, $M_X(0) = 1$. Thus

$$M_X(t) = \begin{cases} \frac{1}{4}\Big(\frac{e^{3t} - e^{-t}}{t}\Big) & \text{if } t \neq 0 \\ 1 & \text{if } t = 0. \end{cases}$$

Since X is uniform over $(-1, 3)$, $E(X) = \frac{-1+3}{2} = 1$ and $\text{Var}(X) = \frac{\big[3-(-1)\big]^2}{12} = \frac{4}{3}$.
(b) By the definition of derivative,

$$E(X) = M_X'(0) = \lim_{h \to 0} \frac{M_X(h) - M_X(0)}{h} = \lim_{h \to 0} \frac{1}{h}\Big(\frac{e^{3h} - e^{-h}}{4h} - 1\Big)$$

$$= \lim_{h \to 0} \frac{e^{3h} - e^{-h} - 4h}{4h^2} = \lim_{h \to 0} \frac{3e^{3h} + e^{-h} - 4}{8h} = \lim_{h \to 0} \frac{9e^{3h} - e^{-h}}{8} = 1,$$

where the fifth and sixth equalities follow from L'Hôpital's rule.

3. Note that

$$M_X(t) = E\big(e^{tX}\big) = \sum_{x=1}^{\infty} e^{tx} \cdot 2\Big(\frac{1}{3}\Big)^x = 2\sum_{x=1}^{\infty} e^{tx} \cdot e^{-x\ln 3} = 2\sum_{x=1}^{\infty} e^{x(t-\ln 3)}.$$

Restricting the domain of $M_X(t)$ to the set $\{t\colon t < \ln 3\}$ and using the geometric series theorem, we get

$$M_X(t) = 2\Big(\frac{e^{t-\ln 3}}{1-e^{t-\ln 3}}\Big) = \frac{2e^t}{3-e^t}.$$

(Note that $e^{-\ln 3} = 1/3$.) Differentiating $M_X(t)$, we obtain

$$M_X'(t) = \frac{6e^t}{\big(3-e^t\big)^2},$$

which gives $E(X) = M_X'(0) = 3/2$.

4. For $t = 0$, $M_X(0) = 1$. For $t \neq 0$, using integration by parts, we obtain

$$M_X(t) = \int_0^1 2xe^{tx}\,dx = \frac{2e^t}{t} - \frac{2e^t}{t^2} + \frac{2}{t^2}.$$

5. **(a)** For $t = 0$, $M_X(0) = 1$. For $t \neq 0$,

$$M_X(t) = \int_0^1 e^{tx} \cdot 6x(1-x)\,dx = 6\int_0^1 xe^{tx}\,dx - 6\int_0^1 x^2e^{tx}\,dx$$

$$= 6\Big(\frac{e^t}{t} - \frac{e^t}{t^2} + \frac{1}{t^2}\Big) - 6\Big(\frac{e^t}{t} - \frac{2e^t}{t^2} + \frac{2e^t}{t^3} - \frac{2}{t^3}\Big) = \frac{12(1-e^t)}{t^3} + \frac{6(1+e^t)}{t^2}.$$

(b) By the definition of derivative,

$$E(X) = M_X'(0) = \lim_{t\to 0}\frac{M_X(t) - M_X(0)}{t} = \lim_{t\to 0}\frac{\dfrac{12(1-e^t)}{t^3} + \dfrac{6(1+e^t)}{t^2} - 1}{t}$$

$$= \lim_{t\to 0}\frac{12(1-e^t) + 6t(1+e^t) - t^3}{t^4} = \frac{1}{2},$$

where the last equality is calculated by applying L'Hôpital's rule four times.

6. Let A be the set of possible values of X. Clearly, $M_X(t) = \sum_{x\in A} e^{tx}p(x)$, where $p(x)$ is the probability mass function of X. Therefore,

$$M'_X(t) = \sum_{x \in A} x e^{tx} p(x),$$
$$M''_X(t) = \sum_{x \in A} x^2 e^{tx} p(x),$$
$$\vdots$$
$$M^{(n)}_X(t) = \sum_{x \in A} x^n e^{tx} p(x).$$

Therefore,

$$M^{(n)}_X(0) = \sum_{x \in A} x^n p(x) = E(X^n).$$

7. **(a)** By definition,

$$M_X(t) = E\big(e^{tX}\big) = \sum_{x=0}^{\infty} e^{tx} \frac{e^{-\lambda}\lambda^x}{x!} = e^{-\lambda} \sum_{x=0}^{\infty} \frac{(\lambda e^t)^x}{x!} = e^{-\lambda} \exp(\lambda e^t) = \exp\big[\lambda(e^t - 1)\big].$$

(b) From

$$M'_X(t) = \lambda e^t \exp\big[\lambda(e^t - 1)\big]$$

and

$$M''_X(t) = \big(\lambda e^t\big)^2 \exp\big[\lambda(e^t - 1)\big] + \lambda e^t \exp\big[\lambda(e^t - 1)\big],$$

we obtain $E(X) = M'_X(0) = \lambda$ and $E(X^2) = M''_X(0) = \lambda^2 + \lambda$. Therefore,

$$\text{Var}(X) = (\lambda^2 + \lambda) - \lambda^2 = \lambda.$$

8. The probability density function of X is given by

$$f(x) = \begin{cases} \dfrac{1}{b-a} & \text{if } a < x < b \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, for $t \neq 0$,

$$M_X(t) = E\big(e^{tX}\big) = \int_a^b \frac{1}{b-a} e^{tx}\, dx = \frac{1}{b-a}\Big(\frac{e^{tb} - e^{ta}}{t}\Big),$$

whereas for $t = 0$, $M_X(0) = 1$. Thus

$$M_X(t) = \begin{cases} \dfrac{1}{b-a}\Big(\dfrac{e^{tb} - e^{ta}}{t}\Big) & \text{if } t \neq 0 \\ 1 & \text{if } t = 0. \end{cases}$$

9. The probability mass function of a geometric random variable X, $p(x)$ with parameter p is given by

$$p(x) = pq^{x-1}, \quad q = 1 - p, \quad x = 1, 2, 3, \dots .$$

Thus

$$M_X(t) = \sum_{x=1}^{\infty} pq^{x-1}e^{tx} = \frac{p}{q}\sum_{x=1}^{\infty}\big(qe^t\big)^x.$$

Now by the geometric series theorem, $\sum_{x=1}^{\infty}\big(qe^t\big)^x$ converges to $\big(qe^t\big)/\big(1-qe^t\big)$ if $qe^t < 1$ or, equivalently, if $t < -\ln q$. Restricting the domain of $M_X(t)$ to the set $\{t: t < -\ln q\}$, we obtain

$$M_X(t) = \frac{p}{q}\sum_{x=1}^{\infty}\big(qe^t\big)^x = \frac{p}{q}\cdot\frac{qe^t}{1-qe^t} = \frac{pe^t}{1-qe^t}.$$

Now

$$M_X'(t) = \frac{pe^t}{(1-qe^t)^2} \quad \text{and} \quad M_X''(t) = \frac{pe^t + pqe^{2t}}{(1-qe^t)^3}.$$

Therefore,

$$E(X) = M_X'(0) = \frac{p}{(1-q)^2} = \frac{1}{p}.$$

and

$$E(X^2) = M_X''(0) = \frac{p(1+q)}{(1-q)^3} = \frac{1+q}{p^2}.$$

Thus

$$\text{Var}(X) = E(X^2) - \big[E(X)\big]^2 = \frac{1+q}{p^2} - \frac{1}{p^2} = \frac{q}{p^2}.$$

10. Let X be a discrete random variable with the probability mass function $p(x) = x/21$, $x = 1, 2, 3, 4, 5, 6$. The moment-generating function of X is the given function.

11. X is a discrete random variable with the set of possible values $\{1, 3, 4, 5\}$ and probability mass function

x	1	3	4	5
$p(x)$	5/15	4/15	2/15	4/15.

12. We have that

$$M_{2X+1}(t) = E\big[e^{(2X+1)t}\big] = e^tE\big(e^{2tX}\big) = e^tM_X(2t) = \frac{e^t}{1-2t}, \quad t < \frac{1}{2}.$$

13. Note that

$$M_X'(t) = \frac{24}{(2-t)^4}, \quad M_X''(t) = \frac{96}{(2-t)^5}.$$

Therefore,

$$E(X) = M_X'(0) = \frac{24}{16} = \frac{3}{2}, \quad E(X^2) = M_X''(0) = \frac{96}{32} = 3,$$

and hence $\text{Var}(X) = 3 - (9/4) = 3/4$.

14. Since for odd r's, $M_X^{(r)}(t) = (e^t - e^{-t})/6$ and for even r's, $M_X^{(r)}(t) = (e^t + e^{-t})/6$, we have that $E(X^r) = 0$ if r is odd and $E(X^r) = 1/3$ if r is even.

15. For a random variable X, we must have $M_X(0) = 1$. Since $t/(1-t)$ is 0 at 0, it cannot be a moment-generating function.

16. **(a)** The distribution of X is binomial with parameters 7 and 1/4.
(b) The distribution of X is geometric with parameter 1/2.
(c) The distribution of X is gamma with parameters r and 2.
(d) The distribution of X is Poisson with parameter $\lambda = 3$.

17. Since

$$M_X(t) = \Big(\frac{1}{3}e^t + \frac{2}{3}\Big)^4,$$

X is a binomial random variable with parameters 4 and 1/3; therefore,

$$P(X \leq 2) = \sum_{i=0}^{2} \binom{4}{i} \Big(\frac{1}{3}\Big)^i \Big(\frac{2}{3}\Big)^{4-i} = \frac{8}{9}.$$

18. By relation (11.2),

$$M_X(t) = \sum_{n=0}^{\infty} \frac{2^n}{n!} t^n = \sum_{n=0}^{\infty} \frac{(2t)^n}{n!} = e^{2t}.$$

This shows that $X = 2$ with probability 1.

19. We know that for $t \neq 0$,

$$M_X(t) = \frac{e^t - 1}{t(1-0)} = \frac{e^t - 1}{t}.$$

Therefore, for $t \neq 0$,

$$M_{aX+b}(t) = E\big[e^{t(aX+b)}\big] = e^{bt} E\big[e^{atX}\big] = e^{bt} M_X(at)$$
$$= e^{bt} \cdot \frac{e^{at} - 1}{at} = \frac{e^{(a+b)t} - e^{bt}}{\big[(a+b) - b\big]t},$$

which is the moment-generating function of a uniform random variable over $(b, a+b)$.

20. Let $\mu_n = E(Z^n)$; then

$$M_X(t) = \sum_{n=0}^{\infty} \frac{M_X^n(0)}{n!} t^n = \sum_{n=0}^{\infty} \frac{\mu_n}{n!} t^n. \tag{44}$$

Now $e^t = \sum_{n=0}^{\infty} (t^n/n!)$. Therefore,

$$e^{t^2/2} = \sum_{n=0}^{\infty} \frac{(t^2/2)^n}{n!} = \sum_{n=0}^{\infty} \frac{t^{2n}}{2^n\, n!} = \sum_{n=0}^{\infty} \frac{(2n)!}{2^n\, n!} \frac{t^{2n}}{(2n)!}.$$

comparing this relation with (44), we obtain $E(Z^{2n+1}) = 0$, $\forall n \geq 0$ and $E(Z^{2n}) = \dfrac{(2n)!}{2^n\, n!}$, $\forall n \geq 1$.

21. By definition,

$$M_X(t) = \frac{\lambda^r}{\Gamma(r)} \int_0^{\infty} e^{tx} x^{r-1} e^{-\lambda x}\, dx = \frac{\lambda^r}{\Gamma(r)} \int_0^{\infty} e^{(t-\lambda)x} x^{r-1}\, dx.$$

This integral converges if $t < \lambda$. Therefore, if we restrict the range of $M_X(t)$ to $t < \lambda$, by the substitution $u = (\lambda - t)x$, we obtain

$$M_X(t) = \frac{\lambda^r}{\Gamma(r)} \int_0^{\infty} \frac{e^{-u} u^{r-1}}{(\lambda - t)^r}\, du = \frac{\lambda^r}{\Gamma(r)} \cdot \frac{\Gamma(r)}{(\lambda - t)^r} = \Big(\frac{\lambda}{\lambda - t}\Big)^r.$$

Now $M'_X(t) = r\lambda^r(\lambda - t)^{-r-1}$; thus $E(X) = M'_X(0) = r/\lambda$. Also

$$M''_X(t) = r(r+1)\lambda^r(\lambda - t)^{-r-2};$$

therefore, $E(X^2) = M''_X(0) = \big[r(r+1)\big]/\lambda^2$, and hence

$$\text{Var}(X) = \frac{r(r+1)}{\lambda^2} - \Big(\frac{r}{\lambda}\Big)^2 = \frac{r}{\lambda^2}.$$

22. **(a)** Let F be the distribution function of X. We have that

$$P(-X \leq t) = P(X \geq -t) = \int_{-t}^{\infty} f(x)\, dx.$$

Letting $u = -x$ and noting that $f(-u) = f(u)$, we obtain

$$P(-X \leq t) = \int_{t}^{-\infty} f(-u)\,(-\,du) = \int_{-\infty}^{t} f(u)\, du = F(t).$$

This shows that the distribution function of $-X$ is also F.
(b) Clearly,

$$M_X(-t) = \int_{-\infty}^{\infty} e^{-tx} f(x)\, dx.$$

Letting $u = -x$, we get

$$M_X(-t) = \int_{-\infty}^{\infty} e^{tu} f(-u)\, du = \int_{-\infty}^{\infty} e^{tu} f(u)\, du = M_X(t).$$

A second way to explain this is to note that $M_X(-t)$ is the moment-generating function of $-X$. Since X and $-X$ are identically distributed, we must have that $M_X(t) = M_X(-t)$.

23. Note that

$$M_X(t) = E\big(e^{tX}\big) = \sum_{x=1}^{\infty} \frac{6}{\pi^2 x^2} e^{tx} = \frac{6}{\pi^2} \sum_{x=1}^{\infty} \frac{e^{tx}}{x^2}.$$

Now by the ratio test,

$$\lim_{x\to\infty} \frac{e^{t(x+1)}/(x+1)^2}{e^{tx}/x^2} = \lim_{x\to\infty} \frac{x^2}{x^2+2x+1} e^t = e^t$$

which is > 1 for $t \in (0, \infty)$. Therefore, $\sum_{x=1}^{\infty} \dfrac{e^{tx}}{x^2}$ diverges on $(0, \infty)$ and thus on no interval of the form $(-\delta, \delta)$, $\delta > 0$, $M_X(t)$ exists.

24. For $t < 1/2$, (11.2) implies that

$$\begin{aligned} M_X(t) &= \sum_{n=0}^{\infty} \frac{E(X^n)}{n!} t^n = \sum_{n=0}^{\infty} (n+1)(2t)^n = \frac{1}{2} \sum_{n=0}^{\infty} \frac{d}{dt} (2t)^{n+1} \\ &= \frac{1}{2} \frac{d}{dt} \Big[\sum_{n=0}^{\infty} (2t)^{n+1} \Big] = \frac{1}{2} \cdot \frac{d}{dt} \Big[\sum_{n=0}^{\infty} (2t)^n - 1 \Big] = \frac{1}{2} \cdot \frac{d}{dt} \Big[\frac{1}{1-2t} - 1 \Big] \\ &= \frac{1}{(1-2t)^2} = \Big[\frac{1/2}{(1/2)-t} \Big]^2. \end{aligned}$$

We see that for $t < 1/2$, $M_X(t)$ exists; furthermore, it is the moment-generating function of a gamma random variable with parameters $r = 2$ and $\lambda = 1/2$.

25. **(a)** At the end of the first period, with probability 1, the investment will grow to

$$A + A\frac{X}{k} = A\Big(1 + \frac{X}{k}\Big);$$

at the end of the second period, with probability 1, it will grow to

$$A\Big(1 + \frac{X}{k}\Big) + A\Big(1 + \frac{X}{k}\Big) \cdot \frac{X}{k} = A\Big(1 + \frac{X}{k}\Big)^2;$$

and, in general, at the end of the nth period, with probability 1, it will grow to $A\Big(1+\dfrac{X}{k}\Big)^n$.

(b) Dividing a year into k equal periods allows the banks to compound interest quarterly, monthly, or daily. If we increase k, we can compound interest every minute, second, or even fraction of a second. For an infinitesimal $\varepsilon > 0$, suppose that the interest is compounded at the end of each period of length ε. If $\varepsilon \to 0$, then the interest is compounded continuously. Since a year is $1/\varepsilon$ periods, each of length ε, the interest rate per period of length ε is the random variable $X/(1/\varepsilon) = \varepsilon X$. Suppose that at time t, the investment has grown to $A(t)$. Then at $t + \varepsilon$, with probability 1, the investment will be

$$A(t + \varepsilon) = A(t) + A(t) \cdot \varepsilon X.$$

This implies that

$$P\Big(\frac{A(t+\varepsilon)-A(t)}{\varepsilon}=XA(t)\Big)=1.$$

Letting $\varepsilon \to 0$, yields

$$P\Big(\lim_{\varepsilon\to 0}\frac{A(t+\varepsilon)-A(t)}{\varepsilon}=XA(t)\Big)=1$$

or, equivalently, with probability 1,

$$A'(t)=XA(t).$$

(c) Part (b) implies that, with probability 1,

$$\frac{A'(t)}{A(t)}=X.$$

Integrating both sides of this equation, we obtain that, with probability 1,

$$\ln[A(t)]=tX+C,$$

or

$$A(t)=e^{tX+c}.$$

Considering the fact that $A(0)=A$, this equation yields $A=e^c$. Therefore, with probability 1,

$$A(t)=e^{tX}\cdot e^{c}=Ae^{tX}.$$

This shows that if the interest rate is compounded continuously, then an initial investment of A dollars will grow, in t years, with probability 1, to the random variable Ae^{tX}, whose expected value is

$$E(Ae^{tX})=AE(e^{tX})=AM_X(t).$$

We have shown the following:

> If money is invested in a bank at an annual rate X, where X is a random variable, and if the bank compounds interest continuously, then, on average, the money will grow by a factor of $M_X(t)$, the moment-generating function of the interest rate.

26. Since X_i and X_j are binomial with parameters (n, p_i) and (n, p_j),

$$E(X_i)=np_i, \qquad E(X_j)=np_j,$$

$$\sigma_{X_i}=\sqrt{np_i(1-p_i)}, \qquad \sigma_{X_j}=\sqrt{np_j(1-p_j)}.$$

To find $E(X_i X_j)$, note that

$$\begin{aligned} M(t_1, t_2) &= E\big(e^{t_1 X_i + t_2 X_j}\big) \\ &= \sum_{x_i=0}^{n} \sum_{x_j=0}^{n-x_i} e^{t_1 x_i + t_2 x_j} P(X_i = x_i,\ X_j = x_j) \\ &= \sum_{x_i=0}^{n} \sum_{x_j=0}^{n-x_i} e^{t_1 x_i + t_2 x_j} \cdot \frac{n!}{x_i!\, x_j!\, (n - x_i - x_j)!} p_i^{x_i} p_j^{x_j} (1 - p_i - p_j)^{n - x_i - x_j} \\ &= \sum_{x_i=0}^{n} \sum_{x_j=0}^{n-x_i} \frac{n!}{x_i!\, x_j!\, (n - x_i - x_j)!} \big(e^{t_1} p_i\big)^{x_i} \big(e^{t_2} p_j\big)^{x_j} (1 - p_i - p_j)^{n - x_i - x_j} \\ &= \big(p_i e^{t_1} + p_j e^{t_2} + 1 - p_i - p_j\big)^n, \end{aligned}$$

where the last equality follows from multinomial expansion (Theorem 2.6). Therefore,

$$\frac{\partial^2 M}{\partial t_1\, \partial t_2}(t_1, t_2) = n(n-1) p_i p_j e^{t_1} e^{t_2} \big(p_i e^{t_1} + p_j e^{t_2} + 1 - p_i - p_j\big)^{n-2},$$

and so

$$E(X_i X_j) = \frac{\partial^2 M}{\partial t_1 \partial t_2}(0, 0) = n(n-1) p_i p_j.$$

Thus

$$\rho(X_i,\ X_j) = \frac{n(n-1) p_i p_j - (np_i)(np_j)}{\sqrt{np_i(1-p_i)} \cdot \sqrt{np_j(1-p_j)}} = -\sqrt{\frac{p_i p_j}{(1-p_i)(1-p_j)}}.$$

11.2 SUMS OF INDEPENDENT RANDOM VARIABLES

1. $M_{\alpha X}(t) = E\big(e^{t\alpha X}\big) = M_X(t\alpha) = \exp\big[\alpha\mu t + (1/2)\alpha^2\sigma^2 t^2\big].$

2. Since

$$M_{X_1+X_2+\cdots+X_n}(t) = M_{X_1}(t) M_{X_2}(t) \cdots M_{X_n}(t) = \Big[\frac{pe^t}{1-(1-p)e^t}\Big]^n,$$

$X_1 + X_2 + \cdots + X_n$ is negative binomial with parameters (n, p).

3. Since

$$M_{X_1+X_2+\cdots+X_n}(t) = M_{X_1}(t) M_{X_2}(t) \cdots M_{X_n}(t) = \Big(\frac{\lambda}{\lambda - t}\Big)^n,$$

$X_1 + X_2 + \cdots + X_n$ is gamma with parameters n and λ.

4. For $1 \leq i \leq n$, let X_i be negative binomial with parameters r_i and p. We have that

$$M_{X_1+X_2+\cdots+X_n}(t) = M_{X_1}(t)M_{X_2}(t)\cdots M_{X_n}(t)$$

$$= \Big[\frac{pe^t}{1-(1-p)e^t}\Big]^{r_1}\Big[\frac{pe^t}{1-(1-p)e^t}\Big]^{r_2}\cdots\Big[\frac{pe^t}{1-(1-p)e^t}\Big]^{r_n}$$

$$= \Big[\frac{pe^t}{1-(1-p)e^t}\Big]^{r_1+r_2+\cdots+r_n}.$$

Thus $X_1 + X_2 + \cdots + X_r$ is negative binomial with parameters $r_1 + r_2 + \cdots + r_n$ and p.

5. Since

$$\begin{aligned} M_{X_1+X_2+\cdots+X_n}(t) &= M_{X_1}(t)M_{X_2}(t)\cdots M_{X_n}(t) \\ &= \Big(\frac{\lambda}{\lambda-t}\Big)^{r_1}\Big(\frac{\lambda}{\lambda-t}\Big)^{r_2}\cdots\Big(\frac{\lambda}{\lambda-t}\Big)^{r_n} \\ &= \Big(\frac{\lambda}{\lambda-t}\Big)^{r_1+r_2+\cdots+r_n}, \end{aligned}$$

$X_1 + X_2 + \cdots + X_n$ is gamma with parameters $r_1 + r_2 + \cdots + r_n$ and λ.

6. By Theorem 11.4, the total number of underfilled bottles is binomial with parameters 180 and 0.15. Therefore, the desired probability is

$$\binom{180}{27}(0.15)^{27}(0.85)^{153} = 0.083.$$

7. For $j < i$, $P(X = i \mid X + Y = j) = 0$. For $j \geq i$,

$$P(X = i \mid X + Y = j) = \frac{P(X = i,\ Y = j - i)}{P(X + Y = j)} = \frac{P(X = i)P(Y = j - i)}{P(X + Y = j)}$$

$$= \frac{\dbinom{n}{i}p^i(1-p)^{n-i}\cdot\dbinom{m}{j-i}p^{j-i}(1-p)^{m-(j-i)}}{\dbinom{n+m}{j}p^j(1-p)^{n+m-j}} = \frac{\dbinom{n}{i}\dbinom{m}{j-i}}{\dbinom{n+m}{j}}.$$

Interpretation: Given that in $n + m$ trials exactly j successes have occurred, the probability mass function of the number of successes in the first n trials is hypergeometric. This should be intuitively clear.

8. Since $X+Y+Z$ is Poisson with parameter $\lambda_1+\lambda_2+\lambda_3$ and $X+Z$ is Poisson with parameter $\lambda_1+\lambda_3$, we have that

$$P(Y=y \mid X+Y+Z=t)=\frac{P(Y=y,\ X+Z=t-y)}{P(X+Y+Z=t)}$$

$$=\frac{\dfrac{e^{-\lambda_2}\lambda_2^y}{y!}\cdot\dfrac{e^{-(\lambda_1+\lambda_3)}(\lambda_1+\lambda_3)^{t-y}}{(t-y)!}}{\dfrac{e^{-(\lambda_1+\lambda_2+\lambda_3)}(\lambda_1+\lambda_2+\lambda_3)^t}{t!}}$$

$$=\binom{t}{y}\Big(\frac{\lambda_2}{\lambda_1+\lambda_2+\lambda_3}\Big)^y\Big(\frac{\lambda_1+\lambda_3}{\lambda_1+\lambda_2+\lambda_3}\Big)^{t-y}.$$

9. Let X be the remaining calling time of the person in the booth. Let Y be the calling time of the person ahead of Mr. Watkins. By the memoryless property of exponential, X is exponential with parameter 1/8. Since Y is also exponential with parameter 1/8, assuming that X and Y are independent, the waiting time of Mr. Watkins, $X+Y$, is gamma with parameters 2 and 1/8. Therefore,

$$P(X+Y\geq 12)=\int_{12}^{\infty}\frac{1}{64}xe^{-x/8}\,dx=\frac{5}{2}e^{-3/2}=0.558.$$

10. By Theorem 11.7, $X+Y\sim N(5,9)$, $X-Y\sim N(-3,9)$, and $3X+4Y\sim N(19,130)$. Thus

$$P(X+Y>0)=P\Big(\frac{X+Y-5}{3}>\frac{0-5}{3}\Big)=1-\Phi(-1.67)=\Phi(1.67)=0.9525,$$

$$P(X-Y<2)=P\Big(\frac{X-Y+3}{3}<\frac{2+3}{3}\Big)=\Phi(1.67)=0.9525,$$

and

$$P(3X+4Y>20)=P\Big(\frac{3X+4Y-19}{\sqrt{130}}>\frac{20-19}{\sqrt{130}}\Big)=1-\Phi(0.9)=0.4641.$$

11. Theorem 11.7 implies that $\bar{X}\sim N(110,1.6)$, where $\bar{X}$ is the average of the IQ's of the randomly selected students. Therefore,

$$P(\bar{X}\geq 112)=P\bigg(\frac{\bar{X}-110}{\sqrt{1.6}}\geq\frac{112-110}{\sqrt{1.6}}\bigg)=1-\Phi(1.58)=0.0571.$$

12. Let $\bar{X}_1$ be the average of the accounts selected at store 1 and $\bar{X}_2$ be the average of the accounts selected at store 2. We have that

$$\bar{X}_1\sim N\Big(90,\frac{900}{10}\Big)=N(90,90)\quad\text{and}\quad\bar{X}_2\sim N\Big(100,\frac{2500}{15}\Big)=N\Big(100,\frac{500}{3}\Big).$$

Therefore, $\bar{X}_1 - \bar{X}_2 \sim N\Big(-10, \dfrac{770}{3}\Big)$ and so

$$P(\bar{X}_1 > \bar{X}_2) = P(\bar{X}_1 - \bar{X}_2 > 0) = P\bigg(\frac{\bar{X}_1 - \bar{X}_2 + 10}{\sqrt{770/3}} > \frac{0+10}{\sqrt{770/3}}\bigg)$$
$$= 1 - \Phi(0.62) = 0.2676.$$

13. By Exercise 6, Section 10.5, X and Y are sums of independent standard normal random variables. Hence $\alpha X + \beta Y$ is a linear combination of independent standard normal random variables. Thus, by Theorem 11.7, $\alpha X + \beta Y$ is normal.

14. By Exercise 13, $X - Y$ is normal; its mean is $71 - 60 = 11$, its variance is

$$\begin{aligned}\text{Var}(X - Y) &= \text{Var}(X) + \text{Var}(Y) - 2\text{Cov}(X, Y)\\ &= \text{Var}(X) + \text{Var}(Y) - 2\rho(X, Y)\sigma_X\sigma_Y\\ &= 9 + (2.7)^2 - 2(0.45)(3)(2.7) = 9.\end{aligned}$$

Therefore,

$$P(X - Y \geq 8) = P\Big(\frac{X - Y - 11}{3} \geq \frac{8 - 11}{3}\Big) = 1 - \Phi(-1) = \Phi(1) = 0.8413.$$

15. Let $\bar{X}$ be the average of the weights of the 12 randomly selected athletes. Let $X_1, X_2, \ldots, X_{12}$ be the weights of these athletes. Since

$$\bar{X} \sim N\Big(225, \frac{25^2}{12}\Big) = N\Big(225, \frac{625}{12}\Big),$$

we have that

$$\begin{aligned}P(X_1 + X_2 + \cdots + X_{12} \leq 2700) &= P\Big(\bar{X} \leq \frac{2700}{12}\Big) = P(\bar{X} \leq 225)\\ &= P\bigg(\frac{\bar{X} - 225}{\sqrt{625/12}} \leq \frac{225 - 225}{\sqrt{625/12}}\bigg) = \Phi(0) = \frac{1}{2}.\end{aligned}$$

16. Let $\bar{X}_1$ and $\bar{X}_2$ be the averages of the final grades of the probability and calculus courses Dr. Olwell teaches, respectively. We have that

$$\bar{X}_1 \sim N\Big(65, \frac{418}{22}\Big) = N(65, 19) \quad \text{and} \quad \bar{X}_2 \sim N\Big(72, \frac{448}{28}\Big) = N(72, 16).$$

Therefore, $\bar{X}_1 - \bar{X}_2 \sim N(-7, 35)$ and hence the desired probability is

$$\begin{aligned}P\big(|\bar{X}_1 - \bar{X}_2| \geq 2\big) &= P(\bar{X}_1 - \bar{X}_2 \geq 2) + P(\bar{X}_1 - \bar{X}_2 \leq -2)\\ &= P\bigg(\frac{\bar{X}_1 - \bar{X}_2 + 7}{\sqrt{35}} \geq \frac{2+7}{\sqrt{35}}\bigg) + P\bigg(\frac{\bar{X}_1 - \bar{X}_2 + 7}{\sqrt{35}} \leq \frac{-2+7}{\sqrt{35}}\bigg)\\ &= 1 - \Phi(1.52) + \Phi(0.85) = 1 - 0.9352 + 0.8023 = 0.8671.\end{aligned}$$

17. Let X and Y be the lifetimes of the mufflers of the first and second cars, respectively.
(a) To calculate the desired probability, $P(|X - Y| \geq 1.5)$, note that by symmetry,

$$P\big(|X - Y| \geq 1.5\big) = 2P(X - Y \geq 1.5).$$

Now $X - Y \sim N(0, 2)$, hence

$$P\big(|X - Y| \geq 1.5\big) = 2P\bigg(\frac{X - Y - 0}{\sqrt{2}} \geq \frac{1.5 - 0}{\sqrt{2}}\bigg) = 2\big[1 - \Phi(1.06)\big] = 0.289.$$

(b) Let Z be the lifetime of the first muffler the family buys. By symmetry, the desired probability is

$$2P(Y > X + Z) = 2P(Y - X - Z > 0).$$

Now $Y - X - Z \sim N(-3, 3)$. Hence

$$2P(Y - X - Z > 0) = 2P\bigg(\frac{Y - X - Z + 3}{\sqrt{3}} > \frac{0 + 3}{\sqrt{3}}\bigg) = 2\big[1 - \Phi(1.73)\big] = 0.0836.$$

18. Let n be the maximum number of passengers who can use the elevator and $X_1, X_2, \ldots, X_n$ be the weights of n random passengers. We must have

$$P(X_1 + X_2 + \cdots X_n > 3000) < 0.0003$$

or, equivalently,

$$P(X_1 + X_2 + \cdots + X_n \leq 3000) > 0.9997.$$

Let $\bar{X}$ be the mean of the weights of the n random passengers. We must have

$$P\bigg(\bar{X} \leq \frac{3000}{n}\bigg) > 0.9997.$$

Since $\bar{X} \sim N\Big(155, \dfrac{625}{n}\Big)$, we must have

$$P\Big(\frac{\bar{X} - 155}{25/\sqrt{n}} \leq \frac{(3000/n) - 155}{25/\sqrt{n}}\Big) > 0.9997,$$

or

$$\Phi\Big(\frac{3000}{25\sqrt{n}} - \frac{155\sqrt{n}}{25}\Big) > 0.9997.$$

Using Table 2 of the Appendix, this gives

$$\frac{3000}{25\sqrt{n}} - \frac{155\sqrt{n}}{25} \geq 3.49$$

or, equivalently,

$$155n + 87.25\sqrt{n} - 3000 \leq 0.$$

Since the roots of the quadratic equation $155n + 87.25\sqrt{n} - 3000 = 0$ are (approximately) $\sqrt{n} = 4.127$ and $\sqrt{n} = -4.69$, the inequality is valid if and only if

$$\big(\sqrt{n} + 4.69\big)\big(\sqrt{n} - 4.127\big) \leq 0.$$

But $\sqrt{n} + 4.69 > 0$, so the inequality is valid if and only if $\sqrt{n} - 4.127 \leq 0$ or $n \leq 17.032$. Therefore the answer is $n = 17$.

19. By Remark 9.3, the marginal joint probability mass function of $X_1, X_2, \ldots, X_k$ is multinomial with parameters n and $(p_1, p_2, \ldots, p_k, 1 - p_1 - p_2 - \cdots - p_k)$. Thus, letting $p = p_1 + p_2 + \cdots + p_k$ and $x = x_1 + x_2 + \cdots + x_k$, we have that

$$p(x_1, x_2, \ldots, x_k) = \frac{n!}{x_1!\, x_2! \,\cdots\, x_k!\,(n-x)!} p_1^{x_1} p_2^{x_2} \cdots p_k^{x_k} (1-p)^{n-x}.$$

This gives

$$\begin{aligned}
&P(X_1 + X_2 + \cdots + X_k = i)\\
&\quad = \sum_{x_1+x_2+\cdots+x_k=i} \frac{n!}{x_1!\, x_2! \,\cdots\, x_k!\,(n-i)!} p_1^{x_1} p_2^{x_2} \cdots p_k^{x_k} (1-p)^{n-i}\\
&\quad = \frac{n!}{i!\,(n-i)!}(1-p)^{n-i} \sum_{x_1+x_2+\cdots+x_k=i} \frac{i!}{x_1!\, x_2! \,\cdots\, x_k!} p_1^{x_1} p_2^{x_2} \cdots p_k^{x_k}\\
&\quad = \binom{n}{i}(1-p)^{n-i}(p_1 + p_2 + \cdots + p_k)^i\\
&\quad = \binom{n}{i} p^i (1-p)^{n-i}.
\end{aligned}$$

This shows that $X_1 + X_2 + \cdots + X_k$ is binomial with parameters n and $p = p_1 + p_2 + \cdots + p_k$.

20. First note that if Y_1 and Y_2 are two exponential random variables each with rate λ, $\min(Y_1, Y_2)$ is exponential with rate 2λ. Now let $A_1, A_2, \ldots, A_{11}$ be the customers in the line ahead of Kim. Due to the memoryless property of exponential random variables, X_1, the time until A_1's turn to make a call is exponential with rate $2(1/3) = 2/3$. The time until A_2's turn to call is $X_1 + X_2$, where X_2 is exponential with rate $2(1/3) = 2/3$. Continuing this argument and considering the fact that Kim is the 12th person waiting in the line, we have that the time until Kim's turn to make a phone call is $X_1 + X_2 + \cdots + X_{12}$, where $\{X_1, X_2, \ldots, X_{12}\}$ is an independent and identically distributed sequence of exponential random variables each with rate $2/3$. Hence the distribution of the waiting time of Kim is gamma with parameters $(12, 2/3)$. Her expected waiting time is $12(2/3) = 18$.

11.3 MARKOV AND CHEBYSHEV INEQUALITIES

1. Let X be the lifetime (in months) of a randomly selected dollar bill. We are given that $E(X) = 22$. By Markov inequality,

$$P(X \geq 60) \leq \frac{22}{60} = 0.37.$$

This shows that at most 37% of the one-dollar bills last 60 or more months; that is, at least five years.

2. We have that $P(X \geq 2) = 2/5$. Hence, by Markov's inequality,

$$\frac{2}{5} = P(X \geq 2) \leq \frac{E(X)}{2}.$$

This gives $E(X) \geq 4/5$.

3. **(a)** $P(X \geq 11) \leq \dfrac{E(X)}{11} = \dfrac{5}{11} = 0.4545.$

(b) $P(X \geq 11) = P(X - 5 \geq 6) \leq P\big(|X - 5| \geq 6\big) \leq \dfrac{\sigma^2}{36} = \dfrac{42 - 25}{36} = 0.472.$

4. Let X be the lifetime of the randomly selected light bulb; we have

$$P(X \leq 700) \leq P\big(|X - 800| \geq 100\big) \leq \frac{2500}{10,000} = 0.25.$$

5. Let X be the number of accidents that will occur tomorrow. Then

(a) $P(X \geq 5) \leq \dfrac{2}{5} = 0.4.$

(b) $P(X \geq 5) = 1 - \displaystyle\sum_{i=0}^{4} \frac{e^{-2} 2^i}{i!} = 0.053.$

(c) $P(X \geq 5) = P(X - 2 \geq 3) \leq P\big(|X - 2| \geq 3\big) \leq \dfrac{2}{9} = 0.222$

6. Let X be the IQ of a randomly selected student from this campus; we have

$$P(X > 140) \leq P\big(|X - 110| > 30\big) \leq \frac{15}{900} = 0.017.$$

Therefore, less than 1.7% of these students have an IQ above 140.

7. Let X be the waiting period from the time Helen orders the book until she receives it. We want to find a so that $P(X < a) \geq 0.95$ or, equivalently, $P(X \geq a) \leq 0.05$. But

$$P(X \geq a) = P(X - 7 \geq a - 7) \leq P\big(|X - 7| \geq a - 7\big) \leq \frac{4}{(a - 7)^2}.$$

So we should determine the value of a for which $4/(a - 7)^2 \leq 0.05$; it is easily seen that $a \geq 15.9$ or $a = 16$. Therefore, Helen should order the book 16 days earlier.

8. By Markov's inequality, $P(X \geq 2\mu) \leq \dfrac{\mu}{2\mu} = \dfrac{1}{2}$.

9. $P(X > 2\mu) = P(X - \mu > \mu) \leq P\big(|X - \mu| \geq \mu\big) \leq \dfrac{\mu}{\mu^2} = \dfrac{1}{\mu}$.

10. We have that

$$\begin{aligned} P(38 < \bar{X} < 46) &= P(-4 < \bar{X} - 42 < 4) = P\big(|\bar{X} - 42| < 4\big) \\ &= 1 - P\big(|\bar{X} - 42| \geq 4\big). \end{aligned}$$

By (11.3),

$$P\big(|\bar{X} - 42| \geq 4\big) \leq \frac{60}{16(25)} = \frac{3}{20}.$$

Hence

$$P(38 < \bar{X} < 46) \geq 1 - \frac{3}{20} = \frac{17}{20} = 0.85.$$

11. For $i = 1, 2, \dots, n$, let X_i be the IQ of the ith student selected at random. We want to find n, so that

$$P\Big(-3 < \frac{X_1 + X_2 + \cdots + X_n}{n} - \mu < 3\Big) \geq 0.92$$

or, equivalently,

$$P(|\bar{X} - \mu| \geq 3) \leq 0.08.$$

Since $E(X_i) = \mu$ and $\text{Var}(X_i) = 150$, by (11.3),

$$P(|\bar{X} - \mu| \geq 3) \leq \frac{150}{3^2 \cdot n}.$$

Therefore, all we need to do is to find n for which $150/(9n) \leq 0.08$. This gives $n \geq 150/[9(0.08)] = 208.33$. Thus the psychologist should choose a sample of size 209.

12. Let $X_1, X_2, \dots, X_n$ be the random sample, μ be the expected value of the distribution, and σ^2 be the variance of the distribution. We want to find n so that

$$P(|\bar{X} - \mu| < 2\sigma) \geq 0.98$$

or, equivalently,

$$P(|\bar{X} - \mu| \geq 2\sigma) < 0.02.$$

By (11.3),

$$P(|\bar{X} - \mu| \geq 2\sigma) \leq \frac{\sigma^2}{(2\sigma)^2 \cdot n} = \frac{1}{4n}.$$

Therefore, all we need to do is to make sure that $1/(4n) \leq 0.02$. This gives $n \geq 12.5$. So a sample of size 13 gives a mean which is within 2 standard deviations from the expected value with a probability of at least 0.98.

13. Call a random observation success, if the operator is busy. Call it failure, if he is free. In (11.5), let $\varepsilon = 0.05$ and $\alpha = 0.04$; we have

$$n \geq \frac{1}{4(0.05)^2(0.04)} = 2500.$$

Therefore, at least 2500 independent observations should be made to ensure that $(1/n)\sum_{i=1}^{n}$ estimates p, the proportion of time that the airline operator is busy, with a maximum error of 0.05 with probability 0.96 or higher.

14. By (11.5),

$$n \geq \frac{1}{4(0.05)^2(0.06)} = 1666.67.$$

Therefore, it suffices to flip the coin $n = 1667$ times independently.

15. $P\big(|X-\mu| \geq \alpha\big) = P\big(|X-\mu|^{2n} \geq \alpha^{2n}\big) \leq \dfrac{E\big[(X-\mu)^{2n}\big]}{\alpha^{2n}}.$

16. By Markov's inequality, $P(X > t) = P\big(e^{kX} > e^{kt}\big) \leq \dfrac{E\big(e^{kX}\big)}{e^{kt}}.$

17. By the Corollary of Cauchy-Schwarz Inequality (Theorem 10.3),

$$\big[E(X-Y)\big]^2 \leq E\big[(X-Y)^2\big] = 0.$$

This gives that $E(X-Y) = 0$. Therefore,

$$\text{Var}(X-Y) = E\big[(X-Y)^2\big] - \big[E(X-Y)\big]^2 = 0.$$

We have shown that $X-Y$ is a random variable with mean 0 and variance 0; by Example 11.16, $P(X-Y=0) = 1$. So with probability 1, $X = Y$.

18. If $Y = X$ with probability 1, Theorem 10.5 implies that $\rho(X,Y) = 1$. Suppose that $\rho(X,Y) = 1$; we show that X=Y with probability 1. Note that $E(X) = E(Y) = (n+1)/2$, $\text{Var}(X) = \text{Var}(Y) = (n^2-1)/12$, and $\sigma_X = \sigma_Y = \sqrt{(n^2-1)/12}$. These and

$$1 = \rho(X,Y) = \frac{E(XY) - E(X)E(Y)}{\sigma_X \sigma_Y}$$

imply that $E(XY) = (2n^2+3n+1)/6$. Therefore,

$$\begin{aligned} E\big[(X-Y)^2\big] &= E(X^2 - 2XY + Y^2) = E(X^2) + E(Y^2) - 2E(XY) \\ &= \text{Var}(X) + \big[E(X)\big]^2 + \text{Var}(Y) + \big[E(Y)\big]^2 - 2E(XY) \\ &= \frac{n^2-1}{12} + \Big(\frac{n+1}{2}\Big)^2 + \frac{n^2-1}{12} + \Big(\frac{n+1}{2}\Big)^2 - \frac{2n^2+3n+1}{3} = 0. \end{aligned}$$

$E\big[(X-Y)^2\big] = 0$ implies that with probability 1, X=Y (see Exercise 17 above).

19. By Markov's inequality,

$$P\Big(X \geq \frac{1}{t}\ln\alpha\Big) = P(tX \geq \ln\alpha) = P\big(e^{tX} \geq \alpha\big) \leq \frac{E\big(e^{tX}\big)}{\alpha} = \frac{1}{\alpha}M_X(t).$$

20. Using gamma function introduced in Section 7.4,

$$E(X) = \frac{1}{n!}\int_0^\infty x^{n+1}e^{-x}\,dx = \frac{\Gamma(n+2)}{n!} = \frac{(n+1)!}{n!} = n+1,$$

$$E(X^2) = \frac{1}{n!}\int_0^\infty x^{n+2}e^{-x}\,dx = \frac{\Gamma(n+3)}{n!} = \frac{(n+2)!}{n!} = (n+1)(n+2).$$

Hence $\sigma_X^2 = (n+1)(n+2) - (n+1)^2 = n+1$. Now

$$P(0 < X < 2n+2) = 1 - P(X \geq 2n+2),$$

and by Chebyshev's inequality,

$$P(X \geq 2n+2) = P\big(X - (n+1) \geq n+1\big) \leq P\big(\big|X-(n+1)\big| \geq n+1\big)$$
$$\leq \frac{n+1}{(n+1)^2} = \frac{1}{n+1}.$$

Therefore,

$$P(0 < X < 2n+1) \geq 1 - \frac{1}{n+1} = \frac{n}{n+1}.$$

11.4 LAWS OF LARGE NUMBERS

1. Since

$$E(X_i) = \int_0^1 x \cdot 4x(1-x)\,dx = \frac{1}{3},$$

by the strong law of large numbers,

$$P\Big(\lim_{n\to\infty}\frac{X_1+X_2+\cdots+X_n}{n} = \frac{1}{3}\Big) = 1.$$

2. If $X_1 > M$ with probability 1, then $X_2 > M$ with probability 1 since X_1 and X_2 are identically distributed. Therefore, $X_1 + X_2 > 2M > M$ with probability 1. This argument shows that

$$\{X_1 > M\} \subseteq \{X_1 + X_2 > M\} \subseteq \{X_1 + X_2 + X_3 > M\} \subseteq \cdots.$$

Therefore, by the continuity of probability function (Theorem 1.8),

$$\lim_{n\to\infty} P(X_1 + X_2 + \cdots + X_n > M) = P\big(\lim_{n\to\infty} X_1 + X_2 + \cdots + X_n > M\big).$$

By this relation, it suffices to show that $\forall M > 0$,

$$\lim_{n\to\infty} X_1 + X_2 + \cdots + X_n > M \tag{45}$$

with probability 1. Let S be the sample space over which X_i's are defined. Let $\mu = E(X_i)$; we are given that $\mu > 0$. By the central limit theorem,

$$P\Big(\lim_{n\to\infty} \frac{X_1 + X_2 + \cdots X_n}{n} = \mu\Big) = 1.$$

Therefore, letting

$$V = \Big\{\omega \in S\colon \lim_{n\to\infty} \frac{X_1(\omega) + X_2(\omega) + \cdots X_n(\omega)}{n} = \mu\Big\},$$

we have that $P(V) = 1$. To establish (45), it is sufficient to show that $\forall \omega \in V$,

$$\lim_{n\to\infty} X_1(\omega) + X_2(\omega) + \cdots X_n(\omega) = \infty. \tag{46}$$

To do so, applying the definition of limit to

$$\lim_{n\to\infty} \frac{X_1(\omega) + X_2(\omega) + \cdots X_n(\omega)}{n} = \mu,$$

we have that for $\varepsilon = \mu/2$, there exists a positive integer N (depending on ω) such that $\forall n > N$,

$$\Big|\frac{X_1(\omega) + X_2(\omega) + \cdots X_n(\omega)}{n} - \mu\Big| < \varepsilon = \frac{\mu}{2}$$

or, equivalently,

$$-\frac{\mu}{2} < \frac{X_1(\omega) + X_2(\omega) + \cdots X_n(\omega)}{n} - \mu < \frac{\mu}{2}.$$

This yields

$$\frac{X_1(\omega) + X_2(\omega) + \cdots X_n(\omega)}{n} > \frac{\mu}{2}.$$

Thus, for all $n > N$,

$$X_1(\omega) + X_2(\omega) + \cdots X_n(\omega) > \frac{n\mu}{2},$$

which establishes (46).

3. For $0 < \varepsilon < 1$,

$$P\big(|Y_n - 0| > \varepsilon\big) = 1 - P\big(|Y_n - 0| \le \varepsilon\big) = 1 - P(X \le n) = 1 - \int_0^n f(x)\,dx.$$

Therefore,

$$\lim_{n\to\infty} P\big(|Y_n - 0| > \varepsilon\big) = 1 - \int_0^\infty f(x)\,dx = 1 - 1 = 0,$$

showing that Y_n converges to 0 in probability.

4. By the strong law of large numbers, S_n/n converges to μ almost surely. Therefore, S_n/n converges to μ in probability and hence

$$\begin{aligned}\lim_{n\to\infty} P\Big(n(\mu-\varepsilon) \le S_n \le n(\mu+\varepsilon)\Big) &= \lim_{n\to\infty} P\Big(\mu-\varepsilon \le \frac{S_n}{n} \le \mu+\varepsilon\Big)\\ &= \lim_{n\to\infty} P\Big(\Big|\frac{S_n}{n}-\mu\Big| \le \varepsilon\Big)\\ &= 1-\lim_{n\to\infty} P\Big(\Big|\frac{S_n}{n}-\mu\Big| > \varepsilon\Big) = 1-0=1.\end{aligned}$$

5. Suppose that the bank will never be empty of customers again. We will show a contradiction. Let $U_n = T_1 + T_2 + \cdots + T_n$. Then U_n is the time the nth new customer arrives. Let W_i be the service time of the ith new customer *served*. Clearly, $W_1, W_2, W_3, \ldots$ are independent and identically distributed random variables with $E(W_i) = 1/\mu$. Let $Z_n = T_1 + W_1 + W_2 + \cdots + W_n$. Since the bank will never be empty of customers, Z_n is the departure time of the nth new customer *served*. By the strong law of large numbers,

$$\lim_{n\to\infty} \frac{U_n}{n} = \frac{1}{\lambda}$$

and

$$\begin{aligned}\lim_{n\to\infty} \frac{Z_n}{n} &= \lim_{n\to\infty} \Big(\frac{T_1}{n} + \frac{W_1+W_2+\cdots+W_n}{n}\Big)\\ &= \lim_{n\to\infty} \frac{T_1}{n} + \lim_{n\to\infty} \frac{W_1+W_2+\cdots+W_n}{n} = 0+\frac{1}{\mu} = \frac{1}{\mu}.\end{aligned}$$

Clearly, the bank will never remain empty of customers again if and only if $\forall n$,

$$U_{n+1} < Z_n.$$

This implies that

$$\frac{U_{n+1}}{n} < \frac{Z_n}{n}$$

or, equivalently,

$$\frac{n+1}{n} \cdot \frac{U_{n+1}}{n+1} < \frac{Z_n}{n}.$$

Thus

$$\lim_{n\to\infty} \frac{n+1}{n} \cdot \frac{U_{n+1}}{n+1} \le \lim_{n\to\infty} \frac{Z_n}{n} \tag{47}$$

Since $\displaystyle\lim_{n\to\infty} \frac{n+1}{n} = 1$, and with probability 1, $\displaystyle\lim_{n\to\infty} \frac{U_{n+1}}{n+1} = \frac{1}{\lambda}$ and $\displaystyle\lim_{n\to\infty} \frac{Z_n}{n} = \frac{1}{\mu}$, (47) implies that $\dfrac{1}{\lambda} \le \dfrac{1}{\mu}$ or $\lambda \ge \mu$. This is a contradiction to the fact that $\lambda < \mu$. Hence, with probability 1, eventually, for some period, the bank will be empty of customers again.

6. Suppose that the bank will never be empty of customers again. We will show a contradiction. Let $U_n = T_1 + T_2 + \cdots + T_n$. Then U_n is the time the nth new customer arrives. Let R be the sum of the remaining service time of the customer being served and the sums of the service times of the m customers present in the queue at $t = 0$. Let $Z_n = R + S_1 + S_2 + \cdots + S_n$. Since the bank will never be empty of customers, and customers are served on a first-come, first-served basis, we have that $U_1 < R$ and hence Z_n is the departure time of the nth new customer. By the strong law of large numbers,

$$\lim_{n\to\infty} \frac{U_n}{n} = \frac{1}{\lambda}$$

and

$$\lim_{n\to\infty} \frac{Z_n}{n} = \lim_{n\to\infty} \Big(\frac{R}{n} + \frac{S_1 + S_2 + \cdots + S_n}{n}\Big)$$
$$= \lim_{n\to\infty} \frac{R}{n} + \lim_{n\to\infty} \frac{S_1 + S_2 + \cdots + S_n}{n} = 0 + \frac{1}{\mu} = \frac{1}{\mu}.$$

Clearly, the bank will never remain empty of customers if and only if $\forall n$,

$$U_{n+1} < Z_n.$$

This implies that

$$\frac{U_{n+1}}{n} < \frac{Z_n}{n}$$

or, equivalently,

$$\frac{n+1}{n} \cdot \frac{U_{n+1}}{n+1} < \frac{Z_n}{n}.$$

Thus

$$\lim_{n\to\infty} \frac{n+1}{n} \cdot \frac{U_{n+1}}{n+1} \le \lim_{n\to\infty} \frac{Z_n}{n} \tag{48}$$

Since $\lim\limits_{n\to\infty} \dfrac{n+1}{n} = 1$, and with probability 1, $\lim\limits_{n\to\infty} \dfrac{U_{n+1}}{n+1} = \dfrac{1}{\lambda}$ and $\lim\limits_{n\to\infty} \dfrac{Z_n}{n} = \dfrac{1}{\mu}$, (48) implies that $\dfrac{1}{\lambda} \le \dfrac{1}{\mu}$ or $\lambda \ge \mu$. This is a contradiction to the fact that $\lambda < \mu$. Hence, with probability 1, eventually, for some period, the bank will be empty of customers.

7. X_n converges to 0 in probability because for every $\varepsilon > 0$, $P\big(|X_n - 0| \ge \varepsilon\big)$ is the probability that the random point selected from $[0, 1]$ is in $\Big[\dfrac{i}{2^k}, \dfrac{i+1}{2^k}\Big]$. Now $n \to \infty$ implies that $2^k \to \infty$ and the length of the interval $\Big[\dfrac{i}{2^k}, \dfrac{i+1}{2^k}\Big] \to 0$, Therefore, $\lim_{n\to\infty} P\big(|X_n - 0| \ge \varepsilon\big) = 0$. However, X_n does not converge at any point because for all positive natural number N, there are always $m > N$ and $n > N$, such that $X_m = 0$ and $X_n = 1$ making it impossible for $|X_n - X_m|$ to be less than a given $0 < \varepsilon < 1$.

11.5 CENTRAL LIMIT THEOREM

1. Let $X_1, X_2, \dots, X_{150}$ be the random points selected from the interval $(0, 1)$. For $1 \le i \le 150$, X_i is uniform over $(0, 1)$. Therefore, $E(X_i) = \mu = 0.5$ and $\sigma_{X_i} = 1/\sqrt{12}$. We have

$$P\Big(0.48 < \frac{X_1 + X_2 + \cdots + X_{150}}{150} < 0.52\Big) = P(72 < X_1 + X_2 + \cdots + X_{150} < 78)$$

$$= P\bigg(\frac{72 - (150)(0.5)}{\sqrt{150}\big(1/\sqrt{12}\big)} < \frac{X_1 + X_2 + \cdots + X_{150} - (150)(0.5)}{\sqrt{150}\big(1/\sqrt{12}\big)} < \frac{78 - (150)(0.5)}{\sqrt{150}\big(1/\sqrt{12}\big)}\bigg)$$

$$\approx \Phi(0.85) - \Phi(-0.85) = 2\Phi(0.85) - 1 = 2(0.8023) - 1 = 0.6046.$$

2. For $1 \le i \le 35$, let X_i be the score of the ith student selected at random. By the central limit theorem

$$P(460 < \bar{X} < 540) = P\Big(460 < \frac{X_1 + X_2 + \cdots + X_{35}}{35} < 540\Big)$$

$$= P(16100 < X_1 + X_2 + \cdots + X_{35} < 18900)$$

$$= P\bigg(\frac{16100 - 35(500)}{100\sqrt{35}} < \frac{X_1 + X_2 + \cdots + X_{35} - 35(500)}{100\sqrt{35}} < \frac{18900 - 35(500)}{100\sqrt{35}}\bigg)$$

$$= P\Big(-2.37 < \frac{X_1 + X_2 + \cdots + X_{35} - 35(500)}{100\sqrt{35}} < 2.37\Big)$$

$$= \Phi(2.37) - \Phi(-2.37) = 0.9911 - 0.0089 = 0.9822.$$

3. We have that

$$\mu = \int_1^3 \frac{1}{9}x\Big(x + \frac{5}{2}\Big)\,dx = \frac{56}{27} = 2.07,$$

$$E(X^2) = \int_1^3 \frac{1}{9}x^2\Big(x + \frac{5}{2}\Big)\,dx = \frac{125}{27},$$

$$\sigma_X = \sqrt{(125/27) - (56/27)^2} = 0.57.$$

The desired probability is

$$P(2 < \bar{X} < 2.15) = P\Big(2 < \frac{X_1 + X_2 + \cdots + X_{24}}{24} < 2.15\Big)$$

$$= P(48 < X_1 + X_2 + \cdots + X_{24} < 51.6)$$

$$= P\Big(\frac{48 - 24(2.07)}{0.57\sqrt{24}} < \frac{X_1 + X_2 + \cdots + X_{24} - 24(2.07)}{0.57\sqrt{24}} < \frac{51.6 - 24(2.07)}{0.57\sqrt{24}}\Big)$$

$$\approx \Phi(0.69) - \Phi(-0.60) = 0.7549 - 0.2743 = 0.4806.$$

4. Let $X_1, X_2, \ldots, X_n$ be the sample. Since f is an even function, for $1 \le i \le n$,

$$E(X_i) = \int_{-\infty}^{\infty} \frac{1}{2} x e^{-|x|}\,dx = 0$$

$$E(X_i^2) = \int_{-\infty}^{\infty} \frac{1}{2} x^2 e^{-|x|}\,dx = \int_0^{\infty} x^2 e^{-x}\,dx = 2$$

$$\sigma_{X_i} = \sqrt{2 - 0} = \sqrt{2}.$$

By the central limit theorem,

$$P(\bar{X} > 0) = P\Big(\frac{X_1 + X_2 + \cdots + X_n}{n} > 0\Big)$$

$$= P\Big(\frac{X_1 + X_2 + \cdots + X_n - n(0)}{\sqrt{2}\sqrt{n}} > 0\Big) = 1 - \Phi(0) = 0.5.$$

5. Let $\mu = E(X_i)$ and $\sigma = \sigma_{X_i}$. Clearly, $E(S_n) = n\mu$ and $\sigma_{S_n} = \sigma\sqrt{n}$; thus, by the central limit theorem,

$$P\big(E(S_n) - \sigma_{Sn} \le S_n \le E(S_n) + \sigma_{S_n}\big) = P\big(n\mu - \sigma\sqrt{n} \le S_n \le n\mu + \sigma\sqrt{n}\,\big)$$

$$= P\Big(-1 \le \frac{S_n - n\mu}{\sigma\sqrt{n}} \le 1\Big) \approx \Phi(1) - \Phi(-1) = 2\Phi(1) - 1 = 0.6826.$$

6. For $1 \le i \le 300$, let X_i be the amount of the ith expenditure minus Jim's ith record; X_i is approximately uniform over $(-1/2, 1/2)$. Hence $E(X_i) = 0$ and $\sigma_{X_i} = \sqrt{\big[(1/2) - (-1/2)\big]^2/12} = 1/(2\sqrt{3})$. The desired probability is

$$P(-10 < X_1 + X_2 + \cdots + X_{300} < 10)$$

$$= P\Big(\frac{-10 - 300(0)}{\sqrt{300}\big(1/(2\sqrt{3})\big)} < \frac{X_1 + X_2 + \cdots + X_{300} - 300(0)}{\sqrt{300}\big(1/(2\sqrt{3})\big)} < \frac{10 - 300(0)}{\sqrt{300}\big(1/(2\sqrt{3})\big)}\Big)$$

$$\approx \Phi(2) - \Phi(-2) = 0.9772 - 0.0228 = 0.9544.$$

7. Note that *actual value* is a nebulous concept. In this exercise, like everywhere else, we are using it to mean the average of a *very large* number of measurements. Let X_i be the error in the ith measurement; $\mu = E(X_i) = 0$, $\sigma = \sigma_{X_i} = 1/\sqrt{3}$. Hence

$$P\Big(-0.25 < \frac{X_1 + X_2 + \cdots + X_{50}}{50} < 0.25\Big)$$

$$= P(-12.5 < X_1 + X_2 + \cdots + X_{50} < 12.5)$$

$$= P\Big(\frac{-12.5}{\big(1/\sqrt{3}\big)\sqrt{50}} < \frac{X_1 + X_2 + \cdots + X_{50}}{\big(1/\sqrt{3}\big)\sqrt{50}} < \frac{12.5}{\big(1/\sqrt{3}\big)\sqrt{50}}\Big)$$

$$\approx \Phi(3.06) - \Phi(-3.06) = 2\Phi(3.06) - 1 = 0.9778.$$

8. For $1 \le i \le 300$, let $X_i = 2$, if the ith employee attends with his or her spouse; let $X_i = 1$, if the ith employee attends alone; let $X_i = 0$, if the ith employee does not attend. To find the desired quantity, the probability of the event $\sum_{i=1}^{300} X_i \ge 320$, note that

$$\mu = E(X_i) = 2 \cdot \frac{1}{3} + 1 \cdot \frac{1}{3} + 0 \cdot \frac{1}{3} = 1,$$

$$E(X_i^2) = 4 \cdot \frac{1}{3} + 1 \cdot \frac{1}{3} + 0 \cdot \frac{1}{3} = \frac{5}{3},$$

$$\sigma_{X_i}^2 = \frac{5}{3} - 1 = \frac{2}{3}, \qquad \sigma_{X_i} = \sqrt{\frac{2}{3}}.$$

Thus

$$P\Big(\sum_{i=1}^{300} X_i \ge 320\Big) = P\Big(\frac{\sum_{i=1}^{300} X_i - 300}{\sqrt{2/3}\sqrt{300}} \ge \frac{320 - 300}{\sqrt{2/3}\sqrt{300}}\Big) \approx 1 - \Phi(1.41) = 0.0793.$$

9. Direct calculations show that

$$\mu = \int_4^6 x f(x)\,dx = 2/\ln(3/2) = 4.93,$$

$$E(X^2) = \int_4^6 x^2 f(x)\,dx = 10/\ln(3/2)$$

$$\sigma_X = \sqrt{\frac{10}{\ln(3/2)} - \frac{4}{[\ln(3/2)]^2}} = 0.577.$$

We want to find n so that

$$P\big(|\bar{X} - \mu| \le 0.07\big) \ge 0.98$$

or, equivalently,

$$P(-0.07 \le \bar{X} - \mu \le 0.07) \ge 0.98.$$

Since

$$\begin{aligned}
P\Big(-0.07 \le \frac{X_1 + X_2 + \cdots + X_n}{n} - \mu < 0.07\Big)
&= P(-0.07n \le X_1 + X_2 + \cdots + X_n - n\mu \le 0.07n)\\
&= P\Big(\frac{-0.07n}{0.577\sqrt{n}} \le \frac{X_1 + X_2 + \cdots + X_n - n\mu}{0.577\sqrt{n}} \le \frac{0.07n}{0.577\sqrt{n}}\Big)\\
&\approx \Phi\big(0.12\sqrt{n}\big) - \Phi\big(-0.12\sqrt{n}\big) = 2\Phi\big(0.12\sqrt{n}\big) - 1,
\end{aligned}$$

all we need to do is to find n so that

$$2\Phi\big(0.12\sqrt{n}\big) - 1 \ge 0.98,$$

or $\Phi\big(0.12\sqrt{n}\big) \ge 0.99$. By Table 2 of the appendix, this is satisfied if $0.12\sqrt{n} \ge 2.33$, or $n \ge 377.007$. Therefore, for all sample sizes of 378 or larger, the sample mean is within ± 0.07 of the μ.

10. Let

$$X_i = \begin{cases} 0.125 & \text{with probability } 1/2\\ -0.125 & \text{with probability } 1/2. \end{cases}$$

The change in the stock price, per share, after 60 days is $X_1 + X_2 + \cdots + X_{60}$. Clearly, $E(X_i) = 0$ and $\sigma_{X_i} = 0.125$. To find the distribution of $X_1 + X_2 + \cdots + X_{60}$, note that for all t,

$$P\Big(\sum_{i=1}^{60} X_i \le t\Big) = P\Big(\frac{\sum_{i=1}^{60} X_i - 60(0)}{0.125\sqrt{60}} \le \frac{t}{0.125\sqrt{60}}\Big) \approx \Phi\Big(\frac{t}{0.968}\Big).$$

This relation implies that

$$P\Big(\frac{X_1 + X_2 + \cdots + X_{60}}{0.968} \le t\Big) \approx \Phi(t).$$

So $(X_1 + X_2 + \cdots + X_{60})/0.968$ is approximately standard normal and hence

$$X_1 + X_2 + \cdots + X_{60} \sim N(0, 0.968^2).$$

Since the most likely value of a normal random variable with mean 0 is 0, the change in the stock price after 60 days is most likely 0 and hence the most likely value of the holdings of this investor after 60 days is 50,000.

11. Let X_1 be the number of tosses until the first tails. Let X_2 be the number of additional tosses until the second tails; X_3 be the number of tosses after the second tails until the third tails, and so on. Clearly, X_i's are independent geometric random variables, each with parameter

1/2. To find the desired probability, $P(X_1+X_2+\cdots+X_{50}\geq 75)$, note that $E(X_i)=2$ and $\sigma_{X_i}=\dfrac{\sqrt{1-(1/2)}}{1/2}=2\sqrt{1/2}$. Therefore,

$$\begin{aligned}
&P(X_1+X_2+\cdots+X_{50}\geq 75)\\
&\qquad = P\Big(\frac{X_1+X_2+\cdots+X_{50}-50(2)}{\sqrt{50}\cdot 2\sqrt{1/2}}\geq \frac{75-50(2)}{\sqrt{50}\cdot 2\sqrt{1/2}}\Big)\\
&\qquad \approx 1-\Phi(-2.5)=\Phi(2.5)=0.9938.
\end{aligned}$$

12. By Exercise 8, Section 7.4, for each i, $i\geq 1$, the random variable X_i^2 is gamma with parameters $\lambda=1/2$ and $r=1/2$. Therefore,

$$\mu=E(X_i^2)=\frac{r}{\lambda}=1$$

and

$$\sigma^2=\text{Var}(X_i^2)=\frac{r}{\lambda^2}=2.$$

Therefore, by central limit theorem,

$$\begin{aligned}
\lim_{n\to\infty}P\big(S_n\leq n+\sqrt{2n}\big)&=\lim_{n\to\infty}P\Big(\frac{S_n-n}{\sqrt{2n}}\leq 1\Big)\\
&=\lim_{n\to\infty}P\Big(\frac{S_n-n\mu}{\sigma\sqrt{n}}\leq 1\Big)=\Phi(1)=0.8413.
\end{aligned}$$

13. Let $Y_n=\sum_{i=1}^n X_i$; Y_n is Poisson with rate n. On the one hand,

$$P(Y_n\leq n)=\sum_{k=0}^{n}\frac{e^{-n}n^k}{k!}=\frac{1}{e^n}\sum_{k=0}^{n}\frac{n^k}{k!},$$

and on the other hand,

$$\begin{aligned}
\lim_{n\to\infty}P(Y_n\leq n)&=\lim_{n\to\infty}P\Big(\sum_{i=1}^{n}X_i\leq n\Big)\\
&=\lim_{n\to\infty}P\Big(\frac{\sum_{i=1}^n X_i-n}{\sqrt{n}}\leq\frac{n-n}{\sqrt{n}}\Big)=\Phi(0)=\frac{1}{2}.
\end{aligned}$$

So

$$\lim_{n\to\infty}\frac{1}{e^n}\sum_{k=0}^{\infty}\frac{n^k}{k!}=\frac{1}{2}.$$

REVIEW PROBLEMS FOR CHAPTER 11

1. $\bar{X}$, the average wage of a sample of 10 employees is normal with mean \$27000 and standard deviation $\$4900/\sqrt{10} = \$1549.52$. Therefore, the desired probability is

$$P(\bar{X} \geq 30,000) = P\left(\frac{\bar{X} - 27000}{1549.52} \geq \frac{30,000 - 27000}{1549.52}\right) = 1 - \Phi(1.94) = 0.0262.$$

2. $M_X(t)$ is the moment-generating function of a binomial random variable with parameters 10 and 2/3. Therefore, $\text{Var}(X) = 10 \times \frac{2}{3} \times \frac{1}{3} = \frac{20}{9}$ and

$$P(X \geq 8) = \sum_{i=8}^{10} \binom{10}{i} \left(\frac{2}{3}\right)^i \left(\frac{1}{3}\right)^{10-i} = 0.299.$$

3. $M_X(t)$ is the moment-generating function of a discrete random variable X with $P(X = 1) = 1/6$, $P(X = 2) = 1/3$, and $P(X = 3) = 1/2$. Therefore, F, the distribution function of X is given by

$$F(x) = \begin{cases} 0 & t < 1 \\ 1/6 & 1 \leq t < 2 \\ 1/2 & 2 \leq t < 3 \\ 1 & t \geq 3. \end{cases}$$

4. $M_X(t)$ is the moment-generating function of a normal random variable with mean 1 and variance 4.

5. X is a uniform random variable over the interval $(-1/2, 1/2)$.

6. X is a Poisson random variable with parameter $\lambda = 1/2$. Therefore,

$$P(X > 0) = 1 - P(X = 0) = 1 - e^{-1/2} = 0.393.$$

7. Note that

$$M_X^{(n)}(t) = \frac{(-1)^{n+1}(n+1)!}{(1-t)^{n+2}}.$$

Therefore, $E(X^n) = M_X^{(n)}(0) = (-1)^{n+1}(n+1)!$.

8. Let $\bar{X}$ be the average of the heights of 10 randomly selected men and $\bar{Y}$ be the average heights of 6 randomly selected women. Theorem 10.7 implies that $\bar{X} \sim N\left(173, \frac{40}{10}\right)$ and $\bar{Y} \sim N\left(160, \frac{20}{6}\right)$; thus $\bar{X} - \bar{Y} \sim N\left(13, \frac{22}{3}\right)$. Therefore,

$$P(\bar{X} - \bar{Y} \geq 5) = P\left(\frac{\bar{X} - \bar{Y} - 13}{\sqrt{22/3}} \geq \frac{5 - 13}{\sqrt{22/3}}\right) = \Phi(2.95) = 0.9984.$$

9. By definition,

$$\begin{aligned}E\big(e^{tX}\big) &= \int_{-\infty}^{\infty} \frac{1}{2} e^{-|x|} e^{tx}\,dx = \int_{-\infty}^{0} \frac{1}{2} e^{x}\cdot e^{tx}\,dx + \int_{0}^{\infty} \frac{1}{2} e^{-x}\cdot e^{tx}\,dx \\ &= \frac{1}{2}\int_{-\infty}^{0} e^{(1+t)x}\,dx + \frac{1}{2}\int_{0}^{\infty} e^{x(t-1)}\,dx.\end{aligned}$$

Now for these integrals to exist, we must restrict the domain of the moment-generating function of X to $\{t \in \mathbf{R}\colon -1 < t < 1\}$. In this domain,

$$\begin{aligned}M_X(t) = E\big(e^{tX}\big) &= \frac{1}{2(1+t)} e^{(1+t)x}\Big|_{-\infty}^{0} + \frac{1}{2(t-1)} e^{x(t-1)}\Big|_{0}^{\infty} \\ &= \frac{1}{2(1+t)} + \frac{1}{2(1-t)} = \frac{1}{1-t^2}.\end{aligned}$$

10. **(a)** By the law of total probability (Theorem 3.4),

$$\begin{aligned}P(X+Y=n) &= \sum_{i=0}^{n} P(X+Y=n \mid X=i)P(X=i) \\ &= \sum_{i=0}^{n} P(X+Y=n,\ X=i) = \sum_{i=0}^{n} P(Y=n-i,\ X=i) \\ &= \sum_{i=0}^{n} P(X=i)P(Y=n-i).\end{aligned}$$

(b) By part (a),

$$\begin{aligned}P(X+Y=n) &= \sum_{i=0}^{n} \frac{e^{-\lambda}\lambda^{i}}{i!}\cdot\frac{e^{-\mu}\mu^{n-i}}{(n-i)!} = e^{-(\lambda+\mu)}\cdot\frac{1}{n!}\cdot\sum_{i=0}^{n}\binom{n}{i}\lambda^{i}\mu^{n-i} \\ &= \frac{e^{-(\lambda+\mu)}(\lambda+\mu)^{n}}{n!},\end{aligned}$$

where the last equality follows from the binomial expansion (Theorem 2.5).

11. We have

$$\begin{aligned}P\Big(0.95 < \frac{X_1+X_2+\cdots+X_{28}}{28} < 1.05\Big) &= P(26.6 < X_1+X_2+\cdots+X_{28} < 29.4) \\ &= P\bigg(\frac{26.6-28}{2\sqrt{28}} < \frac{X_1+X_2+\cdots+X_{28}-28(1)}{2\sqrt{28}} < \frac{29.4-28}{2\sqrt{28}}\bigg) \\ &\approx \Phi(0.13) - \Phi(-0.13) = 0.5517 - 0.4483 = 0.1034.\end{aligned}$$

12. In (11.5), let $\varepsilon = 0.01$ and $\alpha = 0.06$; we have

$$n \geq \frac{1}{4(0.01)^2(0.06)} = 41,666.67.$$

Therefore, at least 41667 patients should participate in the trial.

13. By (11.4),

$$P\big(|\widehat{p} - p| < 0.05\big) \geq 1 - \frac{p(1-p)}{(0.05)^2 5000} \geq 1 - \frac{1}{4(0.05)^2 5000} = 0.98,$$

since $p(1-p) \leq 1/4$ implies that $-p(1-p) \geq -1/4$.

14. For $i = 1, 2, 3, \ldots, n$, let X_i be the IQ of the ith student of the sample. We want to determine n so that

$$P\Big(-0.2 < \frac{X_1 + X_2 + \cdots + X_n}{n} - \mu < .2\Big) \geq 0.98.$$

Since $E(X_i) = \mu$ and $\text{Var}(X_i) = 170$, by the central limit theorem,

$$P\Big(-0.2 < \frac{\sum_{i=1}^n X_i}{n} - \mu < 0.2\Big) = P\Big(-(0.2)n < \sum_{i=1}^n X_i - n\mu < (0.2)n\Big)$$

$$= P\Big(\frac{-(0.2)n}{\sqrt{170n}} < \frac{\sum_{i=1}^n X_i - n\mu}{\sqrt{170n}} < \frac{(0.2)n}{\sqrt{170n}}\Big)$$

$$\approx \Phi\Big[\frac{(0.2)n}{\sqrt{170n}}\Big] - \Phi\Big[\frac{-(0.2)n}{\sqrt{170n}}\Big] = 2\Phi\Big(\frac{0.2\sqrt{n}}{\sqrt{170}}\Big) - 1 \geq 0.98.$$

Therefore, we should determine n so that $\Phi(0.2\sqrt{n}/\sqrt{170}) \geq 0.98$. From Table 2 of the Appendix, we find $(0.2)\sqrt{n}/\sqrt{170} = 2.33$, which implies that $n = 23072.8250$; therefore, the psychologist should choose a sample of size 23073.

15. Let X_i be the amount chopped off on the ith charge in dollars. Let X be the actual amount Ed has charged to his credit card this month minus the amount his record shows. Clearly, $X = X_1 + X_2 + \cdots + X_{20}$, and for $1 \leq i \leq 20$, X_i is uniform over $(0, 1)$. Thus $E(X_i) = 1/2$ and $\text{Var}(X_i) = 1/12$ and hence $E(X) = 20/2 = 10$ and $\text{Var}(X) = 20/12 = 5/3$. Therefore, by Chebyshev's inequality,

$$P(X > 15) = P(X - 10 > 5) \leq P\big(|X - 10| > 5\big)$$
$$= P\big(|X - E(X)| > 5\big) \leq \frac{5/3}{25} = 0.0667.$$

16. $P(X \geq 45) \leq P\big(|X - 0| \geq 45\big) \leq 15^2/45^2 = 1/9.$

17. Suppose that the ith randomly selected book is X_i centimeters thick. The desired probability is

$$P(X_1 + X_2 + \cdots + X_{31} \le 87) = P\Big(\frac{X_1 + X_2 + \cdots + X_{31} - 3(31)}{1\sqrt{31}} \le \frac{87 - 3(31)}{1\sqrt{31}}\Big)$$
$$\approx \Phi\Big(\frac{87 - 93}{\sqrt{31}}\Big) = \Phi(-1.08) = 1 - 0.8599 = 0.1401.$$

18. For $1 \le i \le 20$, let X_i denote the outcome of the ith roll. We have

$$E(X_i) = \sum_{i=1}^{6} i \cdot \frac{1}{6} = \frac{7}{2}, \qquad E(X_i^2) = \sum_{i=1}^{6} i^2 \cdot \frac{1}{6} = \frac{91}{6}.$$

Thus $\text{Var}(X_i) = \dfrac{91}{6} - \dfrac{49}{4} = \dfrac{35}{12}$, and hence

$$P\Big(65 \le \sum_{i=1}^{20} X_i \le 75\Big) = P\Big(\frac{65 - 70}{\sqrt{35/12} \cdot \sqrt{20}} \le \frac{\sum_{i=1}^{20} X_i - 70}{\sqrt{35/12} \cdot \sqrt{20}} \le \frac{75 - 70}{\sqrt{35/12} \cdot \sqrt{20}}\Big)$$

$$\approx \Phi(0.65) - \Phi(-0.65) = 2\Phi(0.65) - 1 = 0.4844.$$

19. By Markov's inequality, $P(X \ge n\mu) \le \dfrac{\mu}{n\mu} = \dfrac{1}{n}$. So $nP(X \ge n\mu) \le 1$.

20. Let $X = \sum_{i=1}^{26} Xi$. We have that

$$E(X_i) = 26/51 = 0.5098, \qquad E(X_i^2) = E(X_i) = 0.5098,$$
$$\text{Var}(X_i) = 0.5098 - (0.5098)^2 = 0.2499,$$

$$E(X_iX_j) = P(X_i = 1,\ X_j = 1) = P(X_i = 1)P(X_j = 1 \mid X_i = 1) = \frac{26}{51} \cdot \frac{25}{49} = 0.2601,$$

and

$$\text{Cov}(X_i,\ X_j) = E(X_iX_j) - E(X_i)E(X_j) = 0.2601 - (0.5098)^2 = 0.0002.$$

Thus $E(X) = 26(0.5098) = 13.2548$ and

$$\text{Var}(X) = \sum_{i=1}^{26} \text{Var}(X_i) + 2\sum_{i<j}\sum \text{Cov}(X_i,\ X_j)$$
$$= 26(0.2499) + 2\binom{26}{2}(0.0002) = 6.6274.$$

Therefore, by Chebyshev's inequality,

$$P(X \le 10) \le P\big(|X - 13.2548| \ge 3.2548\big) \le \frac{6.6274}{(3.2548)^2} = 0.6256.$$

Chapter 12

Stochastic Processes

12.2 MORE ON POISSON PROCESSES

1. We know that $E\big[N(t)\big] = \text{Var}\big[N(t)\big] = \lambda t$. Hence $E\big[N(t)/t\big] = \lambda$ and $\text{Var}\big[N(t)/t\big] = \lambda/t$. Applying Chebyshev's inequality to $N(t)/t$, we have

$$P\Big(\Big|\frac{N(t)}{t} - \lambda\Big| \geq \varepsilon\Big) \leq \frac{\lambda}{t\varepsilon^2}.$$

As $t \to \infty$, the result follows from this relation.

2. By Wald's equation,

$$E\big[Y(52)\big] = E\big[N(52)\big]E(X_i) = \big[52(2.3)\big](1.2) = 143.52.$$

By Theorem 10.8,

$$\begin{aligned}\text{Var}\big[Y(52)\big] &= E\big[N(52)\big]\text{Var}(X_i) + \big[E(X_i)\big]^2\,\text{Var}\big[N(52)\big]\\ &= \big[52(2.3)\big](0.7)^2 + (1.2)^2\big[52(2.3)\big] = 230.828,\\ \sigma_{Y(52)} &= \sqrt{230.828} = 15.193.\end{aligned}$$

3. Let X_1 be the time between Linda's arrival at the point and the first car passing by her. Let X_2 be the time between the first and second cars passing Linda, and so forth. The X_i's are independent exponential random variables with mean $1/\lambda = 7$. Let N be the first integer for which

$$X_1 \leq 15,\ X_2 \leq 15,\ \ldots,\ X_N \leq 15,\ X_{N+1} > 15.$$

The time Linda has to wait before being able to cross the street is 0 if $N = 0$ (i.e., $X_1 > 15$), and is $S_N = X_1 + X_2 + \cdots + X_N$, otherwise. Therefore,

$$\begin{aligned}E(S_N) = E\big[E(S_N \mid N)\big] &= \sum_{i=0}^{\infty} E(S_N \mid N = i)P(N = i)\\ &= \sum_{i=1}^{\infty} E(S_N \mid N = i)P(N = i),\end{aligned}$$

where the last equality follows since for $N = 0$, we have that $S_N = 0$. Now

$$E(S_N \mid N = i) = E(X_1 + X_2 + \cdots + X_i \mid N = i) = \sum_{j=1}^{i} E(X_j \mid N = i)$$
$$= \sum_{j=1}^{i} E(X_j \mid X_j \le 15),$$

where by Remark 8.1,

$$E(X_j \mid X_j \le 15) = \frac{1}{F(15)} \int_0^{15} t f(t)\,dt;$$

F and f being the probability distribution and density functions of X_i's, respectively. That is, for $t \ge 0$, $F(t) = 1 - e^{-t/7}$, $f(t) = (1/7)e^{-t/7}$. Thus

$$E(X_j \mid X_j \le 15) = \frac{1}{1 - e^{-15/7}} \int_0^{15} \frac{t}{7} e^{-t/7}\,dt = (1.1329)\Big[-(t+7)e^{-t/7} \Big]_0^{15}$$
$$= (1.1329)(4.41898) = 5.00631.$$

This gives $E(S_N \mid N = i) = 5.00631i$. To find $P(N = i)$, note that for $i \ge 1$,

$$P(N = i) = P(X_1 \le 15,\ X_2 \le 15,\ \ldots,\ X_i \le 15,\ X_{i+1} > 15)$$
$$= \big[F(15)\big]^i \big[1 - F(15)\big] = (0.8827)^i (0.1173).$$

Putting all these together, we obtain

$$E(S_N) = \sum_{i=1}^{\infty} E(S_N \mid N = i) P(N = i) = \sum_{i=1}^{\infty} (5.00631i)(0.8827)^i (0.1173)$$
$$= (0.5872) \sum_{i=1}^{\infty} i (0.8827)^i = (0.5872) \cdot \frac{0.8827}{(1 - 0.8827)^2} = 37.6707,$$

where the next to last equality follows from $\sum_{i=1}^{\infty} i r^i = r/(1-r)^2$, $|r| < 1$. Therefore, on average, Linda has to wait approximately 38 seconds before she can cross the street.

4. Label the time point 9:00 A.M. as $t = 0$. Then $t = 4$ corresponds to 1:00 P.M. Let $N(t)$ be the number of fish caught at or prior to t; $\big\{N(t) : t \ge 0\big\}$ is a Poisson process with rate 2. Let $X_1, X_2, \ldots, X_6$ be six uniformly distributed independent random variables over $[0, 4]$. By theorem 12.4, given that $N(4) = 6$, the time that the fisherman caught the first fish is $Y = \min(X_1, X_2, \ldots, X_6)$. Therefore, the desired probability is

$$P(Y < 1) = 1 - P(Y \ge 1) = 1 - P\big(\min(X_1, X_2, \ldots, X_6) \ge 1\big)$$
$$= 1 - P(X_1 \ge 1, X_2 \ge 1, \ldots, X_6 \ge 1)$$
$$= 1 - P(X_1 \ge 1)P(X_2 \ge 1) \cdots P(X_6 \ge 1) = 1 - \Big(\frac{3}{4}\Big)^6 = 0.822.$$

5. Let S_1, S_2, and S_3 be the number of meters of wire manufactured, after the inspector left, until the first, second, and third fractures appeared, respectively. By Theorem 12.4, given that $N(200) = 3$, the joint probability density function of S_1, S_2, and S_3 is

$$f_{S_1,S_2,S_3|N(200)}(t_1, t_2, t_3 \mid 3) = \frac{3!}{8,000,000}, \qquad 0 < t_1 < t_2 < t_3 < 200.$$

Using this, the probability we are interested in, is given by the following triple integral:

$$\begin{aligned} P(S_1 + 60 < S_2,\ S_2 + 60 < S_3) &= \int_0^{80} \int_{t_1+60}^{140} \int_{t_2+60}^{200} \frac{3!}{8,000,000}\, dt_3\, dt_2\, dt_1 \\ &= \frac{3!}{8,000,000} \int_0^{80} \Big[\int_{t_1+60}^{140} (140 - t_2)\, dt_2 \Big] dt_1 \\ &= \frac{6}{8,000,000} \int_0^{80} \Big(3200 - 80t_1 + \frac{1}{2}t_1^2 \Big) dt_1 \\ &= \frac{6}{8,000,000} \Big[\frac{1}{6}t_1^3 - 40t_1^2 + 3200t_1 \Big]_0^{80} \\ &= \frac{8}{125} = 0.064. \end{aligned}$$

6. By (12.8), the conditional probability density function of S_k, given that $N(t) = n$, is

$$f_{S_k|N(t)}(x|n) = \frac{n!}{(n-k)!\,(k-1)!} \cdot \frac{1}{t}\Big(\frac{x}{t}\Big)^{k-1}\Big(1 - \frac{x}{t}\Big)^{n-k}, \quad 0 \le x \le t.$$

Therefore,

$$E\big[S_k \mid N(t) = n\big] = \int_0^t \frac{n!}{(n-k)!\,(k-1)!}\, x \cdot \frac{1}{t}\Big(\frac{x}{t}\Big)^{k-1}\Big(1 - \frac{x}{t}\Big)^{n-k} dx.$$

Letting $x/t = u$, we have $(1/t)\,dx = du$. Thus

$$E\big[S_k \mid N(t) = n\big] = \frac{n!}{(n-k)!\,(k-1)!}\, t \int_0^1 u^k (1-u)^{n-k}\, du.$$

What we want to show follows from the following relations discussed in Section 7.5:

$$\int_0^1 u^k (1-u)^{n-k}\, du = B(k+1, n-k+1) = \frac{\Gamma(k+1)\Gamma(n-k+1)}{\Gamma(n+2)} = \frac{k!\,(n-k)!}{(n+1)!}.$$

7. Let T be the time until the next arrival, and let S be the time until the next departure. By the memoryless property of exponential random variables, T and S are exponential random variables with parameters λ and μ, respectively. They are independent by the definition of an $M/M/1$ queue. Thus

$$P(A) = P(T > t \text{ and } S > T) = P(T > t)P(S > t) = e^{-\lambda t} \cdot e^{-\mu t} = e^{-(\lambda+\mu)t},$$

$$\begin{aligned}P(B) = P(S > T) &= \int_0^\infty P(S > T \mid T = u)\lambda e^{-\lambda u}\,du \\ &= \int_0^\infty P(S > u \mid T = u)\lambda e^{-\lambda u}\,du = \int_0^\infty P(S > u)\lambda e^{-\lambda u}\,du \\ &= \lambda \int_0^\infty e^{-\mu u} \cdot e^{\lambda u}\,du = \frac{\lambda}{\lambda+\mu}.\end{aligned}$$

A similar calculation shows that

$$\begin{aligned}P(AB) = P(S > T > t) &= \int_t^\infty P(S > T \mid T = u)\lambda e^{-\lambda u}\,du \\ &= \int_t^\infty e^{-\mu u} \cdot \lambda e^{-\lambda u}\,du = \frac{\lambda}{\lambda+\mu} e^{-(\lambda+\mu)t} = P(A)P(B).\end{aligned}$$

8. (a) Let X be the number of customers arriving to the queue during a service period S. Then

$$\begin{aligned}P(X = n) &= \int_0^\infty P(X = n \mid S = t)\mu e^{-\mu t}\,dt = \int_0^\infty \frac{e^{-\lambda t}\,(\lambda t)^n}{n!}\mu e^{-\mu t}\,dt \\ &= \frac{\lambda^n \mu}{n!}\int_0^\infty t^n\, e^{-(\lambda+\mu)t}\,dt = \frac{\lambda^n \mu}{n!\,(\lambda+\mu)}\int_0^\infty t^n(\lambda+\mu)e^{-(\lambda+\mu)t}\,dt.\end{aligned}$$

Note that $(\lambda+\mu)e^{-(\lambda+\mu)t}$ is the probability density function of an exponential random variable Z with parameter $\lambda+\mu$. Hence

$$P(X = n) = \frac{\lambda^n \mu}{n!\,(\lambda+\mu)} E(Z^n).$$

By Example 11.4,

$$E(Z^n) = \frac{n!}{(\lambda+\mu)^n}.$$

Therefore,

$$P(X = n) = \frac{\lambda^n \mu}{(\lambda+\mu)^{n+1}} = \Big(1 - \frac{\lambda}{\lambda+\mu}\Big)^n \Big(\frac{\mu}{\lambda+\mu}\Big), \quad n \geq 0.$$

This is the probability mass function of a geometric random variable with parameter $\mu/(\lambda+\mu)$.

(b) Due to the memoryless property of exponential random variables, the remaining service time of the customer being served is also exponential with parameter μ. Hence we want to find the number of new customers arriving during a period, which is the sum of $n+1$ independent exponential random variables. Since during each of these service times the number of new arrivals is geometric with parameter $\mu/(\lambda+\mu)$, during the entire period under consideration, the distribution of the total number of new customers arriving is the sum of $n+1$ independent geometric random variables each with parameter $\mu/(\lambda+\mu)$, which is negative binomial with parameters $n+1$ and $\mu/(\lambda+\mu)$.

9. It is straightforward to check that $M(t)$ is stationary, orderly, and possesses independent increments. Clearly, $M(0) = 0$. Thus $\{M(t)\colon t \geq 0\}$ is a Poisson process. To find its rate, note that, for $0 \leq k < \infty$,

$$\begin{aligned}
P\big(M(t) = k\big) &= \sum_{n=k}^{\infty} P\big(M(t) = k \mid N(t) = n\big) P\big(N(t) = n\big) \\
&= \sum_{n=k}^{\infty} \binom{n}{k} p^k (1-p)^{n-k} \cdot \frac{e^{-\lambda t}(\lambda t)^n}{n!} \\
&= \frac{e^{-\lambda t} p^k}{k!\,(1-p)^k} \sum_{n=k}^{\infty} \frac{\big[\lambda t(1-p)\big]^n}{(n-k)!} \\
&= \frac{e^{-\lambda t} p^k}{k!\,(1-p)^k} \cdot \big[\lambda t(1-p)\big]^k \sum_{n=k}^{\infty} \frac{\big[\lambda t(1-p)\big]^{n-k}}{(n-k)!} \\
&= \frac{e^{-\lambda t} p^k}{k!} (\lambda t)^k e^{\lambda t(1-p)} = \frac{(\lambda p t)^k}{k!} e^{-\lambda p t}.
\end{aligned}$$

This shows that the parameter of $\{M(t)\colon t \geq 0\}$ is λp.

10. Note that $P\big(V_i = \min(V_1, V_2, \dots, V_k)\big)$ is the probability that the first shock occurring to the system is of type i. Suppose that the first shock occurs to the system at time u. If we label the time point u as $t = 0$, then from that point on, by stationarity and the independent-increments property, probabilistically, the behavior of these Poisson processes is identical to the system considered prior to u. So the probability that the second shock is of type i is identical to the probability that the first shock is of type i, and so on. Hence they are all equal to $P\big(V_i = \min(V_1, V_2, \dots, V_k)\big)$. To find this probability, note that, for $1 \leq j \leq k$, V_j's, are independent exponential random variables, and the probability density function of V_j is $\lambda_j e^{-\lambda_j t}$. Thus $P(V_j > u) = e^{-\lambda_j u}$. By conditioning on V_i, we have

$$\begin{aligned}
&P\big(V_i = \min(V_1, \dots, V_k)\big) \\
&\quad = \int_0^{\infty} P\big(\min(V_1, \dots, V_k) = V_i \mid V_i = u\big) \lambda_i e^{-\lambda_i u}\,du \\
&\quad = \lambda_i \int_0^{\infty} P\big(\min(V_1, \dots, V_k) = u \mid V_i = u\big) e^{-\lambda_i u}\,du \\
&\quad = \lambda_i \int_0^{\infty} P(V_1 \geq u, \dots, V_{i-1} \geq u, V_{i+1} \geq u, \dots, V_k \geq u \mid V_i = u) e^{-\lambda_i u}\,du \\
&\quad = \lambda_i \int_0^{\infty} P(V_1 \geq u, \dots, V_{i-1} \geq u, V_{i+1} \geq u, \dots, V_k \geq u) e^{-\lambda_i u}\,du \\
&\quad = \lambda_i \int_0^{\infty} P(V_1 \geq u) \cdots P(V_{i-1} \geq u) P(V_{i+1} \geq u) \cdots P(V_k \geq u) e^{-\lambda_i u}\,du
\end{aligned}$$

$$= \lambda_i \int_0^\infty e^{-\lambda_1 u} \cdots e^{-\lambda_{i-1} u} \cdot e^{-\lambda_{i+1} u} \cdots e^{-\lambda_k u} \cdot e^{-\lambda_i u}\, du$$

$$= \lambda_i \int_0^\infty e^{-(\lambda_1 + \cdots + \lambda_k)u}\, du = \lambda_i \int_0^\infty e^{-\lambda u}\, du = \frac{\lambda_i}{\lambda}.$$

12.3 MARKOV CHAINS

1. $\{X_n : n = 1, 2, \dots\}$ is not a Markov chain. For example, $P(X_4 = 1)$ depends on all the values of X_1, X_2, and X_3, and not just X_3. That is, whether or not the fourth person selected is female depends on the genders of all three persons selected prior to the fourth and not only on the gender of the third person selected.

2. For $j \geq 0$,

$$P(X_n = j) = \sum_{i=0}^{\infty} P(X_n = j \mid X_0 = i)P(X_0 = i) = \sum_{i=0}^{\infty} p_{ij}^n p(i),$$

where p_{ij}^n is the ijth entry of the matrix $\boldsymbol{P}^n$.

3. The transition probability matrix of this Markov chain is

$$\boldsymbol{P} = \begin{pmatrix} 0 & 1/2 & 0 & 0 & 0 & 1/2 \\ 1/2 & 0 & 1/2 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 1/2 & 0 & 0 \\ 0 & 0 & 1/2 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 1/2 & 0 & 1/2 \\ 1/2 & 0 & 0 & 0 & 1/2 & 0 \end{pmatrix}.$$

By calculating $\boldsymbol{P}^4$ and $\boldsymbol{P}^5$, we will find that, (a) the probability that in 4 transitions the Markov chain returns to 1 is $P_{11}^4 = 3/8$; (b) the probability that, in 5 transitions, the Markov chain enters 2 or 6 is

$$p_{12}^5 + p_{16}^5 = \frac{11}{32} + \frac{11}{32} = \frac{11}{16}.$$

4. ***Solution* 1:** Starting at 0, the process eventually enters 1 or 2 with equal probabilities. Since 2 is absorbing, "never entering 1" is equivalent to eventually entering 2 directly from 0. The probability of that is 1/2.

***Solution* 2:** Let Z be the number of transitions until the first visit to 1. Note that state 2 is absorbing. If the process enters 2, it will always remain there. Hence $Z = n$ if and only if the first $n-1$ transitions are from 0 to 0, and the nth transition is from 0 to 1, implying that

$$P(Z=n)=\Big(\frac{1}{2}\Big)^{n-1}\Big(\frac{1}{4}\Big), \qquad n=1,2,\ldots .$$

The probability that the process ever enters 1 is

$$P(Z<\infty)=\sum_{n=1}^{\infty}\Big(\frac{1}{2}\Big)^{n-1}\Big(\frac{1}{4}\Big)=\frac{1/4}{1-(1/2)}=\frac{1}{2}.$$

Therefore, the probability that the process never enters 1 is $1-(1/2)=1/2$.

5. **(a)** By the Markovian property, given the present, the future is independent of the past. Thus the probability that tomorrow Emmett will not take the train to work is, simply, $p_{21}+p_{23}=1/2+1/6=2/3$.

(b) The desired probability is

$$p_{21}p_{11}+p_{21}p_{13}+p_{23}p_{31}+p_{23}p_{33}=1/4.$$

6. Let X_n denote the number of balls in urn I after n transfers. The stochastic process $\{X_n : n = 0, 1, \ldots\}$ is a Markov chain with state space $\{0, 1, \ldots, 5\}$ and transition probability matrix

$$\boldsymbol{P}=\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0\\ 1/5 & 0 & 4/5 & 0 & 0 & 0\\ 0 & 2/5 & 0 & 3/5 & 0 & 0\\ 0 & 0 & 3/5 & 0 & 2/5 & 0\\ 0 & 0 & 0 & 4/5 & 0 & 1/5\\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

Direct calculations show that

$$\boldsymbol{P}^{(6)}=\boldsymbol{P}^{6}=\begin{pmatrix} \frac{241}{3125} & 0 & \frac{2044}{3125} & 0 & \frac{168}{625} & 0\\[4pt] 0 & \frac{5293}{15625} & 0 & \frac{9492}{15625} & 0 & \frac{168}{3125}\\[4pt] \frac{1022}{15625} & 0 & \frac{9857}{15625} & 0 & \frac{4746}{15625} & 0\\[4pt] 0 & \frac{4746}{15625} & 0 & \frac{9857}{15625} & 0 & \frac{1022}{15625}\\[4pt] \frac{168}{3125} & 0 & \frac{9492}{15625} & 0 & \frac{5293}{15625} & 0\\[4pt] 0 & \frac{168}{625} & 0 & \frac{2044}{3125} & 0 & \frac{241}{3125} \end{pmatrix}.$$

Hence, by Theorem 12.5,

$$P(X_6 = 4) = 0 \cdot \frac{168}{625} + \frac{1}{15} \cdot 0 + \frac{2}{15} \cdot \frac{4746}{15625} + \frac{3}{15} \cdot 0 + \frac{4}{15} \cdot \frac{5293}{15625} + \frac{5}{15} \cdot 0 = 0.1308.$$

7. By drawing a transition graph, it is readily seen that this Markov chain consists of the recurrent classes $\{0, 3\}$ and $\{2, 4\}$ and the transient class $\{1\}$.

8. Let Z_n be the outcome of the nth toss. Then

$$X_{n+1} = \max(X_n, Z_{n+1})$$

shows that $\{X_n \colon n = 1, 2, \dots\}$ is a Markov chain. Its state space is $\{1, 2, \dots, 6\}$, and its transition probability matrix is given by

$$\boldsymbol{P} = \begin{pmatrix} 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 \\ 0 & 2/6 & 1/6 & 1/6 & 1/6 & 1/6 \\ 0 & 0 & 3/6 & 1/6 & 1/6 & 1/6 \\ 0 & 0 & 0 & 4/6 & 1/6 & 1/6 \\ 0 & 0 & 0 & 0 & 5/6 & 1/6 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

It is readily seen that no two states communicate with each other. Therefore, we have six classes of which $\{1\}$, $\{2\}$, $\{3\}$, $\{4\}$, $\{5\}$, are transient, and $\{6\}$ is recurrent (in fact, absorbing).

9. This can be achieved more easily by drawing a transition graph. An example of a desired matrix is as follows:

$$\begin{pmatrix} 0 & 0 & 1/2 & 0 & 1/2 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1/3 & 2/3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2/5 & 0 & 3/5 \\ 0 & 0 & 0 & 0 & 1/2 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3/5 & 0 & 2/5 \\ 0 & 0 & 0 & 0 & 1/3 & 0 & 2/3 & 0 \end{pmatrix}.$$

10. For $1 \le i \le 7$, starting from state i, let x_i be the probability that the Markov chain will eventually be absorbed into state 4. We are interested in x_6. Applying the law of total probability repeatedly, we obtain the following system of linear equations:

$$\begin{cases} x_1 = (0.3)x_1 + (0.7)x_2 \\ x_2 = (0.3)x_1 + (0.2)x_2 + (0.5)x_3 \\ x_3 = (0.6)x_4 + (0.4)x_5 \\ x_4 = 1 \\ x_5 = x_3 \\ x_6 = (0.1)x_1 + (0.3)x_2 + (0.1)x_3 + (0.2)x_5 + (0.2)x_6 + (0.1)x_7 \\ x_7 = 0. \end{cases}$$

Solving this system of equations, we obtain

$$\begin{cases} x_1 = x_2 = x_3 = x_4 = x_5 = 1 \\ x_6 = 0.875 \\ x_7 = 0. \end{cases}$$

Therefore, the probability is 0.875 that, starting from state 6, the Markov chain will eventually be absorbed into state 4.

11. Let π_1, π_2, and π_3 be the long-run probabilities that the sportsman devotes to horseback riding, sailing, and scuba diving, respectively. Then, by Theorem 12.7, π_1, π_2, and π_3 are obtained from solving the system of equations.

$$\begin{pmatrix} \pi_1 \\ \pi_2 \\ \pi_3 \end{pmatrix} = \begin{pmatrix} 0.20 & 0.32 & 0.60 \\ 0.30 & 0.15 & 0.13 \\ 0.50 & 0.53 & 0.27 \end{pmatrix} \begin{pmatrix} \pi_1 \\ \pi_2 \\ \pi_3 \end{pmatrix}$$

along with $\pi_1 + \pi_2 + \pi_3 = 1$. The matrix equation above gives us the following system of equations

$$\begin{cases} \pi_1 = 0.20\pi_1 + 0.32\pi_2 + 0.60\pi_3 \\ \pi_2 = 0.30\pi_1 + 0.15\pi_2 + 0.13\pi_3 \\ \pi_3 = 0.50\pi_1 + 0.53\pi_2 + 0.27\pi_3. \end{cases}$$

By choosing any two of these equations along with $\pi_1 + \pi_2 + \pi_3 = 1$, we obtain a system of three equations in three unknowns. Solving that system yields $\pi_1 = 0.38856$, $\pi_2 = 0.200056$, and $\pi_3 = 0.411383$. Hence the long-run probability that on a randomly selected vacation day the sportsman sails is approximately 0.20.

12. For $n \geq 1$, let

$$X_n = \begin{cases} 1 & \text{if the } n\text{th fish caught is trout} \\ 0 & \text{if the } n\text{th fish caught is not trout.} \end{cases}$$

Then $\{X_n : n = 1, 2, \dots\}$ is a Markov chain with state space $\{0, 1\}$ and transition probability matrix

$$\begin{pmatrix} 10/11 & 1/11 \\ 8/9 & 1/9 \end{pmatrix}$$

Let π_0 be the fraction of fish in the lake that are not trout, and π_1 be the fraction of fish in the lake that are trout. Then, by Theorem 12.7, π_0 and π_1 satisfy

$$\begin{pmatrix} \pi_0 \\ \pi_1 \end{pmatrix} = \begin{pmatrix} 10/11 & 8/9 \\ 1/11 & 1/9 \end{pmatrix} \begin{pmatrix} \pi_0 \\ \pi_1 \end{pmatrix},$$

which gives us the following system of equations

$$\begin{cases} \pi_0 = (10/11)\pi_0 + (8/9)\pi_1 \\ \pi_1 = (1/11)\pi_0 + (1/9)\pi_1. \end{cases}$$

By choosing any one of these equations along with the relation $\pi_0 + \pi_1 = 1$, we obtain a system of two equations in two unknown. Solving that system yields $\pi_0 = 88/97 \approx 0.907$ and $\pi_1 = 9/97 \approx 0.093$. Therefore, approximately 9.3% of the fish in the lake are trout.

13. Let

$$X_n = \begin{cases} 1 & \text{if the } n\text{th card is drawn by player I} \\ 2 & \text{if the } n\text{th card is drawn by player II} \\ 3 & \text{if the } n\text{th card is drawn by player III.} \end{cases}$$

$\{X_n : n = 1, 2, \dots\}$ is a Markov chain with probability transition matrix

$$\boldsymbol{P} = \begin{pmatrix} 48/52 & 4/52 & 0 \\ 0 & 39/52 & 13/52 \\ 12/52 & 0 & 40/52 \end{pmatrix}.$$

Let π_1, π_2, and π_3 be the proportion of cards drawn by players I, II, and III, respectively. π_1, π_2, and π_3 are obtained from

$$\begin{pmatrix} \pi_1 \\ \pi_2 \\ \pi_3 \end{pmatrix} = \begin{pmatrix} 12/13 & 0 & 3/13 \\ 1/13 & 3/4 & 0 \\ 0 & 1/4 & 10/13 \end{pmatrix} \begin{pmatrix} \pi_1 \\ \pi_2 \\ \pi_3 \end{pmatrix}$$

and $\pi_1 + \pi_2 + \pi_3 = 1$, which gives $\pi_1 = 39/64 \approx 0.61$, $\pi_2 = 12/64 \approx 0.19$, and $\pi_3 = 13/64 \approx 0.20$.

14. For $1 \le i \le 9$, let π_i be the probability that the mouse is in cell i, $1 \le i \le 9$, at a random time in the future. Then π_i's satisfy

$$\begin{pmatrix}\pi_1\\ \pi_2\\ \pi_3\\ \pi_4\\ \pi_5\\ \pi_6\\ \pi_7\\ \pi_8\\ \pi_9\end{pmatrix} = \begin{pmatrix} 0 & 1/3 & 0 & 1/3 & 0 & 0 & 0 & 0 & 0\\ 1/2 & 0 & 1/2 & 0 & 1/4 & 0 & 0 & 0 & 0\\ 0 & 1/3 & 0 & 0 & 0 & 1/3 & 0 & 0 & 0\\ 1/2 & 0 & 0 & 0 & 1/4 & 0 & 1/2 & 0 & 0\\ 0 & 1/3 & 0 & 1/3 & 0 & 1/3 & 0 & 1/3 & 0\\ 0 & 0 & 1/2 & 0 & 1/4 & 0 & 0 & 0 & 1/2\\ 0 & 0 & 0 & 1/3 & 0 & 0 & 0 & 1/3 & 0\\ 0 & 0 & 0 & 0 & 1/4 & 0 & 1/2 & 0 & 1/2\\ 0 & 0 & 0 & 0 & 0 & 1/3 & 0 & 1/3 & 0 \end{pmatrix} \begin{pmatrix}\pi_1\\ \pi_2\\ \pi_3\\ \pi_4\\ \pi_5\\ \pi_6\\ \pi_7\\ \pi_8\\ \pi_9\end{pmatrix}.$$

Solving this system of equations along with $\sum_{i=1}^{9} \pi_1$, we obtain

$$\pi_1 = \pi_3 = \pi_7 = \pi_9 = 1/12,$$
$$\pi_2 = \pi_4 = \pi_6 = \pi_8 = 1/8,$$
$$\pi_5 = 1/6.$$

15. Let X_n denote the number of balls in urn I after n transfers. The stochastic process $\{X_n : n = 0, 1, \dots\}$ is a Markov chain with state space $\{0, 1, \dots, 5\}$ and transition probability matrix

$$\boldsymbol{P} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0\\ 1/5 & 0 & 4/5 & 0 & 0 & 0\\ 0 & 2/5 & 0 & 3/5 & 0 & 0\\ 0 & 0 & 3/5 & 0 & 2/5 & 0\\ 0 & 0 & 0 & 4/5 & 0 & 1/5\\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

Clearly, $\{X_n : n = 0, 1, \dots\}$ is an irreducible recurrent Markov chain; since it is finite-state, it is positive recurrent. However, $\{X_n : n = 0, 1, \dots\}$ is not aperiodic, and the period of each state is 2. Hence the limiting probabilities do not exist. For $0 \le i \le 5$, let π_i be the fraction of time urn I contains i balls. Then with this interpretation, π_i's satisfy the following equations

$$\begin{pmatrix}\pi_0\\ \pi_1\\ \pi_2\\ \pi_3\\ \pi_4\\ \pi_5\end{pmatrix} = \begin{pmatrix} 0 & 1/5 & 0 & 0 & 0 & 0\\ 1 & 0 & 2/5 & 0 & 0 & 0\\ 0 & 4/5 & 0 & 3/5 & 0 & 0\\ 0 & 0 & 3/5 & 0 & 4/5 & 0\\ 0 & 0 & 0 & 2/5 & 0 & 1\\ 0 & 0 & 0 & 0 & 1/5 & 0 \end{pmatrix} \begin{pmatrix}\pi_0\\ \pi_1\\ \pi_2\\ \pi_3\\ \pi_4\\ \pi_5\end{pmatrix},$$

$\sum_{i=0}^{5} \pi_i = 1$. Solving these equations, we obtain

$$\pi_0 = \pi_5 = 1/31,$$
$$\pi_1 = \pi_4 = 5/31,$$
$$\pi_2 = \pi_3 = 10/31.$$

Therefore, the fraction of time an urn is empty is $\pi_0 + \pi_5 = 2/31$. Hence the expected number of balls transferred between two consecutive times that an urn becomes empty is $31/2 = 15.5$.

16. ***Solution* 1:** Let X_n be the number of balls in urn I immediately before the nth game begins. Then $\{X_n : n = 1, 2, \dots\}$ is a Markov chain with state space $\{0, 1, \dots, 7\}$ and transition probability matrix

$$\boldsymbol{P} = \begin{pmatrix} 3/4 & 1/4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1/4 & 1/2 & 1/4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1/4 & 1/2 & 1/4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1/4 & 1/2 & 1/4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/4 & 1/2 & 1/4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/4 & 1/2 & 1/4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/4 & 1/2 & 1/4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1/4 & 3/4 \end{pmatrix}.$$

Since the transition probability matrix is doubly stochastic; that is, the sum of each column is also 1, for $i = 0, 1, \dots, 7$, π_i, the long-run probability that the number of balls in urn I immediately before a game begins is 1/8 (see Example 12.35). This implies that the long-run probability mass function of the number of balls in urn I or II is 1/8 for $i = 0, 1, \dots, 7$.

***Solution* 2:** Let X_n be the number of balls in the urn selected at step 1 of the nth game. Then $\{X_n : n = 1, 2, \dots\}$ is a Markov chain with state space $\{0, 1, \dots, 7\}$ and transition probability matrix

$$\boldsymbol{P} = \begin{pmatrix} 1/2 & 0 & 0 & 0 & 0 & 0 & 0 & 1/2 \\ 1/4 & 1/4 & 0 & 0 & 0 & 0 & 1/4 & 1/4 \\ 0 & 1/4 & 1/4 & 0 & 0 & 1/4 & 1/4 & 0 \\ 0 & 0 & 1/4 & 1/4 & 1/4 & 1/4 & 0 & 0 \\ 0 & 0 & 0 & 1/2 & 1/2 & 0 & 0 & 0 \\ 0 & 0 & 1/4 & 1/4 & 1/4 & 1/4 & 0 & 0 \\ 0 & 1/4 & 1/4 & 0 & 0 & 1/4 & 1/4 & 0 \\ 1/4 & 1/4 & 0 & 0 & 0 & 0 & 1/4 & 1/4 \end{pmatrix}.$$

Since the transition probability matrix is doubly stochastic; that is, the sum of each column is also 1, for $i = 0, 1, \dots, 7$, π_i, the long-run probability that the number of balls in the urn selected at step 1 of a game is 1/8 (see Example 12.35). This implies that the long-run probability mass function of the number of balls in urn I or II is 1/8 for $i = 0, 1, \dots, 7$.

17. For $i \geq 0$, state i is directly accessible from 0. On the other hand, i is accessible from $i + 1$. These two facts make it possible for all states to communicate with each other. Therefore, the Markov chain has only one class. Since 0 is recurrent and aperiodic (note that $p_{00} > 0$ makes 0 aperiodic), all states are recurrent and aperiodic. Let π_k be the long-run probability that a computer selected at the end of a semester will last at least k additional semesters. Solving

$$\begin{pmatrix} \pi_0 \\ \pi_1 \\ \pi_2 \\ \vdots \end{pmatrix} = \begin{pmatrix} p_1 & 1 & 0 & 0 & \dots \\ p_2 & 0 & 1 & 0 & \dots \\ p_3 & 0 & 0 & 1 & \dots \\ \vdots & & & & \end{pmatrix} \begin{pmatrix} \pi_0 \\ \pi_1 \\ \pi_2 \\ \vdots \end{pmatrix}$$

along with $\sum_{i=0}^{\infty} \pi_i = 1$, we obtain

$$\pi_0 = \frac{1}{1 + \sum_{i=1}^{\infty}(1 - p_1 - p_2 - \cdots - p_i)},$$

$$\pi_k = \frac{1 - p_1 - p_2 - \cdots - p_k}{1 + \sum_{i=1}^{\infty}(1 - p_1 - p_2 - \cdots - p_i)}, \quad k \geq 1.$$

18. Let DN denote the state at which the last movie Mr. Gorfin watched was not a drama, but the one before that was a drama. Define DD, ND, and NN similarly, and label the states DD, DN, ND, and NN by 0, 1, 2, and 3, respectively. Let $X_n = 0$ if the nth and $(n-1)$st movies Mr. Gorfin watched were both dramas. Define $X_n = 1$, 2, and 3 similarly. Then $\{X_n \colon n = 1, 2, \dots\}$ is a Markov chain with state space $\{0, 1, 2, 3\}$ and transition probability matrix

$$\boldsymbol{P} = \begin{pmatrix} 7/8 & 1/8 & 0 & 0 \\ 0 & 0 & 1/2 & 1/2 \\ 1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 1/8 & 7/8 \end{pmatrix}.$$

(a) If the first two movies Mr. Gorfin watched last weekend were dramas, the probability that the fourth one is a drama is $p_{00}^2 + p_{02}^2$. Since

$$\boldsymbol{P}^2 = \begin{pmatrix} 49/64 & 7/64 & 1/16 & 1/16 \\ 1/4 & 1/4 & 1/16 & 7/16 \\ 7/16 & 1/16 & 1/4 & 1/4 \\ 1/16 & 1/16 & 7/64 & 49/64 \end{pmatrix},$$

the desired probability is $(49/64) + (1/16) = 53/64$.

(b) Let π_0 denote the long-run probability that Mr. Gorfin watches two dramas in a row. Define π_1, π_2, and π_3 similarly. We have that,

$$\begin{pmatrix} \pi_0 \\ \pi_1 \\ \pi_2 \\ \pi_3 \end{pmatrix} = \begin{pmatrix} 7/8 & 0 & 1/2 & 0 \\ 1/8 & 0 & 1/2 & 0 \\ 0 & 1/2 & 0 & 1/8 \\ 0 & 1/2 & 0 & 7/8 \end{pmatrix} \begin{pmatrix} \pi_0 \\ \pi_1 \\ \pi_2 \\ \pi_3 \end{pmatrix}.$$

Solving this system along with $\pi_0 + \pi_1 + \pi_2 + \pi_3 = 1$, we obtain $\pi_0 = 2/5$, $\pi_1 = 1/10$, $\pi_2 = 1/10$, and $\pi_3 = 2/5$. Hence the probability that Mr. Gorfin watches two dramas in a row is 2/5.

19. Clearly,

$$X_{n+1}=\begin{cases}0 & \text{if the } (n+1)\text{st outcome is 6}\\ 1+X_n & \text{otherwise.}\end{cases}$$

This relation shows that $\{X_n\colon n=1,2,\dots\}$ is a Markov chain. Its transition probability matrix is given by

$$\boldsymbol{P}=\begin{pmatrix}1/6 & 5/6 & 0 & 0 & 0 & \dots\\ 1/6 & 0 & 5/6 & 0 & 0 & \dots\\ 1/6 & 0 & 0 & 5/6 & 0 & \dots\\ 1/6 & 0 & 0 & 0 & 5/6 & \dots\\ \vdots & & & & & \end{pmatrix}.$$

It is readily seen that all states communicate with 0. Therefore, by transitivity of the communication property, all states communicate with each other. Therefore, the Markov chain is irreducible. Clearly, 0 is recurrent. Since $p_{00}>0$, it is aperiodic as well. Hence all states are recurrent and aperiodic. On the other hand, starting at 0, the expected number of transitions until the process returns to 0 is 6. This is because the number of tosses until the next 6 obtained is a geometric random variable with probability of success $p=1/6$, and hence expected value $1/p=6$. Therefore, 0, and hence all other states are positive recurrent. Next, a simple probabilistic argument shows that,

$$\pi_i=\Big(\frac{5}{6}\Big)^i\Big(\frac{1}{6}\Big),\quad i=0,1,2,\dots.$$

This can also be shown by solving the following system of equations:

$$\begin{cases}\begin{pmatrix}\pi_0\\ \pi_1\\ \pi_2\\ \pi_3\\ \vdots\end{pmatrix}=\begin{pmatrix}1/6 & 1/6 & 1/6 & 1/6 & \dots\\ 5/6 & 0 & 0 & 0 & \dots\\ 0 & 5/6 & 0 & 0 & \dots\\ 0 & 0 & 5/6 & 0 & \dots\\ \vdots & & & & \end{pmatrix}\begin{pmatrix}\pi_0\\ \pi_1\\ \pi_2\\ \pi_3\\ \vdots\end{pmatrix}\\ \pi_0+\pi_1+\pi_2+\cdots=1.\end{cases}$$

20. **(a)** Let

$$X_n=\begin{cases}1 & \text{if Alberto wins the } n\text{th game}\\ 0 & \text{if Alberto loses the } n\text{th game.}\end{cases}$$

Then $\{X_n\colon n=1,2,\dots\}$ is a Markov chain with state space $\{0,1\}$. Its transition probability matrix is $\boldsymbol{P}=\begin{pmatrix}1-p & p\\ p & 1-p\end{pmatrix}$. Using induction, we will now show that

$$\boldsymbol{P}^{(n)}=\boldsymbol{P}^n=\begin{pmatrix}\frac{1}{2}+\frac{1}{2}(1-2p)^n & \frac{1}{2}-\frac{1}{2}(1-2p)^n\\ \frac{1}{2}-\frac{1}{2}(1-2p)^n & \frac{1}{2}+\frac{1}{2}(1-2p)^n\end{pmatrix}.$$

Clearly, for $n = 1$, $\boldsymbol{P}^{(1)} = \boldsymbol{P}$. Suppose that

$$\boldsymbol{P}^{(n)} = \begin{pmatrix} \frac{1}{2} + \frac{1}{2}(1-2p)^n & \frac{1}{2} - \frac{1}{2}(1-2p)^n \\ \frac{1}{2} - \frac{1}{2}(1-2p)^n & \frac{1}{2} + \frac{1}{2}(1-2p)^n \end{pmatrix}.$$

We will show that

$$\boldsymbol{P}^{n+1} = \begin{pmatrix} \frac{1}{2} + \frac{1}{2}(1-2p)^{n+1} & \frac{1}{2} - \frac{1}{2}(1-2p)^{n+1} \\ \frac{1}{2} - \frac{1}{2}(1-2p)^{n+1} & \frac{1}{2} + \frac{1}{2}(1-2p)^{n+1} \end{pmatrix}.$$

To do so, note that

$$\boldsymbol{P}^{(n+1)} = \begin{pmatrix} p_{00} & p_{01} \\ p_{10} & p_{11} \end{pmatrix} \begin{pmatrix} p_{00}^n & p_{01}^n \\ p_{10}^n & p_{11}^n \end{pmatrix} = \begin{pmatrix} p_{00}p_{00}^n + p_{01}p_{10}^n & p_{00}p_{01}^n + p_{01}p_{11}^n \\ p_{10}p_{00}^n + p_{11}p_{10}^n & p_{10}p_{01}^n + p_{11}p_{11}^n \end{pmatrix}.$$

Thus

$$\begin{aligned} p_{11}^{n+1} &= p_{10}p_{01}^n + p_{11}p_{11}^n = p\Big[\frac{1}{2} - \frac{1}{2}(1-2p)^n\Big] + (1-p)\Big[\frac{1}{2} + \frac{1}{2}(1-2p)^n\Big] \\ &= \frac{1}{2}\big[p + (1-p)\big] + \frac{1}{2}(1-2p)^n\big[-p + (1-p)\big] = \frac{1}{2} + \frac{1}{2}(1-2p)^{n+1}. \end{aligned}$$

This establishes what we wanted to show. The proof that $p_{00}^{n+1} = \frac{1}{2} + \frac{1}{2}(1-2p)^{n+1}$ is identical to what we just showed. We have

$$P_{01}^{n+1} = 1 - P_{00}^{n+1} = 1 - \Big[\frac{1}{2} + \frac{1}{2}(1-2p)^n\Big] = \frac{1}{2} - \frac{1}{2}(1-2p)^n.$$

Similarly,

$$p_{10}^{n+1} = 1 - p_{11}^{n+1} = \frac{1}{2} - \frac{1}{2}(1-2p)^n.$$

(b) Let π_0 and π_1 be the long-run probabilities that Alberto loses and wins a game, respectively. Then

$$\begin{pmatrix} \pi_0 \\ \pi_1 \end{pmatrix} = \begin{pmatrix} 1-p & p \\ p & 1-p \end{pmatrix} \begin{pmatrix} \pi_0 \\ \pi_1 \end{pmatrix},$$

and $\pi_0 + \pi_1 = 1$ imply that $\pi_0 = \pi_1 = 1/2$. Therefore, the expected number of games Alberto will play between two consecutive wins is $1/\pi_1 = 2$.

21. For each $j \geq 0$, $\lim_{n\to\infty} p_{ij}^n$ exists and is independent of i if the following system of equations, in $\pi_0, \pi_1, \ldots$, have a unique solution.

$$\begin{cases} \begin{pmatrix} \pi_0 \\ \pi_1 \\ \pi_2 \\ \pi_3 \\ \vdots \end{pmatrix} = \begin{pmatrix} 1-p & 1-p & 0 & 0 & 0 & 0 & \ldots \\ p & 0 & 1-p & 0 & 0 & 0 & \ldots \\ 0 & p & 0 & 1-p & 0 & 0 & \ldots \\ 0 & 0 & p & 0 & 1-p & 0 & \ldots \\ \vdots & & & & & & \end{pmatrix} \begin{pmatrix} \pi_0 \\ \pi_1 \\ \pi_2 \\ \pi_3 \\ \vdots \end{pmatrix} \\ \pi_0 + \pi_1 + \pi_2 + \cdots = 1. \end{cases}$$

From the matrix equation, we obtain

$$\pi_i = \Big(\frac{p}{1-p}\Big)^i \pi_0, \qquad i = 0, 1, \ldots .$$

For these quantities to satisfy $\sum_{i=0}^{\infty} \pi_i = 1$, we need the geometric series $\sum\limits_{i=0}^{\infty} \Big(\frac{p}{1-p}\Big)^i$ to converge. Hence we must have $p < 1 - p$, or $p < 1/2$. Therefore, for $p < 1/2$, this irreducible, aperiodic Markov chain which is positively recurrent has limiting probabilities. Note that, for $p < 1/2$,

$$\pi_0 \sum_{i=0}^{\infty} \Big(\frac{p}{1-p}\Big)^i = 1$$

yields $\pi_0 = 1 - \dfrac{p}{1-p}$. Thus the limiting probabilities are

$$\pi_i = \Big(\frac{p}{1-p}\Big)^i \Big(1 - \frac{p}{1-p}\Big), \qquad i = 0, 1, 2, \ldots .$$

22. Let Y_n be Carl's fortune after the nth game. Let X_n be Stan's fortune after the nth game. Let $Z_n = Y_n - X_n$. The $\{Z_n : n = 0, 1, \ldots\}$ is a random walk with state space $\{0, \pm 2, \pm 4, \ldots\}$. We have that $Z_0 = 0$, and at each step either the process moves two units to the right with probability 0.46 or two units to the left with probability 0.54. Let A be the event that, starting at 0, the random walk will eventually enter 2; $P(A)$ is the desired quantity. By the law of total probability,

$$\begin{aligned} P(A) &= P(A \mid Z_1 = 2)P(Z_1 = 2) + P(A \mid Z_1 = -2)P(Z_1 = -2) \\ &= 1 \cdot (0.46) + \big[P(A)\big]^2 \cdot (0.54). \end{aligned}$$

To show that $P(A \mid Z_1 = -2) = \big[P(A)\big]^2$, let E be the event of, starting from -2, eventually entering 0. It should be clear that $P(E) = P(A)$. By independence of E and A, we have

$$P(A \mid Z = -2) = P(EA) = P(E)P(A) = \big[P(A)\big]^2.$$

We have shown that $P(A)$, the quantity we are interested in, satisfies

$$(0.54)\big[P(A)\big]^2 - P(A) + 0.46 = 0.$$

This is a quadratic equation in $P(A)$. Solving it gives $P(A) = 23/27 \approx 0.85$.

23. We will use induction on m. For $m = 1$, the relation is, simply, the Markovian property, which is true. Suppose that the relation is valid for $m - 1$. We will show that it is also valid for m. We have

$$\begin{aligned}
&P(X_{n+m} = j \mid X_0 = i_0,\ X_1 = i_1, \dots, X_n = i_n)\\
&\quad = \sum_{i\in\mathcal{S}} P(X_{n+m} = j \mid X_0 = i_0, \dots, X_n = i_n,\ X_{n+m-1} = i)\\
&\qquad\qquad\qquad\qquad P(X_{n+m-1} = i \mid X_0 = i_0, \dots, X_n = i_n)\\
&\quad = \sum_{i\in\mathcal{S}} P(X_{n+m} = j \mid X_{n+m-1} = i)P(X_{n+m-1} = i \mid X_n = i_n)\\
&\quad = \sum_{i\in S} P(X_{n+m} = j \mid X_{n+m-1} = i,\ X_n = i_n)P(X_{n+m-1} = i \mid X_n = i_n)\\
&\quad = P(X_{n+m} = j \mid X_n = i_n),
\end{aligned}$$

where the following relations are valid from the definition of Markov chain: given the present state, the process is independent of the past.

$$P(X_{n+m} = j \mid X_0 = i_0, \dots, X_n = i_n,\ X_{n+m-1} = i) = P(X_{n+m} = j \mid X_{n+m-1} = i),$$

$$P(X_{n+m} = j \mid X_{n+m-1} = i) = P(X_{n+m} = j \mid X_{n+m-1} = i,\ X_n = i_n).$$

24. Let (0, 0), the origin, be denoted by O. It should be clear that, for all $n \geq 0$, $P_{OO}^{2n+1} = 0$. Now, for $n \geq 1$, let Z_1, Z_2, Z_3, and Z_4 be the number of transitions to the right, left, up, and down, respectively. The joint probability mass function of Z_1, Z_2, Z_3, and Z_4 is multinomial. We have

$$\begin{aligned}
P_{OO}^{2n} &= \sum_{i=0}^{n} P(Z_1 = i, Z_2 = i, Z_3 = n - i, Z_4 = n - i)\\
&= \sum_{i=0}^{n} \frac{(2n)!}{i!\,i!\,(n-i)!\,(n-i)!}\Big(\frac{1}{4}\Big)^i\Big(\frac{1}{4}\Big)^i\Big(\frac{1}{4}\Big)^{n-i}\Big(\frac{1}{4}\Big)^{n-i}\\
&= \sum_{i=0}^{n} \frac{(2n)!}{n!\,n!}\cdot\frac{n!}{i!\,(n-i)!}\cdot\frac{n!}{i!\,(n-i)!}\Big(\frac{1}{4}\Big)^{2n}\\
&= \Big(\frac{1}{4}\Big)^{2n}\binom{2n}{n}\sum_{i=0}^{n}\binom{n}{i}^2.
\end{aligned}$$

By Example 2.28, $\sum_{i=0}^{n}\binom{n}{i}^2 = \binom{2n}{n}$. Thus $P^n_{OO} = \left(\frac{1}{4}\right)^{2n}\binom{2n}{n}^2$. Now, by Theorem 2.7 (Stirling's formula),

$$\left(\frac{1}{4}\right)^{2n}\binom{2n}{n}^2 = \left(\frac{1}{4}\right)^{2n}\cdot\left[\frac{(2n)!}{n!\,n!}\right]^2 \sim \frac{1}{4^{2n}}\cdot\left[\frac{\sqrt{4\pi n}\,(2n)^{2n}\,e^{-2n}}{(\sqrt{2\pi n}\cdot n^n\cdot e^{-n})^2}\right]^2 = \frac{1}{\pi n}.$$

Therefore, $\sum_{n=1}^{\infty} P^n_{OO} = \sum_{n=1}^{\infty}\left(\frac{1}{4}\right)^{2n}\binom{2n}{n}^2$ is convergent if and only if $\sum_{n=1}^{\infty}\frac{1}{\pi n}$ is convergent. Since $\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}$ is divergent, $\sum_{n=1}^{\infty} P^n_{OO}$ is divergent, showing that the state (0, 0) is recurrent.

25. Clearly, $P(X_{n+1} = 1 \mid X_n = 0) = 1$. For $i \geq 1$, given $X_n = i$, either $X_{n+1} = i + 1$ in which case we say that a transition to the *right* has occurred, or $X_{n+1} = i - 1$ in which case we say that a transition to the *left* has occurred. For $i \geq 1$, given $X_n = i$, when the nth transition occurs, let S be the remaining service time of the customer being served or the service time of a new customer, whichever applies. Let T be the time from the nth transition until the next arrival. By the memoryless property of exponential random variables, S and T are exponential random variables with parameters μ and λ, respectively. For $i \geq 1$,

$$\begin{aligned}P(X_{n+1} = i+1 \mid X_n = i) &= P(T < S) = \int_0^{\infty} P(S > T \mid T = t)\lambda e^{-\lambda t}\,dt\\ &= \int_0^{\infty} P(S > t)\lambda e^{-\lambda t}\,dt = \int_0^{\infty} e^{-\mu t}\cdot\lambda e^{-\lambda t}\,dt = \frac{\lambda}{\lambda+\mu}.\end{aligned}$$

Therefore,

$$P(X_{n+1} = i-1 \mid X_n = i) = P(T > S) = 1 - \frac{\lambda}{\lambda+\mu} = \frac{\mu}{\lambda+\mu}.$$

These calculations show that knowing X_n, the next transition does not depend on the values of X_j for $j < n$. Therefore, $\{X_n : n = 1, 2, \dots\}$ is a Markov chain, and its transition probability matrix is given by

$$\boldsymbol{P} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & \dots\\ \frac{\mu}{\lambda+\mu} & 0 & \frac{\lambda}{\lambda+\mu} & 0 & 0 & \dots\\ 0 & \frac{\mu}{\lambda+\mu} & 0 & \frac{\lambda}{\lambda+\mu} & 0 & \dots\\ 0 & 0 & \frac{\mu}{\lambda+\mu} & 0 & \frac{\lambda}{\lambda+\mu} & \dots\\ \vdots & & & & & \end{pmatrix}.$$

Since all states are accessible from each other, this Markov chain is irreducible. Starting from 0, for the Markov chain to return to 0, it needs to make as many transitions to the left as it makes to the right. Therefore, $P_{00}^n > 0$ only for positive even integers. Since the greatest common divisor of such integers is 2, the period of 0, and hence the period of all other states is 2.

26. The ijth element of $\boldsymbol{PQ}$ is the product of the ith row of $\boldsymbol{P}$ with the jth column of $\boldsymbol{Q}$. Thus it is $\sum_{\ell} p_{i\ell}q_{\ell j}$. To show that the sum of each row of $\boldsymbol{PQ}$ is 1, we will now calculate the sum of the elements of the ith row of $\boldsymbol{PQ}$, which is

$$\sum_j \sum_{\ell} p_{i\ell}q_{\ell j} = \sum_{\ell} \sum_j p_{i\ell}q_{\ell j} = \sum_{\ell} \Big(p_{i\ell} \sum_j q_{\ell j}\Big) = \sum_{\ell} p_{i\ell} = 1.$$

Note that $\sum_j q_{\ell j} = 1$ and $\sum_{\ell} p_{i\ell} = 1$ since the sum of the elements of the ℓth row of $\boldsymbol{Q}$ and the sum of the elements of the ith row of $\boldsymbol{P}$ are 1.

27. If state j is accessible from state i, there is a path

$$i = i_1,\ i_2,\ i_3,\ \dots,\ i_n = j$$

from i to j. If $n \leq K$, we are done. If $n > K$, by the pigeonhole principle, there must exist k and ℓ $(k < \ell)$ so that $i_k = i_\ell$. Now the path

$$i = i_1,\ i_2,\ \dots,\ i_k,\ i_{k+1},\ \dots,\ i_\ell,\ i_{\ell+1},\ \dots,\ i_n = j$$

can be reduced to

$$i = i_1,\ i_2,\ \dots,\ i_k,\ i_{\ell+1},\ \dots,\ i_n = j$$

which is still a path from i to j but in fewer steps. Repeating this procedure, we can eliminate all of the states that appear more than once from the path and yet reach from i to j with a positive probability. After all such eliminations are made, we obtain a path

$$i = i_1,\ i_{m_1},\ i_{m_2},\ \dots,\ i_n = j$$

in which the states $i_1, i_{m_1}, i_{m_2}, \dots, i_n$ are distinct states. Since there are K states altogether, this path has at most K states.

28. Let $I = \{n \geq 1\colon p_{ii}^n > 0\}$ and $J = \{n \geq 1\colon p_{jj}^n > 0\}$. Then $d(i)$, the period of i, is the greatest common divisor of the elements of I, and $d(j)$, the period of j, is the greatest common divisor of the elements of J. If $d(i) \neq d(j)$, then one of $d(i)$ and $d(j)$ is smaller than the other one. We will prove the theorem for the case in which $d(j) < d(i)$. The proof for the case in which $d(i) < d(j)$ follows by symmetry. Suppose that for positive integers n and m, $p_{ij}^n > 0$ and $p_{ji}^m > 0$. Let $k \in J$; then $p_{jj}^k > 0$. We have

$$p_{ii}^{n+m} \geq p_{ij}^n p_{ji}^m > 0,$$

and

$$p_{ii}^{n+k+m} \geq p_{ij}^{n} p_{jj}^{k} p_{ji}^{m} > 0.$$

By these inequalities, we have that $d(i)$ divides $n + m$ and $n + k + m$. Hence it divides $(n + k + m) - (n + m) = k$. We have shown that, if $k \in J$, then $d(i)$ divides k. This means that $d(i)$ divides all members of J. It contradicts the facts that $d(j)$ is the greatest common divisor of J and $d(j) < d(i)$. Therefore, we must have $d(i) = d(j)$.

29. The stochastic process $\{X_n : n = 1, 2, \dots\}$ is a Markov chain with state space $\{0, 1, \dots, k-1\}$. For $0 \leq i \leq k - 2$, a transition is only possible from state i to 0 or $i + 1$. The only transition from $k - 1$ is to 0. Let Z be the number of weeks it takes Liz to play again with Bob from the time they last played. The event $Z > i$ occurs if and only if Liz has not played with Bob since i Sundays ago, and the earliest she will play with him is next Sunday. Now the probability is i/k that Liz will play with Bob if last time they played was i Sundays ago; hence

$$P(Z > i) = 1 - \frac{i}{k}, \quad i = 1, 2, \dots, k - 1.$$

Using this fact, for $0 \leq i \leq k - 2$, we obtain

$$\begin{aligned}
p_{i(i+1)} &= P(X_{n+1} = i + 1 \mid X_n = i) = \frac{P(X_n = i,\ X_{n+1} = i + 1)}{P(X_n = i)} \\
&= \frac{P(Z > i + 1)}{P(Z > i)} = \frac{1 - \dfrac{i+1}{k}}{1 - \dfrac{i}{k}} = \frac{k - i - 1}{k - i}, \\
p_{i0} &= P(X_{n+1} = 0 \mid X_n = i) = 1 - \frac{k - i - 1}{k - i} = \frac{1}{k - i}, \\
p_{(k-1)0} &= P(X_{n+1} = 0 \mid X_n = k - 1) = 1.
\end{aligned}$$

Hence the transition probability matrix of $\{X_n : n = 1, 2, \dots\}$ is given by

$$
\boldsymbol{P}=\begin{pmatrix}
\frac{1}{k} & 1-\frac{1}{k} & 0 & 0 & 0 & \dots & 0 & 0\\
\frac{1}{k-1} & 0 & 1-\frac{1}{k-1} & 0 & 0 & \dots & 0 & 0\\
\frac{1}{k-2} & 0 & 0 & 1-\frac{1}{k-2} & 0 & \dots & 0 & 0\\
\frac{1}{k-3} & 0 & 0 & 0 & 1-\frac{1}{k-3} & \dots & 0 & 0\\
\vdots & & & & & & & \\
\frac{1}{2} & 0 & 0 & 0 & 0 & \dots & 0 & \frac{1}{2}\\
1 & 0 & 0 & 0 & 0 & \dots & 0 & 0
\end{pmatrix}.
$$

It should be clear that the Markov chain under consideration is irreducible, aperiodic, and positively recurrent. For $0 \le i \le k-1$, let π_i be the long-run probability that Liz says no to Bob for i consecutive weeks. $\pi_0, \pi_1, \dots, \pi_{k-1}$ are obtained from solving the following matrix equation along with $\sum_{i=0}^{k-1} \pi_i = 1$.

$$
\begin{pmatrix}\pi_0\\ \pi_1\\ \pi_2\\ \pi_3\\ \vdots\\ \pi_{k-2}\\ \pi_{k-1}\end{pmatrix}
=\begin{pmatrix}
\frac{1}{k} & \frac{1}{k-1} & \frac{1}{k-2} & \frac{1}{k-3} & \dots & \frac{1}{2} & 1\\
1-\frac{1}{k} & 0 & 0 & 0 & \dots & 0 & 0\\
0 & 1-\frac{1}{k-1} & 0 & 0 & \dots & 0 & 0\\
0 & 0 & 1-\frac{1}{k-2} & 0 & \dots & 0 & 0\\
0 & 0 & 0 & 1-\frac{1}{k-3} & \dots & 0 & 0\\
\vdots & & & & & & \\
0 & 0 & 0 & 0 & \dots & \frac{1}{2} & 0
\end{pmatrix}
\begin{pmatrix}\pi_0\\ \pi_1\\ \pi_2\\ \pi_3\\ \vdots\\ \pi_{k-2}\\ \pi_{k-1}\end{pmatrix}.
$$

The matrix equation gives

$$
\pi_i = \frac{k-i}{k}\pi_0, \qquad i = 1, 2, \dots, k-1.
$$

Using $\sum_{i=0}^{k-1} \pi_i = 1$, we obtain

$$\pi_0 \sum_{i=0}^{k-1} \frac{k-i}{k} = 1$$

or, equivalently,

$$\frac{\pi_0}{k}\Big[\sum_{i=0}^{k-1} k - \sum_{i=0}^{k-1} i\Big] = 1.$$

This implies that

$$\frac{\pi_0}{k}\Big[k^2 - \frac{(k-1)k}{2}\Big] = 1,$$

which gives $\pi_0 = 2/(k+1)$. Hence

$$\pi_i = \frac{2(k-i)}{k(k+1)}, \qquad i = 0, 1, 2, \dots, k-1.$$

30. Let X_i be the amount of money player A has after i games. Clearly, $X_0 = a$ and $\{X_n : n = 0, 1, \dots\}$ is a Markov chain with state space $\{0, 1, \dots, a, a+1, \dots, a+b\}$. For $0 \le i \le a+b$, let $m_i = E(T \mid X_0 = i)$. Let F be the event that A wins the first game. Then, for $1 \le i \le a+b-1$,

$$E(T \mid X_0 = i) = E(T \mid X_0 = i, F)P(F \mid X_0 = i) + E(T \mid X_0 = i, F^c)P(F^c \mid X_0 = i).$$

This gives

$$m_i = (1 + m_{i+1})\frac{1}{2} + (1 + m_{i-1})\frac{1}{2}, \quad 1 \le i \le a+b-1,$$

or, equivalently,

$$2m_i = 2 + m_{i+1} + m_{i-1}, \quad 1 \le i \le a+b-1.$$

Now rewrite this relation as

$$m_{i+1} - m_i = -2 + m_i - m_{i-1}, \quad 1 \le i \le a+b-1,$$

and, for $1 \le i \le a+b$, let

$$y_i = m_i - m_{i-1}.$$

Then

$$y_{i+1} = -2 + y_i, \quad 1 \le i \le a+b-1,$$

and, for $1 \le i \le a+b$,

$$m_i = y_1 + y_2 + \cdots + y_i.$$

Clearly, $m_0 = 0$, $m_{a+b} = 0$, $y_1 = m_1$, and

$$\begin{aligned} y_2 &= -2 + y_1 = -2 + m_1, \\ y_3 &= -2 + y_2 = -2 + (-2 + m_1) = -4 + m_1 \\ &\vdots \\ y_i &= -2(i-1) + m_1, \quad 1 \le i \le a+b. \end{aligned}$$

Hence, for $1 \le i \le a+b$,

$$\begin{aligned} m_i &= y_1 + y_2 + \cdots + y_i \\ &= i m_1 - 2\big[1 + 2 + \cdots + (i-1)\big] \\ &= i m_1 - i(i-1) = i(m_1 - i + 1). \end{aligned}$$

This and $m_{a+b} = 0$ imply that

$$(a+b)(m_1 - a - b + 1) = 0,$$

or $m_1 = a + b - 1$. Therefore,

$$m_i = i(a + b - i),$$

and hence the desired quantity is

$$E(T \mid X_0 = a) = m_a = ab.$$

31. Let q be a positive solution of the equation $x = \sum_{i=0}^{\infty} \alpha_i x^i$. Then $q = \sum_{i=0}^{\infty} \alpha_i q^i$. We will show that $\forall n \ge 0$, $P(X_n = 0) \le q$. This implies that

$$p = \lim_{n\to\infty} P(X_n = 0) \le q.$$

To establish that $P(X_n = 0) \le q$, we use induction. For $n = 0$, $P(X_0 = 0) = 0 \le q$ is trivially true. Suppose that $P(X_n = 0) \le q$. We have

$$P(X_{n+1} = 0) = \sum_{i=0}^{\infty} P(X_{n+1} = 0 \mid X_1 = i)P(X_1 = i).$$

It should be clear that

$$P(X_{n+1} = 0 \mid X_1 = i) = \big[P(X_n = 0 \mid X_0 = 1)\big]^i.$$

However, since $P(X_0 = 1) = 1$,

$$P(X_n = 0 \mid X_0 = 1) = P(X_n = 0).$$

Therefore,

$$P(X_{n+1} = 0 \mid X_1 = i) = \big[P(X_n = 0)\big]^i.$$

Thus

$$P(X_{n+1} = 0) = \sum_{i=0}^{\infty} \big[P(X_n = 0)\big]^i P(X_1 = i) \le \sum_{i=0}^{\infty} q^i \alpha_i = q.$$

This establishes the theorem.

32. Multiplying $\boldsymbol{P}$ successively, we obtain

$$p_{12} = \frac{1}{13}$$
$$p_{12}^2 = \Big(\frac{9}{13}\Big)\Big(\frac{1}{13}\Big) + \frac{1}{13},$$
$$p_{12}^3 = \Big(\frac{9}{13}\Big)^2\Big(\frac{1}{13}\Big) + \Big(\frac{9}{13}\Big)\Big(\frac{1}{13}\Big) + \frac{1}{13},$$

and in general,

$$p_{12}^n = \frac{1}{13}\Big[\Big(\frac{9}{13}\Big)^{n-1} + \Big(\frac{9}{13}\Big)^{n-2} + \cdots + 1\Big]$$
$$= \frac{1}{13} \cdot \frac{1 - \Big(\dfrac{9}{13}\Big)^n}{1 - \dfrac{9}{13}} = \frac{1}{4}\Big[1 - \Big(\frac{9}{13}\Big)^n\Big].$$

Hence the desired probability is $\lim_{n\to\infty} p_{12}^n = 1/4$.

33. We will use induction. Let $n = 1$; then, for $1 + j - i$ to be nonnegative, we must have $i - 1 \le j$. For the inequality $\dfrac{1 + j - i}{2} \le 1$ to be valid, we must have $j \le i + 1$. Therefore, $i - 1 \le j \le i + 1$. But, for $j = i$, $1 + j - i$ is not even. Therefore, if $1 + j - i$ is an even nonnegative integer satisfying $\dfrac{1 + j - i}{2} \le 1$, we must have $j = i - 1$ or $j = i + 1$. For $j = i - 1$,

$$\frac{n + j - i}{2} = \frac{1 + i - 1 - i}{2} = 0 \quad \text{and} \quad \frac{n - j + i}{2} = \frac{1 - i + 1 + i}{2} = 1.$$

Hence

$$P(X_1 = i - 1 \mid X_0 = i) = 1 - p = \binom{1}{0} p^0 (1 - p)^1,$$

showing that the relation is valid. For $j = i + 1$,

$$\frac{n + j - i}{2} = \frac{1 + i + 1 - i}{2} = 1 \quad \text{and} \quad \frac{n - j + i}{2} = \frac{1 - i - 1 + i}{2} = 0.$$

Hence

$$P(X_1 = i + 1 \mid X_0 = i) = p = \binom{1}{1} p^1 (1 - p)^0,$$

showing that the relation is valid in this case as well. Since, for a simple random walk, the only possible transitions from i are to states $i + 1$ and $i - 1$, in all other cases

$$P(X_1 = j \mid X_0 = i) = 0.$$

We have established the theorem for $n = 1$. Now suppose that it is true for n. We will show it for $n + 1$ by conditioning on X_n:

$$\begin{aligned}
P(X_{n+1} = j \mid X_0 = i) &= P(X_{n+1} = j \mid X_0 = i,\ X_n = j-1)P(X_n = j-1 \mid X_0 = i) \\
&\quad + P(X_{n+1} = j \mid X_0 = i,\ X_n = j+1)P(X_n = j+1 \mid X_0 = i) \\
&= P(X_{n+1} = j \mid X_n = j-1)P(X_n = j-1 \mid X_0 = i) \\
&\quad + P(X_{n+1} = j \mid X_n = j+1)P(X_n = j+1 \mid X_0 = i) \\
&= p \cdot \binom{n}{\frac{n+j-1-i}{2}} p^{(n+j-1-i)/2}(1-p)^{(n-j+1+i)/2} \\
&\quad + (1-p)\binom{n}{\frac{n+j+1-i}{2}} p^{(n+j+1-i)/2}(1-p)^{(n-j-1+i)/2} \\
&= \left[\binom{n}{\frac{n-1+j-i}{2}} + \binom{n}{\frac{n+1+j-i}{2}}\right] p^{(n+1+j-i)/2}(1-p)^{(n+1-j+i)/2} \\
&= \binom{n+1}{\frac{n+1+j-i}{2}} p^{(n+1+j-i)/2}(1-p)^{(n+1-j+i)/2}.
\end{aligned}$$

12.4 CONTINUOUS-TIME MARKOV CHAINS

1. By Chapman-Kolmogorov equations,

$$\begin{aligned}
p_{ij}(t+h) - p_{ij}(t) &= \sum_{k=0}^{\infty} p_{ik}(h)p_{kj}(t) - p_{ij}(t) \\
&= \sum_{k \neq i} p_{ik}(h)p_{kj}(t) + p_{ii}(h)p_{ij}(t) - p_{ij}(t) \\
&= \sum_{k \neq i} p_{ik}(h)p_{kj}(t) + p_{ij}(t)\big[p_{ii}(h) - 1\big].
\end{aligned}$$

Thus

$$\frac{p_{ij}(t+h) - p_{ij}(t)}{h} = \sum_{k \neq i} \frac{p_{ik}(h)}{h} p_{kj}(t) - p_{ij}(t)\frac{1 - p_{ii}(h)}{h}.$$

Letting $h \to 0$, by (12.13) and (12.14), we have

$$p'_{ij}(t) = \sum_{k \neq i} q_{ik}\, p_{kj}(t) - \nu_i\, p_{ij}(t).$$

2. Clearly, $\{X(t): t \geq \}$ is a continuous-time Markov chain. Its balance equations are as follows:

State	Input rate to	=	Output rate from
f	$\mu\pi_0$	=	$\lambda\pi_f$
0	$\lambda\pi_f + \mu\pi_1 + \mu\pi_2 + \mu\pi_3$	=	$\mu\pi_0 + \lambda\pi_0$
1	$\lambda\pi_0$	=	$\lambda\pi_1 + \mu\pi_1$
2	$\lambda\pi_1$	=	$\lambda\pi_2 + \mu\pi_2$
3	$\lambda\pi_2$	=	$\mu\pi_3.$

Solving these equations along with

$$\pi_f + \pi_0 + \pi_1 + \pi_2 + \pi_3 = 1$$

we obtain

$$\pi_f = \frac{\mu^2}{\lambda(\lambda+\mu)}, \qquad \pi_0 = \frac{\mu}{\lambda+\mu},$$

$$\pi_1 = \frac{\lambda\mu}{(\lambda+\mu)^2}, \qquad \pi_2 = \frac{\lambda^2\mu}{(\lambda+\mu)^3},$$

$$\pi_3 = \Big(\frac{\lambda}{\lambda+\mu}\Big)^3.$$

3. The fact that $\{X(t): t \geq 0\}$ is a continuous-time Markov chain should be clear. The balance equations are

State	Input rate to	=	Output rate from	
$(0,0)$	$\mu\pi_{(1,0)} + \lambda\pi_{(0,1)}$	=	$\lambda\pi_{(0,0)} + \mu\pi_{(0,0)}$	
$(n,0)$	$\mu\pi_{(n+1,0)} + \lambda\pi_{(n-1,0)}$	=	$\lambda\pi_{(n,0)} + \mu\pi_{(n,0)},$	$n \geq 1$
$(0,m)$	$\lambda\pi_{(0,m+1)} + \mu\pi_{(0,m-1)}$	=	$\lambda\pi_{(0,m)} + \mu\pi_{(0,m)}$	$m \geq 1.$

4. Let $X(t)$ be the number of customers in the system at time t. Then the process $\{X(t): t \geq 0\}$ is a birth and death process with $\lambda_n = \lambda$, $n \geq 0$, and $\mu_n = n\mu$, $n \geq 1$. To find π_0, the probability that the system is empty, we will first calculate the sum in (12.18). We have

$$\sum_{n=1}^{\infty} \frac{\lambda_0\lambda_1\cdots\lambda_{n-1}}{\mu_1\mu_2\cdots\mu_n} = \sum_{n=1}^{\infty} \frac{\lambda^n}{n!\,\mu^n} = \sum_{n=1}^{\infty} \frac{1}{n!}\Big(\frac{\lambda}{\mu}\Big)^n = -1 + \sum_{n=0}^{\infty} \frac{1}{n!}\Big(\frac{\lambda}{\mu}\Big)^n = -1 + e^{\lambda/\mu}.$$

Hence, by (12.18),

$$\pi_0 = \frac{1}{1 - 1 + e^{\lambda/\mu}} = e^{-\lambda/\mu}.$$

By (12.17),

$$\pi_n = \frac{\lambda^n \pi_0}{n!\mu^n} = \frac{(\lambda/\mu)^n e^{-\lambda/\mu}}{n!}, \quad n = 0, 1, 2, \dots .$$

This shows that the long-run number of customers in such an $M/M/\infty$ queueing system is Poisson with parameter λ/μ. The average number of customers in the system is, therefore, λ/μ.

5. Let $X(t)$ be the number of operators busy serving customers at time t. Clearly, $\{X(t) \colon t \geq 0\}$ is a finite-state birth and death process with state space $\{0, 1, \dots, c\}$, birth rates $\lambda_n = \lambda$, $n = 0, 1, \dots, c$, and death rates $\mu_n = n\mu$, $n = 0, 1, \dots, c$. Let π_0 be the proportion of time that all operators are free. Let π_c be the proportion of time all of them are busy serving customers.

(a) π_c is the desired quantity. By (12.22),

$$\pi_0 = \frac{1}{1 + \displaystyle\sum_{n=1}^{c} \frac{\lambda^n}{n!\,\mu^n}} = \frac{1}{\displaystyle\sum_{n=0}^{c} \frac{1}{n!}\Big(\frac{\lambda}{\mu}\Big)^n}.$$

By (12.21),

$$\pi_c = \frac{\dfrac{1}{c!}(\lambda/\mu)^c}{\sum_{n=0}^{c} \dfrac{1}{n!}(\lambda/\mu)^n}.$$

This formula is called **Erlang's loss formula**.

(b) We want to find the smallest c for which

$$\frac{1/c!}{\sum_{n=0}^{c}(1/n!)} \leq 0.004.$$

For $c = 5$, the left side is 0.00306748. For $c = 4$, it is 0.01538462. Therefore, the airline must hire at least five operators to reduce the probability of losing a call to a number less than 0.004.

6. No, it is not because it is possible for the process to enter state 0 directly from state 2. In a birth and death process, from a state i, transitions are only possible to the states $i-1$ and $i+1$.

7. For $n \geq 0$, let H_n be the time, starting from n, until the process enters state $n+1$ for the first time. Clearly, $E(H_0) = 1/\lambda$ and, by Lemma 12.2,

$$E(H_n) = \frac{1}{\lambda} + E(H_{n-1}), \quad n \geq 1.$$

Hence

$$E(H_0) = \frac{1}{\lambda},$$
$$E(H_1) = \frac{1}{\lambda} + \frac{1}{\lambda} = \frac{2}{\lambda},$$
$$E(H_2) = \frac{1}{\lambda} + \frac{2}{\lambda} = \frac{3}{\lambda}.$$

Continuing this process, we obtain,

$$E(H_n) = \frac{n+1}{\lambda}, \quad n \geq 0.$$

The desired quantity is

$$\sum_{n=i}^{j-1} E(H_n) = \sum_{n=i}^{j-1} \frac{n+1}{\lambda} = \frac{1}{\lambda}\big[(i+1) + (i+2) + \cdots + j\big]$$
$$= \frac{1}{\lambda}\big[(1+2+\cdots+j) - (1+2+\cdots+i)\big]$$
$$= \frac{1}{\lambda}\Big[\frac{j(j+1)}{2} - \frac{i(i+1)}{2}\Big] = \frac{j(j+1) - i(i+1)}{2\lambda}.$$

8. Suppose that a birth occurs each time that an out-of-order machine is repaired and begins to operate, and a death occurs each time that a machine breaks down. The fact that $\{X(t) \colon t \geq 0\}$ is a birth and death process with state space $\{0, 1, \ldots, m\}$ should be clear. The birth and death rates are

$$\lambda_n = \begin{cases} k\lambda & n = 0, 1, \ldots, m-k \\ (m-n)\lambda & n = m-k+1, m-k+2, \ldots, m, \end{cases}$$
$$\mu_n = \quad n\mu \qquad n = 0, 1, \ldots, m.$$

9. The Birth rates are

$$\begin{cases} \lambda_0 = \lambda \\ \lambda_n = \alpha_n \lambda, \quad n \geq 1. \end{cases}$$

The death rates are

$$\begin{cases} \mu_0 = 0 \\ \mu_n = \mu + (n-1)\gamma, \quad n \geq 1. \end{cases}$$

10. Let $X(t)$ be the population size at time t. Then $\{X(t) \colon t \geq 0\}$ is a birth and death process with birth rates $\lambda_n = n\lambda + \gamma$, $n \geq 0$, and death rates $\mu_n = n\mu$, $n \geq 1$. For $i \geq 0$, let H_i

be the time, starting from i, until the population size reaches $i+1$ for the first time. We are interested in $E(H_0)+E(H_1)+E(H_2)$. Note that, by Lemma 12.2,

$$E(H_i)=\frac{1}{\lambda_i}+\frac{\mu_i}{\lambda_i}E(H_{i-1}), \quad i\ge 1.$$

Since $E(H_0)=1/\gamma$,

$$E(H_1)=\frac{1}{\lambda+\gamma}+\frac{\mu}{\lambda+\gamma}\cdot\frac{1}{\gamma}=\frac{\mu+\gamma}{\gamma(\lambda+\gamma)},$$

and

$$E(H_2)=\frac{1}{2\lambda+\gamma}+\frac{2\mu}{2\lambda+\gamma}\cdot\frac{\mu+\gamma}{\gamma(\lambda+\gamma)}=\frac{\gamma(\lambda+\gamma)+2\mu(\mu+\gamma)}{\gamma(\lambda+\gamma)(2\lambda+\gamma)}.$$

Thus the desired quantity is

$$E(H_0)+E(H_1)+E(H_2)=\frac{(\lambda+\gamma)(2\lambda+\gamma)+(\mu+\gamma)(2\lambda+2\mu+\gamma)+\gamma(\lambda+\gamma)}{\gamma(\lambda+\gamma)(2\lambda+\gamma)}.$$

11. Let $X(t)$ be the number of deaths in the time interval $[0,t]$. Since there are no births, by Remark 7.2, it should be clear that $\{X(t)\colon t\ge 0\}$ is a Poisson process with rate μ *as long as the population is not extinct*. Therefore, for $0<j\le i$,

$$p_{ij}(t)=\frac{e^{-\mu t}(\mu t)^{i-j}}{(i-j)!}.$$

Clearly, $p_{00}(t)=1$. For $i>0,\ j=0$, we have

$$p_{i0}(t)=1-\sum_{j=1}^{i}p_{ij}(t)=1-\sum_{j=1}^{i}\frac{e^{-\mu t}(\mu t)^{i-j}}{(i-j)!}=1-\sum_{j=i}^{1}\frac{e^{-\mu t}(\mu t)^{i-j}}{(i-j)!}.$$

Letting $k=i-j$ yields

$$p_{i0}(t)=1-\sum_{k=0}^{i-1}\frac{e^{-\mu t}(\mu t)^{k}}{k!}=\sum_{k=i}^{\infty}\frac{e^{-\mu t}(\mu t)^{k}}{k!}.$$

12. Suppose that a birth occurs whenever a physician takes a break, and a death occurs whenever he or she becomes available to answer patients' calls. Let $X(t)$ be the number of physicians on break at time t. Then $\{X(t)\colon t\ge 0\}$ is a birth and death process with state space $\{0,1,2\}$. Clearly, $X(t)=0$ if at t both of the physicians are available to answer patients' calls, $X(t)=1$ if at t only one of the physicians is available to answer patients' calls, and $X(t)=2$ if at t none of the physicians is available to answer patients' calls. We have that

$$\lambda_0=2\lambda, \quad \lambda_1=\lambda, \quad \lambda_2=0,$$

$$\mu_0 = 0, \quad \mu_1 = \mu, \quad \mu_2 = 2\mu.$$

Therefore,

$$\nu_0 = 2\lambda, \quad \nu_1 = \lambda + \mu, \quad \nu_2 = 2\mu.$$

Also,

$$p_{01} = p_{21} = 1, \quad p_{02} = p_{20} = 0, \quad p_{10} = \frac{\mu}{\lambda + \mu}, \quad p_{12} = \frac{\lambda}{\lambda + \mu}.$$

Therefore,

$$\begin{aligned} &q_{01} = \nu_0 p_{01} = 2\lambda, && q_{10} = \nu_1 p_{10} = \mu, \\ &q_{12} = \nu_1 p_{12} = \lambda, && q_{21} = \nu_2 p_{21} = 2\mu, \\ &q_{02} = q_{20} = 0. \end{aligned}$$

Substituting these quantities in the Kolmogorov backward equations

$$p'_{ij}(t) = \sum_{k \neq i} q_{ik} p_{kj}(t) - \nu_i p_{ij}(t),$$

we obtain

$$\begin{aligned} p'_{00}(t) &= 2\lambda p_{10}(t) - 2\lambda p_{00}(t) \\ p'_{01}(t) &= 2\lambda p_{11}(t) - 2\lambda p_{01}(t) \\ p'_{02}(t) &= 2\lambda p_{12}(t) - 2\lambda p_{02}(t) \\ p'_{10}(t) &= \lambda p_{20}(t) + \mu p_{00}(t) - (\lambda + \mu) p_{10}(t) \\ p'_{11}(t) &= \lambda p_{21}(t) + \mu p_{01}(t) - (\lambda + \mu) p_{11}(t) \\ p'_{12}(t) &= \lambda p_{22}(t) + \mu p_{02}(t) - (\lambda + \mu) p_{12}(t) \\ p'_{20}(t) &= 2\mu p_{10}(t) - 2\mu p_{20}(t) \\ p'_{21}(t) &= 2\mu p_{11}(t) - 2\mu p_{21}(t) \\ p'_{22}(t) &= 2\mu p_{12}(t) - 2\mu p_{22}(t). \end{aligned}$$

13. Let $X(t)$ be the number of customers in the system at time t. Then $\{X(t): n \geq 0\}$ is a birth and death process with $\lambda_n = \lambda$, for $n \geq 0$, and

$$\mu_n = \begin{cases} n\mu & n = 0, 1, \dots, c \\ c\mu & n > c. \end{cases}$$

By (12.21), for $n = 1, 2, \dots c$,

$$\pi_n = \frac{\lambda^n}{n!\, \mu^n} \pi_0 = \frac{1}{n!} \Big(\frac{\lambda}{\mu}\Big)^n \pi_0;$$

for $n > c$,

$$\pi_n = \frac{\lambda^n}{c!\, \mu^c (c\mu)^{n-c}} \pi_0 = \frac{\lambda^n}{c!\, c^{n-c} \mu^n} \pi_0 = \frac{c^c}{c!} \Big(\frac{\lambda}{c\mu}\Big)^n \pi_0 = \frac{c^c}{c!} \rho^n \pi_0.$$

Noting that $\sum_{n=0}^{c}\pi_n+\sum_{n=c+1}^{\infty}\pi_n=1$, we have

$$\pi_0\sum_{n=0}^{c}\frac{1}{n!}\Big(\frac{\lambda}{\mu}\Big)^n+\pi_0\frac{c^c}{c!}\sum_{n=c+1}^{\infty}\rho^n=1.$$

Since $\rho<1$, we have $\sum_{n=c+1}^{\infty}\rho^n=\dfrac{\rho^{c+1}}{1-\rho}$. Therefore,

$$\pi_0=\frac{1}{\displaystyle\sum_{n=0}^{c}\frac{1}{n!}\Big(\frac{\lambda}{\mu}\Big)^n+\frac{c^c}{c!}\sum_{n=c+1}^{\infty}\rho^n}=\frac{c!\,(1-\rho)}{\displaystyle c!\,(1-\rho)\sum_{n=0}^{c}\frac{1}{n!}\Big(\frac{\lambda}{\mu}\Big)^n+c^c\rho^{c+1}}.$$

14. Let $s,t>0$. If $j<i$, then $p_{ij}(s+t)=0$, and

$$\sum_{k=0}^{\infty}p_{ik}(s)p_{kj}(t)=\sum_{k=0}^{i-1}p_{ik}(s)p_{kj}(t)+\sum_{k=i}^{\infty}p_{ik}(s)p_{kj}(t)=0,$$

since $p_{ik}(s)=0$ if $k<i$, and $p_{kj}(t)=0$ if $k\geq i>j$. Therefore, for $j<i$, the Chapman-Kolmogorov equations are valid. Now suppose that $j>i$. Then

$$\begin{aligned}\sum_{k=0}^{\infty}p_{ik}(s)p_{kj}(t)&=\sum_{k=i}^{j}p_{ik}(s)p_{kj}(t)\\&=\sum_{k=i}^{j}\frac{e^{-\lambda s}(\lambda s)^{k-i}}{(k-i)!}\cdot\frac{e^{-\lambda t}(\lambda t)^{j-k}}{(j-k)!}\\&=\frac{e^{-\lambda(t+s)}}{(j-i)!}\sum_{k=i}^{j}\frac{(j-i)!}{k-i)!\,(j-k)!}(\lambda s)^{k-i}(\lambda t)^{j-k}\\&=\frac{e^{-\lambda(t+s)}}{(j-i)!}\sum_{\ell=0}^{j-i}\frac{(j-i)!}{\ell!\,(j-i-\ell)!}(\lambda s)^{\ell}(\lambda t)^{(j-i)-\ell}\\&=\frac{e^{-\lambda(t+s)}}{(j-i)!}\sum_{\ell=0}^{j-i}\binom{j-i}{\ell}(\lambda s)^{\ell}(\lambda t)^{(j-i)-\ell}\\&=\frac{e^{-\lambda(t+s)}}{(j-i)!}(\lambda s+\lambda t)^{j-i}\end{aligned}$$

where the last equality follows by Theorem 2.5, the binomial expansion. Since

$$\frac{e^{-\lambda(t+s)}}{(j-i)!}\big[\lambda(t+s)\big]^{j-i}=p_{ij}(s+t),$$

we have shown that the Chapman-Kolmogorov equations are satisfied.

15. Let $X(t)$ be the number of particles in the shower t units of time after the cosmic particle enters the earth's atmosphere. Clearly, $\{X(t): t \geq 0\}$ is a continuous-time Markov chain with state space $\{1, 2, \dots\}$ and $\nu_i = i\lambda$, $i \geq 1$. (In fact, $\{X(t): t \geq 0\}$ is a pure birth process, but that fact will not help us solve this exercise.) Clearly, for $i \geq 1$, $j \geq 1$,

$$p_{ij} = \begin{cases} 1 & \text{if } j = i+1 \\ 0 & \text{if } j \neq i+1. \end{cases}$$

Hence

$$q_{ij} = \begin{cases} \nu_i & \text{if } j = i+1 \\ 0 & \text{if } j \neq i+1. \end{cases}$$

We are interested in finding $p_{1n}(t)$. This is the desired probability. For $n = 1$, $p_{11}(t)$ is the probability that the cosmic particle does not collide with any air particles during the first t units of time in the earth's atmosphere. Since the time it takes the particle to collide with another particle is exponential with parameter λ, we have $p_{11}(t) = e^{-\lambda t}$. For $n \geq 2$, by the Kolmogorov's forward equation,

$$\begin{aligned} p'_{1n}(t) &= \sum_{k \neq n} q_{kn} p_{1k}(t) - \nu_n p_{1n}(t) \\ &= q_{(n-1)n} p_{1(n-1)}(t) - \nu_n p_{1n}(t) \\ &= \nu_{n-1} p_{1(n-1)}(t) - \nu_n p_{1n}(t). \end{aligned}$$

Therefore,

$$p'_{1n}(t) = (n-1)\lambda p_{1(n-1)}(t) - n\lambda p_{1n}(t). \tag{49}$$

For $n = 2$, this gives

$$p'_{12}(t) = \lambda p_{11}(t) - 2\lambda p_{12}(t)$$

or, equivalently,

$$p'_{12}(t) = \lambda e^{-\lambda t} - 2\lambda p_{12}(t).$$

Solving this first order linear differential equation with boundary condition $p_{12}(0) = 0$, we obtain

$$p_{12}(t) = e^{-\lambda t}(1 - e^{-\lambda t}).$$

For $n = 3$, by (49),

$$p'_{13}(t) = 2\lambda p_{12}(t) - 3\lambda p_{13}(t)$$

or, equivalently,

$$p'_{13}(t) = 2\lambda e^{-\lambda t}(1 - e^{-\lambda t}) - 3\lambda p_{13}(t).$$

Solving this first order linear differential equation with boundary condition $p_{13}(0) = 0$ yields

$$p_{13}(t) = e^{-\lambda t}(1 - e^{-\lambda t})^2.$$

Continuing this process, and using induction, we obtain that

$$p_{1n}(t) = e^{-\lambda t}(1 - e^{-\lambda t})^{n-1} \quad n \geq 1.$$

16. It is straightforward to see that

$$\pi_{(i,j)} = \Big(\frac{\lambda}{\mu_1}\Big)^i\Big(1-\frac{\lambda}{\mu_1}\Big)\Big(\frac{\lambda}{\mu_2}\Big)^j\Big(1-\frac{\lambda}{\mu_2}\Big), \quad i, j \geq 0,$$

satisfy the following balance equations for the tandem queueing system under consideration. Hence, by Example 12.43, $\pi_{(i,j)}$ is the product of an $M/M/1$ system having i customers in the system, and another $M/M/1$ queueing system having j customers in the system. This establishes what we wanted to show.

State	Input rate to	=	Output rate from
$(0, 0)$	$\mu_2\pi_{(0,1)}$	=	$\lambda\pi_{(0,0)}$
$(i, 0), i \geq 1$	$\mu_2\pi_{(i,1)} + \lambda\pi_{(i-1,0)}$	=	$\lambda\pi_{(i,0)} + \mu_1\pi_{(i,0)}$
$(0, j), j \geq 1$	$\mu_2\pi_{(0,j+1)} + \mu_1\pi_{(1,j-1)}$	=	$\lambda\pi_{(0,j)} + \mu_2\pi_{(0,j)}$
$(i, j), i, j \geq 1$	$\mu_2\pi_{(i,j+1)} + \mu_1\pi_{(i+1,j-1)} + \lambda\pi_{(i-1,j)}$	=	$\lambda\pi_{(i,j)} + \mu_1\pi_{(i,j)} + \mu_2\pi_{(i,j)}$.

17. Clearly, $\{X(t): t \geq 0\}$ is a birth and death process with birth rates $\lambda_i = i\lambda$, $i \geq 0$, and death rates $\mu_i = i\mu + \gamma$, $i > 0$; $\mu_0 = 0$. For some $m \geq 1$, suppose that $X(t) = m$. Then, for infinitesimal values of h, by (12.5), the population at $t+h$ is $m+1$ with probability $m\lambda h+o(h)$, it is $m-1$ with probability $(m\mu+\gamma)h + o(h)$, and it is still m with probability

$$1 - m\lambda h - o(h) - (m\mu+\gamma)h - o(h) = 1 - (m\lambda + m\mu + \gamma)h + o(h).$$

Therefore,

$$\begin{aligned} E\big[X(t+h) \mid X(t) = m\big] &= (m+1)\big[m\lambda h + o(h)\big] + (m-1)\big[(m\mu+\gamma)h + o(h)\big] \\ &\quad + m\big[1 - (m\lambda + m\mu + \gamma)h + o(h)\big] \\ &= m + \big[m(\lambda-\mu) - \gamma\big]h + o(h). \end{aligned}$$

This relation implies that

$$E\big[X(t+h) \mid X(t)\big] = X(t) + \big[(\lambda-\mu)X(t) - \gamma\big]h + o(h).$$

Equating the expected values of both sides, and noting that

$$E\Big[E\big[X(t+h) \mid X(t)\big]\Big] = E\big[X(t+h)\big],$$

we obtain

$$E\big[X(t+h)\big] = E\big[X(t)\big] + h(\lambda-\mu)E\big[X(t)\big] - \gamma h + o(h).$$

For simplicity, let $g(t) = E\big[X(t)\big]$. We have shown that

$$g(t+h) = g(t) + h(\lambda-\mu)g(t) - \gamma h + o(h)$$

or, equivalently,

$$\frac{g(t+h)-g(t)}{h} = (\lambda-\mu)g(t) - \gamma + \frac{o(h)}{h}.$$

As $h \to 0$, this gives

$$g'(t) = (\lambda-\mu)g(t) - \gamma.$$

If $\lambda = \mu$, then $g'(t) = -\gamma$. So $g(t) = -\gamma t + c$. Since $g(0) = n$, we must have $c = n$, or $g(t) = -\gamma t + n$. If $\lambda \neq \mu$, to solve the first order linear differential equation,

$$g'(t) = (\lambda-\mu)g(t) - \gamma,$$

let $f(t) = (\lambda-\mu)g(t) - \gamma$. Then

$$\frac{1}{\lambda-\mu} f'(t) = f(t),$$

or

$$\frac{f'(t)}{f(t)} = \lambda - \mu.$$

This yields

$$\ln|f(t)| = (\lambda-\mu)t + c,$$

or

$$f(t) = e^{(\lambda-\mu)t+c} = Ke^{(\lambda-\mu)t},$$

where $K = e^c$. Thus

$$g(t) = \frac{K}{\lambda-\mu}e^{(\lambda-\mu)t} + \frac{\gamma}{\lambda-\mu}.$$

Now $g(0) = n$ implies that $K = n(\gamma-\mu) - \gamma$. Thus

$$g(t) = E\big[X(t)\big] = ne^{(\lambda-\mu)t} + \frac{\gamma}{\lambda-\mu}\big[1 - e^{(\lambda-\mu)t}\big].$$

18. For $n \geq 0$, let E_n be the event that, starting from state n, eventually extinction will occur. Let $\alpha_n = P(E_n)$. Clearly, $\alpha_0 = 1$. We will show that $\alpha_n = 1$, for all n. For $n \geq 1$, starting from n, let Z_n be the state to which the process will move. Then Z_n is a discrete random variable with set of possible values $\{n-1, n+1\}$. Conditioning on Z_n yields

$$P(E_n) = P(E_n \mid Z_n = n-1)P(Z_n = n-1) + P(E_n \mid Z_n = n+1)P(Z_n = n+1).$$

Hence

$$\alpha_n = \alpha_{n-1} \cdot \frac{\mu_n}{\lambda_n + \mu_n} + \alpha_{n+1} \cdot \frac{\lambda_n}{\lambda_n + \mu_n}, \quad n \geq 1,$$

or, equivalently,

$$\lambda_n(\alpha_{n+1} - \alpha_n) = \mu_n(\alpha_n - \alpha_{n-1}), \quad n \geq 1.$$

For $n \geq 0$, let $y_n = \alpha_{n+1} - \alpha_n$. We have

$$\lambda_n y_n = \mu_n y_{n-1}, \quad n \geq 1,$$

or

$$y_n = \frac{\mu_n}{\lambda_n} y_{n-1}, \quad n \geq 1.$$

Therefore,

$$\begin{aligned}
y_1 &= \frac{\mu_1}{\lambda_1} y_0 \\
y_2 &= \frac{\mu_2}{\lambda_2} y_1 = \frac{\mu_1\mu_2}{\lambda_1\lambda_2} y_0 \\
&\ \vdots \\
y_n &= \frac{\mu_1\mu_2\cdots\mu_n}{\lambda_1\lambda_2\cdots\lambda_n} y_0. \quad n \geq 1.
\end{aligned}$$

On the other hand, by $y_n = \alpha_{n+1} - \alpha_n$, $n \geq 0$,

$$\begin{aligned}
\alpha_1 &= \alpha_0 + y_0 = 1 + y_0 \\
\alpha_2 &= \alpha_1 + y_1 = 1 + y_0 + y_1 \\
&\ \vdots \\
\alpha_{n+1} &= 1 + y_0 + y_1 + \cdots + y_n.
\end{aligned}$$

Hence

$$\begin{aligned}
\alpha_{n+1} &= 1 + y_0 + \sum_{k=1}^{n} y_k \\
&= 1 + y_0 + y_0 \sum_{k=1}^{n} \frac{\mu_1\mu_2\cdots\mu_k}{\lambda_1\lambda_2\cdots\lambda_k} \\
&= 1 + y_0 \Big(1 + \sum_{k=1}^{n} \frac{\mu_1\mu_2\cdots\mu_k}{\lambda_1\lambda_2\cdots\lambda_k} \Big) \\
&= 1 + (\alpha_1 - 1) \Big(1 + \sum_{k=1}^{n} \frac{\mu_1\mu_2\cdots\mu_k}{\lambda_1\lambda_2\cdots\lambda_k} \Big).
\end{aligned}$$

Since $\displaystyle\sum_{k=1}^{\infty} \frac{\mu_1\mu_2\cdots\mu_k}{\lambda_1\lambda_2\cdots\lambda_k} = \infty$, the sequence $\displaystyle\sum_{k=1}^{n} \frac{\mu_1\mu_2\cdots\mu_k}{\lambda_1\lambda_2\cdots\lambda_k}$ increases without bound. For α_n's to exist, this requires that $\alpha_1 = 1$, which in turn implies that $\alpha_{n+1} = 1$, for $n \geq 1$.

12.5 BROWNIAN MOTION

1. **(a)** By the independent-increments property of Brownian motions, the desired probability is

$$\begin{aligned} &P\big(-1/2 < Z(10) < 1/2 \mid Z(5) = 0\big) \\ &\qquad = P\big(-1/2 < Z(10) - Z(5) < 1/2 \mid Z(5) = 0\big) \\ &\qquad = P\big(-1/2 < Z(10) - Z(5) < 1/2\big). \end{aligned}$$

Since $Z(10) - Z(5)$ is normal with mean 0 and variance $(10-5)\sigma^2 = 45$, letting $Z \sim N(0, 1)$, we have

$$\begin{aligned} &P\big(-1/2 < Z(10) - Z(5) < 1/2\big) \\ &\qquad = P\Big(\frac{-0.5 - 0}{\sqrt{45}} < Z < \frac{0.5 - 0}{\sqrt{45}}\Big) \\ &\qquad \approx P(-0.07 < Z < 0.07) = \Phi(0.07) - \Phi(-0.07) = 0.056. \end{aligned}$$

(b) In Theorem 12.9, let $t_1 = 5$, $t_2 = 7$, $z_1 = 0$, $z_2 = -1$. We have

$$E\big[Z(6) \mid Z(5) = 0 \text{ and } Z(7) = -1\big] = 0 + \frac{-1-0}{7-5}(6-5) = -0.5,$$

$$\operatorname{Var}\big[Z(6) \mid Z(5) = 0 \text{ and } Z(7) = -1\big] = 9 \cdot \frac{(7-6)(6-5)}{7-5} = 4.5.$$

2. In the subsection of 12.5, *The Maximum of a Brownian Motion*, we have shown that

$$P\big(\max_{0 \le s \le t} X(s) \le u\big) = \begin{cases} 2\Phi\Big(\dfrac{u}{\sigma\sqrt{t}}\Big) - 1 & u \ge 0 \\ 0 & u < 0. \end{cases}$$

We will show that $|X(t)|$ has the same probability distribution function. To do so, note that $X(t) \sim N(0, \sigma^2 t)$ and $X(t)/(\sigma\sqrt{t})$ is standard normal. Thus, for $u \ge 0$,

$$\begin{aligned} P\big(|X(t)| \le u\big) &= P\big(-u \le X(t) \le u\big) = P\big(X(t) \le u\big) - P\big(X(t) < -u\big) \\ &= P\Big(Z \le \frac{u}{\sigma\sqrt{t}}\Big) - P\Big(Z < -\frac{u}{\sigma\sqrt{t}}\Big) \\ &= \Phi\Big(\frac{u}{\sigma\sqrt{t}}\Big) - \Big[1 - \Phi\Big(\frac{u}{\sigma\sqrt{t}}\Big)\Big] = 2\Phi\Big(\frac{u}{\sigma\sqrt{t}}\Big) - 1. \end{aligned}$$

For $u < 0$, $P\big(|X(t)| \le u\big) = 0$. Hence $\max_{0 \le s \le t} X(s)$ and $|X(t)|$ are identically distributed.

3. Let $Z \sim N(0, 1)$. Since $X(t) \sim N(0, \sigma^2 t)$, we have

$$\begin{aligned} P\Big(\frac{|X(t)|}{t} > \varepsilon\Big) &= P\big(|X(t)| > \varepsilon t\big) \\ &= P\big(X(t) > \varepsilon t\big) + P\big(X(t) < -\varepsilon t\big) \\ &= P\Big(Z > \frac{\varepsilon t}{\sigma\sqrt{t}}\Big) + P\Big(Z < -\frac{\varepsilon t}{\sigma\sqrt{t}}\Big) \\ &= P\Big(Z > \frac{\varepsilon\sqrt{t}}{\sigma}\Big) + P\Big(Z < -\frac{\varepsilon\sqrt{t}}{\sigma}\Big) \\ &= 1 - \Phi\big(\varepsilon\sqrt{t}/\sigma\big) + \Phi\big(-\varepsilon\sqrt{t}/\sigma\big) \\ &= 1 - \Phi\big(\varepsilon\sqrt{t}/\sigma\big) + 1 - \Phi\big(\varepsilon\sqrt{t}/\sigma\big) = 2 - 2\Phi\big(\varepsilon\sqrt{t}/\sigma\big). \end{aligned}$$

This implies that

$$\lim_{t\to 0} P\Big(\frac{|X(t)|}{t} > \varepsilon\Big) = 2 - 1 = 1.$$

whereas

$$\lim_{t\to \infty} P\Big(\frac{|X(t)|}{t} > \varepsilon\Big) = 2 - 2 = 0,$$

4. Let F be the probability distribution function of $1/Y^2$. Let $Z \sim N(0, 1)$. We have

$$\begin{aligned} F(t) &= P\big(1/Y^2 \le t\big) = P\big(Y^2 \ge 1/t\big) = P\big(Y \ge 1/\sqrt{t}\,\big) + P\big(Y \le -1/\sqrt{t}\,\big) \\ &= P\Big(Z \ge \frac{\alpha}{\sigma\sqrt{t}}\Big) + P\Big(Z \le -\frac{\alpha}{\sigma\sqrt{t}}\Big) \\ &= 1 - \Phi\Big(\frac{\alpha}{\sigma\sqrt{t}}\Big) + \Phi\Big(-\frac{\alpha}{\sigma\sqrt{t}}\Big) = 2\Big[1 - \Phi\Big(\frac{\alpha}{\sigma\sqrt{t}}\Big)\Big], \end{aligned}$$

which, by (12.35), is also the distribution function of T_α.

5. Clearly, $P(T < x) = 0$ if $x \le t$. For $x > t$, by Theorem 12.10,

$$P(T < x) = P\big(\text{at least one zero in } (t, x)\big) = \frac{2}{\pi}\arccos\sqrt{\frac{t}{x}}.$$

Let F be the distribution function of T. We have shown that

$$F(x) = \begin{cases} 0 & x \le t \\ \dfrac{2}{\pi}\arccos\sqrt{\dfrac{t}{x}} & x \ge t. \end{cases}$$

6. Rewrite $X(t_1) + X(t_2)$ as $X(t_1) + X(t_2) = 2X(t_1) + X(t_2) - X(t_1)$. Now $2X(t_1)$ and $X(t_2) - X(t_1)$ are independent random variables. By Theorem 11.7, $2X(t_1) \sim N(0, 4\sigma^2 t_1)$. Since $X(t_2) - X(t_1) \sim N\big(0, \sigma^2(t_2 - t_1)\big)$, applying Theorem 11.7 once more implies that

$$2X(t_1) + X(t_2) - X(t_1) \sim N\big(0, 4\sigma^2 t_1 + \sigma^2(t_2 - t_1)\big).$$

Hence $X(t_1) + X(t_2) \sim N(0, 3\sigma^2 t_1 + \sigma^2 t_2)$.

7. Let $f(x, y)$ be the joint probability density function of $X(t)$ and $X(t+u)$. Let $f_{X(t+u)|X(t)}(y|a)$ be the conditional probability density function of $X(t+u)$ given that $X(t) = a$. Let $f_{X(t)}(x)$ be the probability density function of $X(t)$. We know that $X(t)$ is normal with mean 0 and variance $\sigma^2 t$. The formula for $f(x, y)$ is given by (12.28). Using these, we obtain

$$f_{X(t+u)|X(t)}(y|a) = \frac{f(a, y)}{f_{X(t)}(a)} = \frac{\dfrac{1}{2\sigma^2\pi\sqrt{tu}}\exp\Big(-\dfrac{1}{2\sigma^2}\Big[\dfrac{a^2}{t} + \dfrac{(y-a)^2}{u}\Big]\Big)}{\dfrac{1}{\sigma\sqrt{2\pi t}}\exp\Big(-\dfrac{a^2}{2\sigma^2 t}\Big)}$$
$$= \frac{1}{\sigma\sqrt{2\pi u}}\exp\Big[-\frac{1}{2\sigma^2 u}(y-a)^2\Big].$$

This shows that the conditional probability density function of $X(t+u)$ given that $X(t) = a$ is normal with mean a and variance $\sigma^2 u$. Hence

$$E\big[X(t+u) \mid X(t) = a\big] = a.$$

This implies that

$$E\big[X(t+u) \mid X(t)\big] = X(t).$$

8. By Example 10.23,

$$E\big[X(t)X(t+u) \mid X(t)\big] = X(t)E\big[X(t+u) \mid X(t)\big].$$

By Exercise 7 above,

$$E\big[X(t+u) \mid X(t)\big] = X(t).$$

Hence

$$\begin{aligned} E\big[X(t)X(t+u)\big] &= E\Big[E\big[X(t)X(t+u) \mid X(t)\big]\Big] \\ &= E\Big[X(t)E\big[X(t+u) \mid X(t)\big]\Big] \\ &= E\big[X(t)\cdot X(t)\big] = E\big[X(t)^2\big] \\ &= \text{Var}\big[X(t)\big] + \big(E\big[X(t)\big]\big)^2 = \sigma^2 t + 0 = \sigma^2 t. \end{aligned}$$

9. For $t > 0$, the probability density function of $Z(t)$ is

$$\phi_t(x) = \frac{1}{\sigma\sqrt{2\pi t}}\exp\Big[-\frac{x^2}{2\sigma^2 t}\Big].$$

Therefore,

$$E\big[V(t)\big] = E\big[|Z(t)|\big] = \int_{-\infty}^{\infty} |x|\phi_t(x)\,dx$$
$$= 2\int_0^{\infty} x\phi_t(x)\,dx = 2\int_0^{\infty} \frac{x}{\sigma\sqrt{2\pi t}} e^{-x^2/(2\sigma^2 t)}\,dx.$$

Making the change of variable $u = \dfrac{x}{\sigma\sqrt{t}}$ yields

$$E\big[V(t)\big] = \sigma\sqrt{\frac{2t}{\pi}}\int_0^{\infty} ue^{-u^2/2}\,du = \sigma\sqrt{\frac{2t}{\pi}}\Big[-e^{-u^2/2}\Big]_0^{\infty} = \sigma\sqrt{\frac{2t}{\pi}}.$$

$$\text{Var}\big[V(t)\big] = E\big[V(t)^2\big] - \big(E\big[V(t)\big]\big)^2 = E\big[Z(t)^2\big] - \frac{2\sigma^2 t}{\pi}$$
$$= \sigma^2 t - \frac{2\sigma^2 t}{\pi} = \sigma^2 t\Big(1 - \frac{2}{\pi}\Big),$$

since

$$E\big[Z(t)^2\big] = \text{Var}\big[Z(t)\big] + \big(E\big[Z(t)\big]\big)^2 = \sigma^2 t + 0 = \sigma^2 t.$$

To find $P\big(V(t) \le z \mid V(0) = z_0\big)$, note that, by (12.27),

$$P\big(V(t) \le z \mid V(0) = z_0\big) = P\big(|Z(t)| \le z \mid V(0) = z_0\big)$$
$$= P\big(-z \le Z(t) \le z \mid V(0) = z_0\big)$$
$$= \int_{-z}^{z} \frac{1}{\sigma\sqrt{2\pi t}} e^{-(u-z_0)^2/(2\sigma^2 t)}\,du.$$

Letting $U \sim N(z_0, \sigma^2 t)$ and $Z \sim N(0, 1)$, this implies that

$$P\big(V(t) \le z \mid V(0) = z_0\big) = P(-z \le U \le z)$$
$$= P\Big(\frac{-z - z_0}{\sigma\sqrt{t}} \le z \le \frac{z - z_0}{\sigma\sqrt{t}}\Big)$$
$$= \Phi\Big(\frac{z - z_0}{\sigma\sqrt{t}}\Big) - \Phi\Big(\frac{-z - z_0}{\sigma\sqrt{t}}\Big)$$
$$= \Phi\Big(\frac{z + z_0}{\sigma\sqrt{t}}\Big) + \Phi\Big(\frac{z - z_0}{\sigma\sqrt{t}}\Big) - 1.$$

10. Clearly, $D(t) = \sqrt{X(t)^2 + Y(t)^2 + Z(t)^2}$. Since $X(t)$, $Y(t)$, and $Z(t)$ are independent and identically distributed normal random variables with mean 0 and variance $\sigma^2 t$, we have

$$E\big[D(t)\big] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \sqrt{x^2+y^2+z^2}\cdot \frac{1}{\sigma\sqrt{2\pi t}}e^{-x^2/(2\sigma^2 t)}\cdot \frac{1}{\sigma\sqrt{2\pi t}}e^{-y^2/(2\sigma^2 t)} \cdot \frac{1}{\sigma\sqrt{2\pi t}}e^{-z^2/(2\sigma^2 t)}\,dx\,dy\,dz$$

$$= \frac{1}{2\pi\sigma^3 t\sqrt{2\pi t}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \sqrt{x^2+y^2+z^2}\cdot e^{-(x^2+y^2+z^2)/(2\sigma^2 t)}\,dx\,dy\,dz.$$

We now make a change of variables to spherical coordinates: $x = \rho\sin\phi\cos\theta$, $y = \rho\sin\phi\sin\theta$, $z = \rho\cos\phi$, $\rho^2 = x^2+y^2+z^2$, $dx\,dy\,dz = \rho^2\sin\phi\,d\rho\,d\phi\,d\theta$, $0 \le \rho < \infty$, $0 \le \phi \le \pi$, and $0 \le \theta \le 2\pi$. We obtain

$$E\big[D(t)\big] = \frac{1}{2\pi\sigma^3 t\sqrt{2\pi t}}\int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} \rho e^{-\rho^2/(2\sigma^2 t)}\cdot\rho^2\sin\phi\,d\rho\,d\phi,\,d\theta$$

$$= \frac{1}{2\pi\sigma^3 t\sqrt{2\pi t}}\int_0^{2\pi}\Big[\int_0^{\pi}\Big(\int_0^{\infty}\rho^3 e^{-\rho^2/(2\sigma^2 t)}\,d\rho\Big)\sin\phi\,d\phi\Big]d\theta$$

$$= \frac{1}{2\pi\sigma^3 t\sqrt{2\pi t}}\int_0^{2\pi}\Big(\int_0^{\pi}\Big[-\sigma^2 t(\rho^2+2\sigma^2 t)e^{-\rho^2/(2\sigma^2 t)}\Big]_0^{\infty}\sin\phi\,d\phi\Big)d\theta$$

$$= \frac{1}{2\pi\sigma^3 t\sqrt{2\pi t}}\cdot 2\sigma^4 t^2\int_0^{2\pi}\Big(\int_0^{\pi}\sin\phi\,d\phi\Big)d\theta = 2\sigma\sqrt{\frac{2t}{\pi}}.$$

11. Noting that $\sqrt{5.29} = 2.3$, we have

$$V(t) = 95e^{-2t+2.3W(t)},$$

where $\big\{W(t)\colon t \ge 0\big\}$ is a standard Brownian motion. Hence $W(t) \sim N(0,t)$. The desired probability is

$$P\big(V(0.75) < 80\big) = P\big(95e^{-2(0.75)+2.3W(0.75)} < 80\big)$$

$$= P\big(e^{2.3W(0.75)} < 3.774\big) = P\big(W(0.75) < 0.577\big)$$

$$= P\Big(\frac{W(0.75)-0}{\sqrt{0.75}} < \frac{0.577}{\sqrt{0.75}}\Big) = P(Z < 0.67) = \Phi(0.67) = 0.7486.$$

REVIEW PROBLEMS FOR CHAPTER 12

1. Label the time point 10:00 as $t = 0$. We are given that $N(180) = 10$ and are interested in $P\big(S_{10} \geq 160 \mid N(180) = 10\big)$. Let $X_1, X_2, \ldots, X_{10}$ be 10 independent random variables uniformly distributed over the interval [0, 180]. Let $Y = \max(X_1, \ldots, X_{10})$. By Theorem 12.4,

$$\begin{aligned} P\Big(S_{10} > 160 \mid N(180) = 10\Big) &= P(Y > 160) = 1 - P(Y \leq 160) \\ &= 1 - P\big(\max(X_1, \ldots, X_{10}) \leq 160\big) \\ &= 1 - P(X_1 \leq 160)P(X_2 \leq 160) \cdots P(X_{10} \leq 160) \\ &= 1 - \Big(\frac{160}{180}\Big)^{10} = 0.692. \end{aligned}$$

2. For all positive integer n, we have that

$$\boldsymbol{P}^{2n} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \boldsymbol{P}^{2n+1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Therefore, $\{X_n : n = 0, 1, \ldots\}$ is not regular.

3. By drawing a transition graph, it can be readily seen that, if states 0, 1, 2, 3, and 4 are renamed 0, 4, 2, 1, and 3, respectively, then the transition probability matrix $\boldsymbol{P_1}$ will change to $\boldsymbol{P_2}$.

4. Let Z be the number of transitions until the first visit to 1. Clearly, Z is a geometric random variable with parameter $p = 3/5$. Hence its expected value is $1/p = 5/3$.

5. By drawing a transition graph, it is readily seen that this Markov chain consists of two recurrent classes {3, 5} and {4}, and two transient classes {1} and {2}.

6. We have that

$$X_{n+1} = \begin{cases} X_n & \text{if the } (n+1)\text{st outcome is not 6} \\ 1 + X_n & \text{if the } (n+1)\text{st outcome is 6.} \end{cases}$$

This shows that $\{X_n : n = 1, 2, \ldots\}$ is a Markov chain with state space $\{0, 1, 2, \ldots\}$. Its transition probability matrix is given by

$$\boldsymbol{P} = \begin{pmatrix} 5/6 & 1/6 & 0 & 0 & 0 & \ldots \\ 0 & 5/6 & 1/6 & 0 & 0 & \ldots \\ 0 & 0 & 5/6 & 1/6 & 0 & \ldots \\ 0 & 0 & 0 & 5/6 & 1/6 & \ldots \\ \vdots & & & & & \end{pmatrix}.$$

All states are transient; no two states communicate with each other. Therefore, we have infinitely many classes; namely, {0}, {1}, {2}, ..., and each one of them is transient.

7. The desired probability is

$$\begin{aligned}
&p_{11}p_{11} + p_{11}p_{12} + p_{12}p_{22} + p_{12}p_{21} + p_{21}p_{11} + p_{21}p_{12} + p_{22}p_{21} + p_{22}p_{22} \\
&\quad = (0.20)^2 + (0.20)(0.30) + (0.30)(0.15) + (0.30)(0.32) \\
&\qquad + (0.32)(0.20) + (0.32)(0.30) + (0.15)(0.32) + (0.15)^2 = 0.4715.
\end{aligned}$$

8. The following is an example of such a transition probability matrix:

$$\boldsymbol{P} = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 & 1/2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/3 & 2/3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

9. For $n \geq 1$, let

$$X_n = \begin{cases} 1 & \text{if the } n\text{th golfball produced is defective} \\ 0 & \text{if the } n\text{th golfball produced is good.} \end{cases}$$

Then $\{X_n : n = 1, 2, \dots\}$ is a Markov chain with state space $\{0, 1\}$ and transition probability matrix $\begin{pmatrix} 15/18 & 3/18 \\ 11/12 & 1/12 \end{pmatrix}$. Let π_0 be the fraction of golfballs produced that are good, and π_1 be the fraction of the balls produced that are defective. Then, by Theorem 12.7, π_0 and π_1 satisfy

$$\begin{pmatrix} \pi_0 \\ \pi_1 \end{pmatrix} = \begin{pmatrix} 15/18 & 11/12 \\ 3/18 & 1/12 \end{pmatrix} \begin{pmatrix} \pi_0 \\ \pi_1 \end{pmatrix},$$

which gives us the following system of equations

$$\begin{cases} \pi_0 = (15/18)\pi_0 + (11/12)\pi_1 \\ \pi_1 = (3/18)\pi_0 + (1/12)\pi_1. \end{cases}$$

By choosing any one of these equations along with the relation $\pi_0 + \pi_1 = 1$, we obtain a system of two equations in two unknowns. Solving that system yields

$$\pi_0 = \frac{11}{13} \approx 0.85 \quad \text{and} \quad \pi_1 = \frac{2}{13} \approx 0.15.$$

Therefore, approximately 15% of the golfballs produced have no logos.

10. Let

$$X_n = \begin{cases} 1 & \text{if the } n\text{th ball is drawn by Carmela} \\ 2 & \text{if the } n\text{th ball is drawn by Daniela} \\ 3 & \text{if the } n\text{th ball is drawn by Lucrezia.} \end{cases}$$

The process $\{X_n : n = 1, 2, \dots\}$ is an irreducible, aperiodic, positive recurrent Markov chain with transition probability matrix

$$\boldsymbol{P} = \begin{pmatrix} 7/31 & 11/31 & 13/31 \\ 7/31 & 11/31 & 13/31 \\ 7/31 & 11/31 & 13/31 \end{pmatrix}.$$

Let π_1, π_2, and π_3 be the long-run proportion of balls drawn by Carmela, Daniela, and Lucrezia, respectively. Intuitively, it should be clear that these quantities are 7/31, 11/31, and 13/31, respectively. However, that can be seen also by solving the following matrix equation along with $\pi_0 + \pi_1 + \pi_3 = 1$.

$$\begin{pmatrix} \pi_1 \\ \pi_2 \\ \pi_3 \end{pmatrix} = \begin{pmatrix} 7/31 & 7/31 & 7/31 \\ 11/31 & 11/31 & 11/31 \\ 13/31 & 13/31 & 13/31 \end{pmatrix} \begin{pmatrix} \pi_1 \\ \pi_2 \\ \pi_3 \end{pmatrix}.$$

11. Let π_1 and π_2 be the long-run probabilities that Francesco devotes to playing golf and playing tennis, respectively. Then, by Theorem 12.7, π_1 and π_2 are obtained from solving the system of equations

$$\begin{pmatrix} \pi_1 \\ \pi_2 \end{pmatrix} = \begin{pmatrix} 0.30 & 0.58 \\ 0.70 & 0.42 \end{pmatrix} \begin{pmatrix} \pi_1 \\ \pi_2 \end{pmatrix}$$

along with $\pi_1 + \pi_2 = 1$. The matrix equation above gives the following system of equations:

$$\begin{cases} \pi_1 = 0.30\pi_1 + 0.58\pi_2 \\ \pi_2 = 0.70\pi_1 + 0.42\pi_2. \end{cases}$$

By choosing any one of these equations along with the relation $\pi_1 + \pi_2 = 1$, we obtain a system of two equations in two unknowns. Solving that system yields $\pi_1 = 0.453125$ and $\pi_2 = 0.546875$. Therefore, the long-run probability that, on a randomly selected day, Francesco plays tennis is approximately 0.55.

12. Suppose that a train leaves the station at $t = 0$. Let X_1 be the time until the first passenger arrives at the station after $t = 0$. Let X_2 be the additional time it will take until a train arrives at the station, X_3 be the time after that until a passenger arrives, and so on. Clearly, X_1, $X_2, \dots$ are the times between consecutive change of states. By the memoryless property of exponential random variables, $\{X_1, X_2, \dots\}$ is a sequence of independent and identically distributed exponential random variables with mean $1/\lambda$. Hence, by Remark 7.2, $\{N(t) : t \geq 0\}$ is a Poisson process with rate λ. Therefore, $N(t)$ is a Poisson random variable with parameter λt.

13. Let $X(t)$ be the number of components working at time t. Clearly, $\{X(t) : t \geq 0\}$ is a continuous-time Markov chain with state space $\{0, 1, 2\}$. Let π_0, π_1, and π_2 be the long-run proportion of time the process is in states 0, 1, and 2, respectively. The balance equations for $\{X(t) : t \geq 0\}$ are as follows:

State	Input rate to	=	Output rate from
0	$\lambda\pi_1$	=	$\mu\pi_0$
1	$2\lambda\pi_2 + \mu\pi_0$	=	$\mu\pi_1 + \lambda\pi_1$
2	$\mu\pi_1$	=	$2\lambda\pi_2$

From these equations, we obtain $\pi_1 = \dfrac{\mu}{\lambda}\pi_0$ and $\pi_2 = \dfrac{\mu^2}{2\lambda^2}\pi_0$. Using $\pi_0 + \pi_1 + \pi_2 = 1$ yields

$$\pi_0 = \frac{2\lambda^2}{2\lambda^2 + 2\lambda\mu + \mu^2}.$$

Hence the desired probability is

$$1 - \pi_0 = \frac{\mu(2\lambda + \mu)}{2\lambda^2 + 2\lambda\mu + \mu^2}.$$

14. Suppose that every time an out-of-order machine is repaired and is ready to operate a birth occurs. Suppose that a death occurs every time that a machine breaks down. The fact that $\big\{X(t)\colon t \geq 0\big\}$ is a birth and death process should be clear. The birth and death rates are

$$\lambda_n = \begin{cases} k\lambda & n = 0, 1, \dots, m + s - k \\ (m + s - n)\lambda & n = m + s - k + 1, m + s - k + 2, \dots, m + s \\ 0 & n \geq m + s; \end{cases}$$

$$\mu_n = \begin{cases} n\mu & n = 0, 1, \dots, m \\ m\mu & n = m + 1, m + 2, \dots, m + s \\ 0 & n > m + s. \end{cases}$$

15. Let $X(t)$ be the number of machines operating at time t. For $0 \leq i \leq m$, let π_i be the long-run proportion of time that there are exactly i machines operating. Suppose that a birth occurs each time that an out-of-order machine is repaired and begins to operate, and a death occurs each time that a machine breaks down. Then $\big\{X(t)\colon t \geq 0\big\}$ is a birth and death process with state space $\{0, 1, \dots, m\}$, and birth and death rates, respectively, given by $\lambda_i = (m - i)\lambda$ and $\mu_i = i\mu$ for $i = 0, 1, \dots, m$. To find π_0, first we will calculate the following sum:

$$\sum_{i=1}^{m} \frac{\lambda_0\lambda_1 \cdots \lambda_{i-1}}{\mu_1\mu_2 \cdots \mu_i} = \sum_{i=1}^{m} \frac{(m\lambda)\big[(m-1)\lambda\big]\big[(m-2)\lambda\big]\cdots\big[(m-i+1)\lambda\big]}{\mu(2\mu)(3\mu)\cdots(i\mu)}$$

$$= \sum_{i=1}^{m} \frac{{}_mP_i\,\lambda^i}{i!\,\mu^i} = \sum_{i=1}^{m} \binom{m}{i}\Big(\frac{\lambda}{\mu}\Big)^i$$

$$= -1 + \sum_{i=0}^{m} \binom{m}{i}\Big(\frac{\lambda}{\mu}\Big)^i 1^{m-i} = -1 + \Big(1 + \frac{\lambda}{\mu}\Big)^m,$$

where ${}_mP_i$ is the number of i-element permutations of a set containing m objects. Hence, by (12.22),

$$\pi_0 = \Big(1 + \frac{\lambda}{\mu}\Big)^{-m} = \Big(\frac{\lambda + \mu}{\mu}\Big)^{-m} = \Big(\frac{\mu}{\lambda + \mu}\Big)^{m}.$$

By (12.21),

$$\begin{aligned}\pi_i &= \frac{\lambda_0\lambda_1\cdots\lambda_{i-1}}{\mu_1\mu_2\cdots\mu_i}\pi_0 = \frac{{}_mP_i\lambda^i}{i!\,\mu^i}\pi_0\\ &= \binom{m}{i}\Big(\frac{\lambda}{\mu}\Big)^i\Big(\frac{\mu}{\lambda+\mu}\Big)^m = \binom{m}{i}\Big(\frac{\lambda}{\mu}\Big)^i\Big(\frac{\mu}{\lambda+\mu}\Big)^i\Big(\frac{\mu}{\lambda+\mu}\Big)^{m-i}\\ &= \binom{m}{i}\Big(\frac{\lambda}{\lambda+\mu}\Big)^i\Big(1 - \frac{\lambda}{\lambda+\mu}\Big)^{m-i}, \quad 0 \le i \le m.\end{aligned}$$

Therefore, in steady-state, the number of machines that are operating is binomial with parameters m and $\lambda/(\lambda + \mu)$.

16. Let $X(t)$ be the number of cars at the center, either being inspected or waiting to be inspected, at time t. Clearly, $\big\{X(t)\colon t \ge 0\big\}$ is a birth and death process with rates $\lambda_n = \lambda/(n + 1)$, $n \ge 0$, and $\mu_n = \mu$, $n \ge 1$. Since

$$\sum_{n=1}^{\infty}\frac{\lambda_0\lambda_1\cdots\lambda_{n-1}}{\mu_1\mu_2\cdots\mu_n} = \sum_{n=1}^{\infty}\frac{\lambda\cdot\dfrac{\lambda}{2}\cdot\dfrac{\lambda}{3}\cdots\dfrac{\lambda}{n}}{\mu^n} = -1 + \sum_{n=0}^{\infty}\frac{1}{n!}\Big(\frac{\lambda}{\mu}\Big)^n = e^{\lambda/\mu} - 1.$$

By (12.18), $\pi_0 = e^{-\lambda/\mu}$. Hence, by (12.17),

$$\pi_n = \frac{\lambda\cdot\dfrac{\lambda}{2}\cdot\dfrac{\lambda}{3}\cdots\dfrac{\lambda}{n}}{\mu^n}e^{-\lambda/\mu} = \frac{(\lambda/\mu)^n e^{-\lambda/\mu}}{n!}, \quad n \ge 0.$$

Therefore, the long-run probability that there are n cars at the center for inspection is Poisson with rate λ/μ.

17. Let $X(t)$ be the population size at time t. Then $\big\{X(t)\colon t \ge 0\big\}$ is a birth and death process with birth rates $\lambda_n = n\lambda$, $n \ge 1$, and death rates $\mu_n = n\mu$, $n \ge 0$. For $i \ge 0$, let H_i be the time, starting from i, until the population size reaches $i + 1$ for the first time. We are interested in $\sum_{i=1}^{4} E(H_i)$. Note that, by Lemma 12.2,

$$E(H_i) = \frac{1}{\lambda_i} + \frac{\mu_i}{\lambda_i}E(H_{i-1}), \quad i \ge 1.$$

Since $E(H_0) = 1/\lambda$,

$$E(H_1) = \frac{1}{\lambda} + \frac{\mu}{\lambda}\cdot\frac{1}{\lambda} = \frac{1}{\lambda} + \frac{\mu}{\lambda^2},$$

$$E(H_2) = \frac{1}{2\lambda} + \frac{2\mu}{2\lambda} \cdot \Big(\frac{1}{\lambda} + \frac{\mu}{\lambda^2}\Big) = \frac{1}{2\lambda} + \frac{\mu}{\lambda^2} + \frac{\mu^2}{\lambda^3},$$

$$E(H_3) = \frac{1}{3\lambda} + \frac{3\mu}{3\lambda}\Big(\frac{1}{2\lambda} + \frac{\mu}{\lambda^2} + \frac{\mu^2}{\lambda^3}\Big) = \frac{1}{3\lambda} + \frac{\mu}{2\lambda^2} + \frac{\mu^2}{\lambda^3} + \frac{\mu^3}{\lambda^4},$$

$$E(H_4) = \frac{1}{4\lambda} + \frac{4\mu}{4\lambda}\Big(\frac{1}{3\lambda} + \frac{\mu}{2\lambda^2} + \frac{\mu^2}{\lambda^3} + \frac{\mu^3}{\lambda^4}\Big) = \frac{1}{4\lambda} + \frac{\mu}{3\lambda^2} + \frac{\mu^2}{2\lambda^3} + \frac{\mu^3}{\lambda^4} + \frac{\mu^4}{\lambda^5}.$$

Therefore, the answer is

$$\sum_{i=1}^{4} E(H_i) = \frac{25\lambda^4 + 34\lambda^3\mu + 30\lambda^2\mu^2 + 24\lambda\mu^3 + 12\mu^4}{12\lambda^5}.$$

18. Let $X(t)$ be the population size at time t. Then $\big\{X(t)\colon t \geq 0\big\}$ is a birth and death process with rates $\lambda_n = \gamma$, $n \geq 0$, and $\mu_n = n\mu$, $n \geq 1$. To find π_i's, we will first calculate the sum in the relation (12.18):

$$\sum_{n=1}^{\infty} \frac{\lambda_0\lambda_1 \cdots \lambda_{n-1}}{\mu_1\mu_2 \cdots \mu_n} = \sum_{n=1}^{\infty} \frac{\gamma^n}{n!\,\mu^n} = -1 + \sum_{n=0}^{\infty} \frac{1}{n!}\Big(\frac{\gamma}{\mu}\Big)^n = -1 + e^{\gamma/\mu}.$$

Thus, by (12.18), $\pi_0 = e^{-\gamma/\mu}$ and, by (12.17), for $i \geq 1$,

$$\pi_i = \frac{\gamma^n}{n!\,\mu^n} e^{-\gamma/\mu} = \frac{(\gamma/\mu)^n e^{-\gamma/\mu}}{n!}.$$

Hence the steady-state probability mass function of the population size is Poisson with parameter γ/μ.

19. By applying Theorem 12.9 to $\big\{Y(t)\colon t \geq 0\big\}$ with $t_1 = 0$, $t_2 = t$, $y_1 = 0$, $y_2 = y$, and $t = s$, we have

$$E\big[Y(s) \mid Y(t) = y\big] = 0 + \frac{y - 0}{t - 0}(s - 0) = \frac{s}{t}y,$$

and

$$\text{Var}\big[Y(s) \mid Y(t) = y\big] = \sigma^2 \cdot \frac{(t - s)(s - 0)}{t - 0} = \sigma^2 (t - s)\frac{s}{t}.$$

20. First, suppose that $s < t$. By Example 10.23,

$$E\big[X(s)X(t) \mid X(s)\big] = X(s)E\big[X(t) \mid X(s)\big].$$

Now, by Exercise 7, Section 12.5,

$$E\big[X(t) \mid X(s)\big] = X(s).$$

Hence

$$\begin{aligned} E\big[X(s)X(t)\big] &= E\Big[E\big[X(s)X(t) \mid X(s)\big]\Big] \\ &= E\Big[X(s)E\big[X(t) \mid X(s)\big]\Big] \\ &= E\big[X(s)X(s)\big] = E\big[X(s)^2\big] \\ &= \text{Var}\big[X(s)\big] + \big(E\big[X(s)\big]\big)^2 \\ &= \sigma^2 s + 0 = \sigma^2 s. \end{aligned}$$

For $t < s$, by symmetry,

$$E\big[X(s)X(t)\big] = \sigma^2 t.$$

Therefore,

$$E\big[X(s)X(t)\big] = \sigma^2 \min(s, t).$$

21. By Theorem 12.10,

$$P(U < x \text{ and } T > y) = P\big(\text{no zeros in } (x, y)\big) = 1 - \frac{2}{\pi} \arccos \sqrt{\frac{x}{y}}.$$

22. Let the current price of the stock, per share, be v_0. Noting that $\sqrt{27.04} = 5.2$, we have

$$V(t) = v_0 e^{3t+5.2W(t)},$$

where $\big\{W(t)\colon t \geq 0\big\}$ is a standard Brownian motion. Hence $W(t) \sim N(0, t)$. The desired probability is calculated as follows:

$$\begin{aligned} P\big(V(2) \geq 2v_0\big) &= P\big(v_0 e^{6+5.2W(2)} \geq 2v_0\big) \\ &= P\big(6 + 5.2W(2) \geq \ln 2\big) = P\big(W(2) \geq -1.02\big) \\ &= P\Big(\frac{W(2) - 0}{\sqrt{2}} \geq -0.72\Big) \\ &= P(Z \geq -0.72) = 1 - P(Z < -0.72) \\ &= 1 - \Phi(-0.72) = 0.7642. \end{aligned}$$