Higher Geometry

Introduction

Brief Historical Sketch

Greeks (Thales, Euclid, Archimedes)
Parallel Postulate
Non-Euclidean Geometries
Projective Geometry Revolution
Axiomatics revisited – Modern Geometries

Modern Applications

Computer Graphics
Bio-medical Applications
Modeling
Cryptology and Coding Theory

Euclid's Definitions

From Book I of *The Elements:*

- A **point** is that which has no part
- A **line** is breadthless length
- The extremities of a line are points
- A **straight line** is a line which lies evenly with the points on itself.

Axiomatic Systems

Vish (Vicious Circle) : Start with any word in a dictionary and continue to look up words used in the definition until some word gets repeated for the first time.

" *Vish illustrates the important principle that any definition of a word must inevitably involve other words, which require further definitions. The only way to avoid a vicious circle is to regard certain primitive concepts as being so simple and obvious that we agree to leave them undefined. Similarly, the proof of any statement uses other statements; and since we must begin somewhere, we agree to leave a few simple statements unproved. These primitive statements are called axioms.*" - Coxeter, **Projective Geometry**, pg. 6.

Vish

Example: Vish using The American Heritage Dictionary

Point : = A dimensionless geometric object having no property but *location*.

Location : = A *place* where something is or might be located.

Place : = A portion of *space*.

Space : = A set of *points* satisfying specified geometric postulates.

Point : = ...

Finite Geometries

Can be traced back to Gino Fano (1892) with some ideas going back to von Staudt (1852).

There are two undefined terms : ***points, lines***. There is also a relation between them called ***on***. This relationship is symmetric so we speak of points being *on* lines and lines being *on* points.

Three Point Geometry

Axioms for the Three Point Geometry:

1. There exist exactly 3 points in this geometry.
2. Two distinct points are **on** exactly one line.
3. Not all the points of the geometry are **on** the same line.
4. Two distinct lines are **on** at least one point.

Theorem 1.1 : Two distinct lines are on exactly one point.

Theorem 1.2 : The three point geometry has exactly three lines.

Theorem 1.1

Two distinct lines are on exactly one point.

To prove this, note that by axiom 4 we need only show that two distinct lines are on at most one point.

Assume, to the contrary, that distinct lines *l* and *m*, meet at points **P** and **Q**. This contradicts axiom 2, which says that the points **P** and **Q** lie on exactly one line. Thus, our assumption is false, and two distinct lines are on at most one point. Proving the theorem.

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3. Not all the points of the geometry are on the same line.
4. Two distinct lines are on at least one point.

Theorem 1.2

The three point geometry has exactly three lines.

Let the line determined by two of the points, say A and B, be denoted by m (Axiom 2). We know that the third point, C, is not on m by Axiom 3. AC is thus a line different from m, and BC is also a line different from m. These two lines can not be equal to each other since that would imply that the three points are on the same line. So there are at least 3 lines.

If there was a fourth line, it would have to meet each of the other lines at a point by Theorem 1.1. As those three lines do not pass through a common point, the fourth line must have at least two points on it contradicting Axiom 2.

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2. Two distinct points are on exactly one line.
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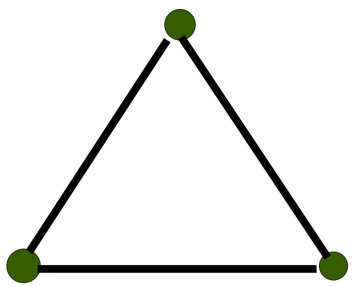
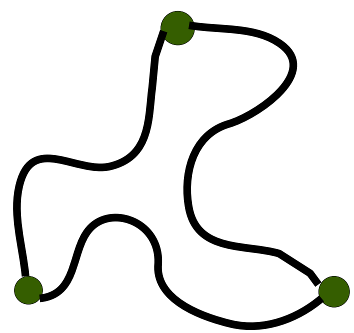
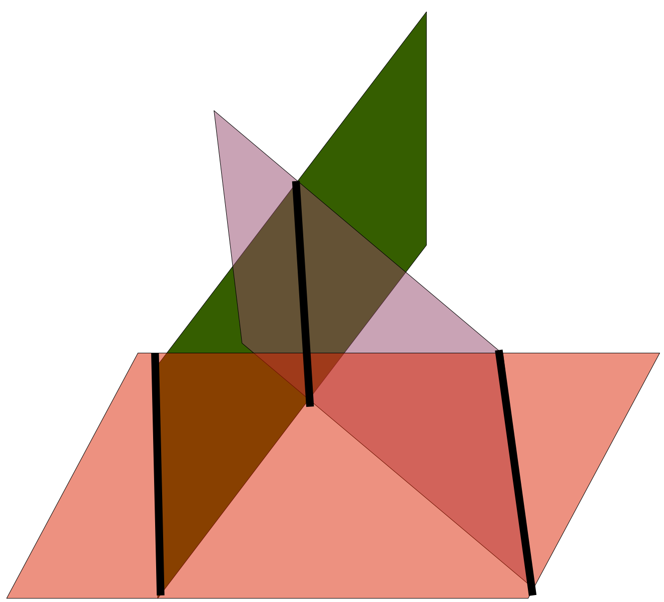
4. Two distinct lines are on at least one point.

Representations





AB AC BC



The Four Line Geometry

The Axioms for the Four Line Geometry:

1. There exist exactly 4 lines.
2. Any two distinct lines have exactly one point on both of them.
3. Each point is on exactly two lines.

Theorem 1.3: The four line geometry has exactly six points.

Theorem 1.4: Each line of the four-line geometry has exactly 3 points on it.

Theorem 1.3

The four line geometry has exactly six points.

There are exactly 6 pairs of lines (4 choose 2), and every pair meets at a point. Since each point lies on only two lines, these six pairs of lines give 6 distinct points.

To prove the statement we need to show that there are no more points than these 6. However, by axiom 3, each point is on two lines of the geometry and every such point has been accounted for -there are no other points.

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1. There exist exactly 4 lines.
2. Any two distinct lines have exactly one point on both of them.
3. Each point is on exactly two lines.

Theorem 1.4

Each line of the four-line geometry has exactly 3 points on it.

Consider any line. The three other lines must each have a point in common with the given line (Axiom 2). These three points are distinct, otherwise Axiom 3 is violated. There can be no other points on the line since if there was, there would have to be another line on the point by Axiom 3 and we can't have that without violating Axiom 1.

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1. There exist exactly 4 lines.
2. Any two distinct lines have exactly one point on both of them.
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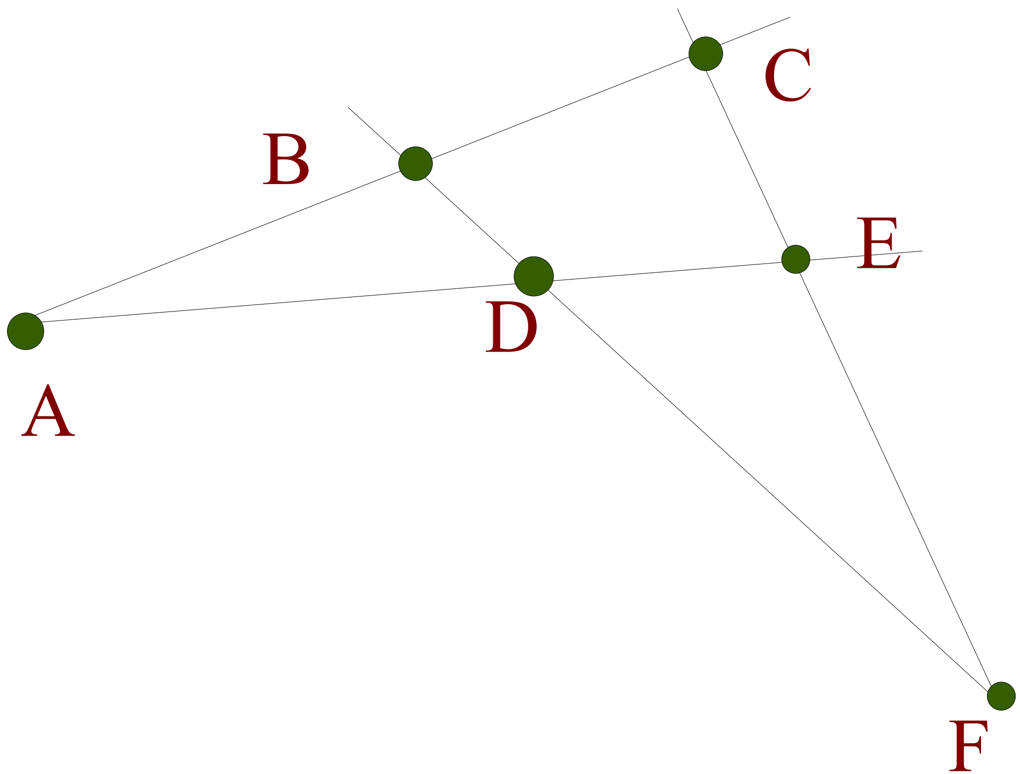
Representations



A	A	F	F
B	D	B	C
C	E	D	E

Plane Duals

The ***plane dual*** of a statement is the statement obtained by interchanging the terms point and line.

Example:

Statement: Two *points* are on a unique *line*.

Plane dual: Two *lines* are on a unique *point*.

or Two lines meet at a unique point.

The plane duals of the axioms for the four-line geometry will give the axioms for the four-point geometry. And the plane duals of Theorems 1.3 and 1.4 will give valid theorems in the four-point geometry.

The Four ~~Line~~ Point Geometry

The Axioms for the Four ~~Line~~ Point Geometry:

1. There exist exactly 4 ~~lines~~ points.

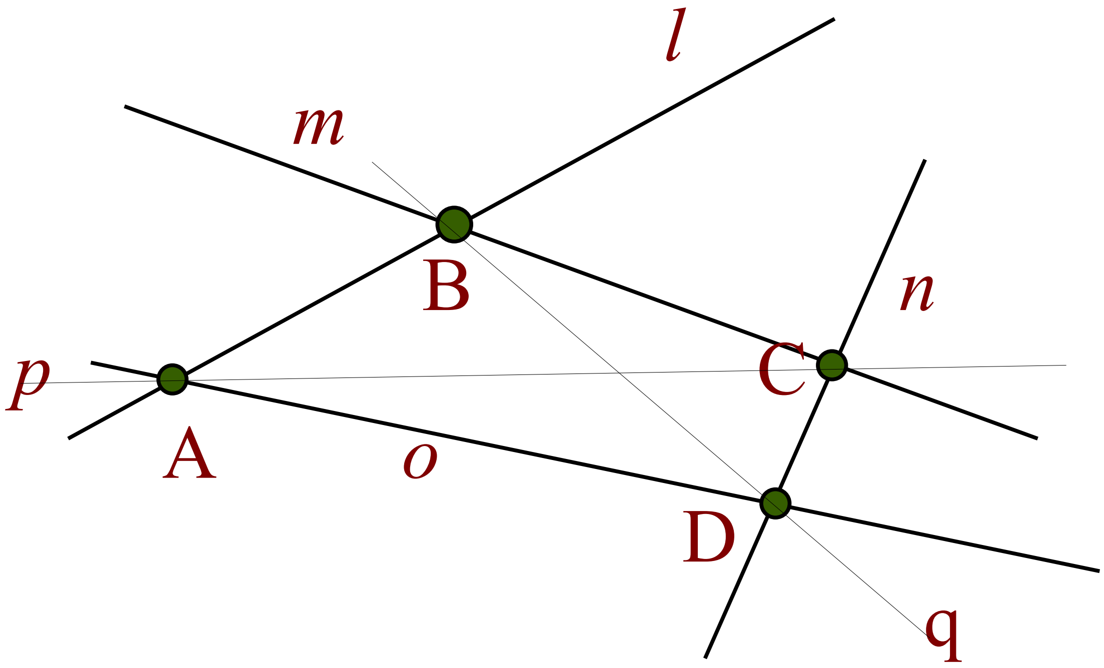
2. Any two distinct ~~lines~~ points have exactly one ~~point~~ line on both of them.

3. Each ~~point~~ line is on exactly two ~~lines~~ points.

Theorem ~~1.3~~ 1.5: The four ~~line~~ point geometry has exactly six ~~points~~ lines.

Theorem ~~1.4~~ 1.6: Each ~~line~~ point of the four-~~line~~ point geometry has exactly 3 ~~points~~ lines on it.

The Four Point Geometry



	A	B	C	D
l	1	1	0	0
m	0	1	1	0
n	0	0	1	1
o	1	0	0	1
p	1	0	1	0
q	0	1	0	1

Incidence Matrix

Fano's Geometry

1. There exists at least one line.
2. Every line of the geometry has exactly 3 points on it.
3. Not all points of the geometry are on the same line.
4. For two distinct points, there exists exactly one line on both of them.
5. Each two lines have at least one point on both of them.

Theorem 1.7: Each two lines have exactly one point in common.

Theorem 1.8 : Fano's geometry consists of exactly seven points and seven lines.

Theorem 1.7

Each two lines have exactly one point in common.

Proof: By Axiom 5 we know that every two lines have at least one point in common, so we must show that they can not have more than one point in common.

Assume that two distinct lines have two distinct points in common. This assumption violates Axiom 4 since these two points would then be on two distinct lines.

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1. There exists at least one line.
2. Every line of the geometry has exactly 3 points on it.
3. Not all points of the geometry are on the same line.
4. For two distinct points, there exists exactly one line on both of them.
5. Each two lines have at least one point on both of them.

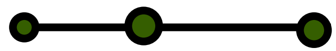
Theorem 1.8

Fano's geometry consists of exactly seven points and seven lines.

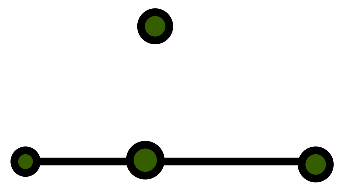
Proof:

Axiom 1

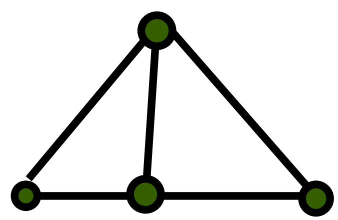
Axiom 2



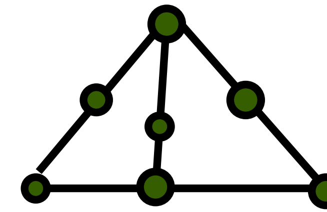
Axiom 3

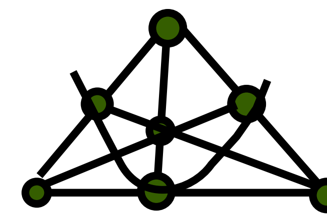


Axiom 4



Axiom 2





At least 7 points and 7 lines

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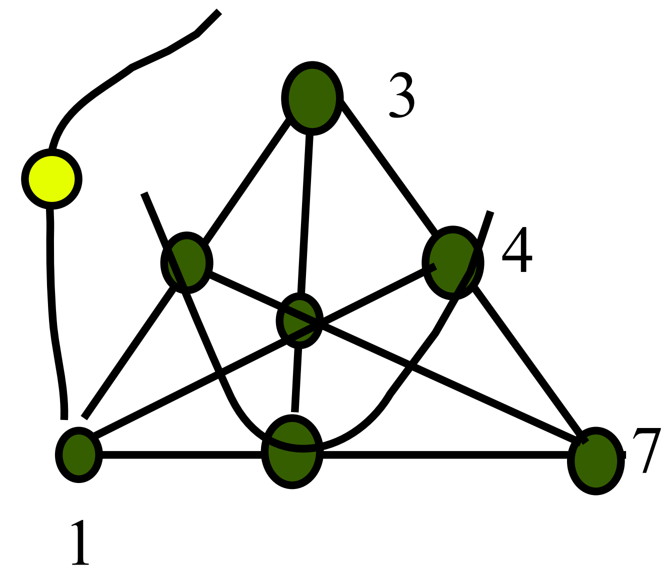
1. There exists at least one line.
2. Every line of the geometry has exactly 3 points on it.
3. Not all points of the geometry are on the same line.
4. For two distinct points, there exists exactly one line on both of them.
5. Each two lines have at least one point on both of them.

Theorem 1.8 (cont.)

Fano's geometry consists of exactly seven points and seven lines.

Proof:

Assume that there is an 8th point. By axiom 4 it must be on a line with point 1. By axiom 5 this line must meet the line containing points 3,4 and 7. But the line can not meet at one of these points otherwise axiom 4 is violated. So the point of intersection would have to be a fourth point on the line 347 which contradicts axiom 2.



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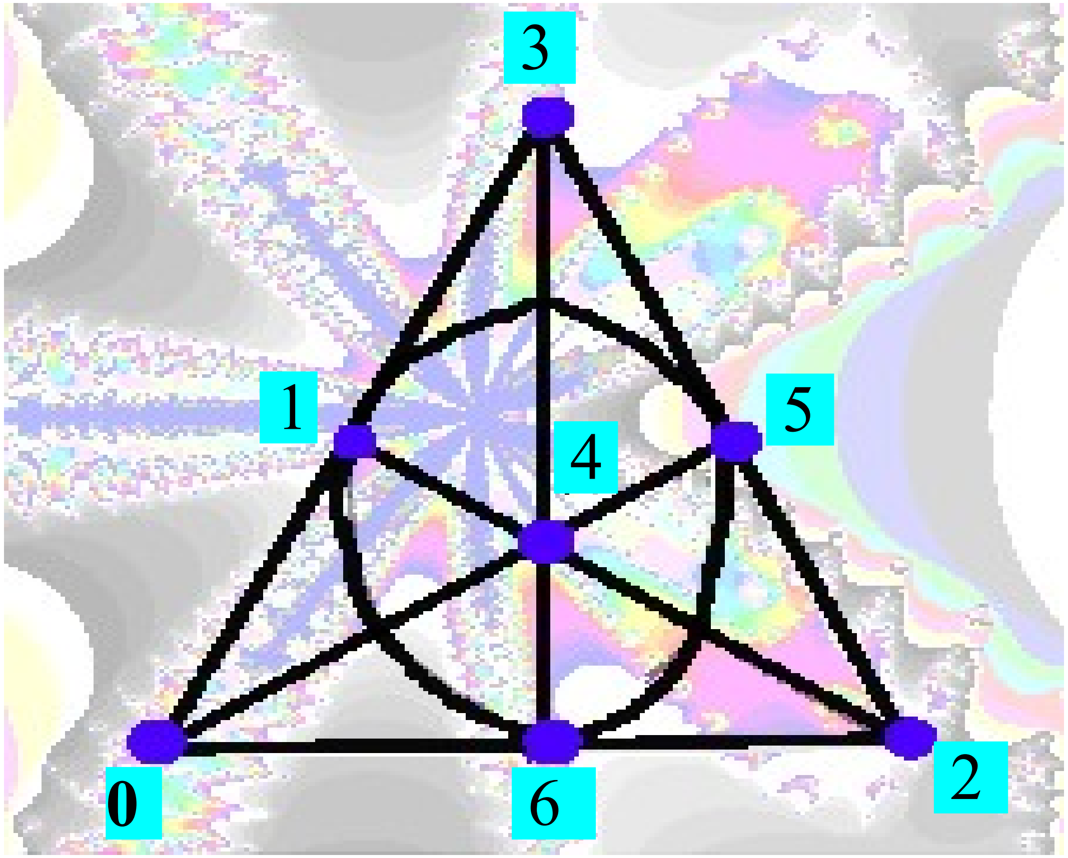
Fano's Geometry



0 1 3
1 2 4
2 3 5
3 4 6
4 5 0
5 6 1
6 0 2

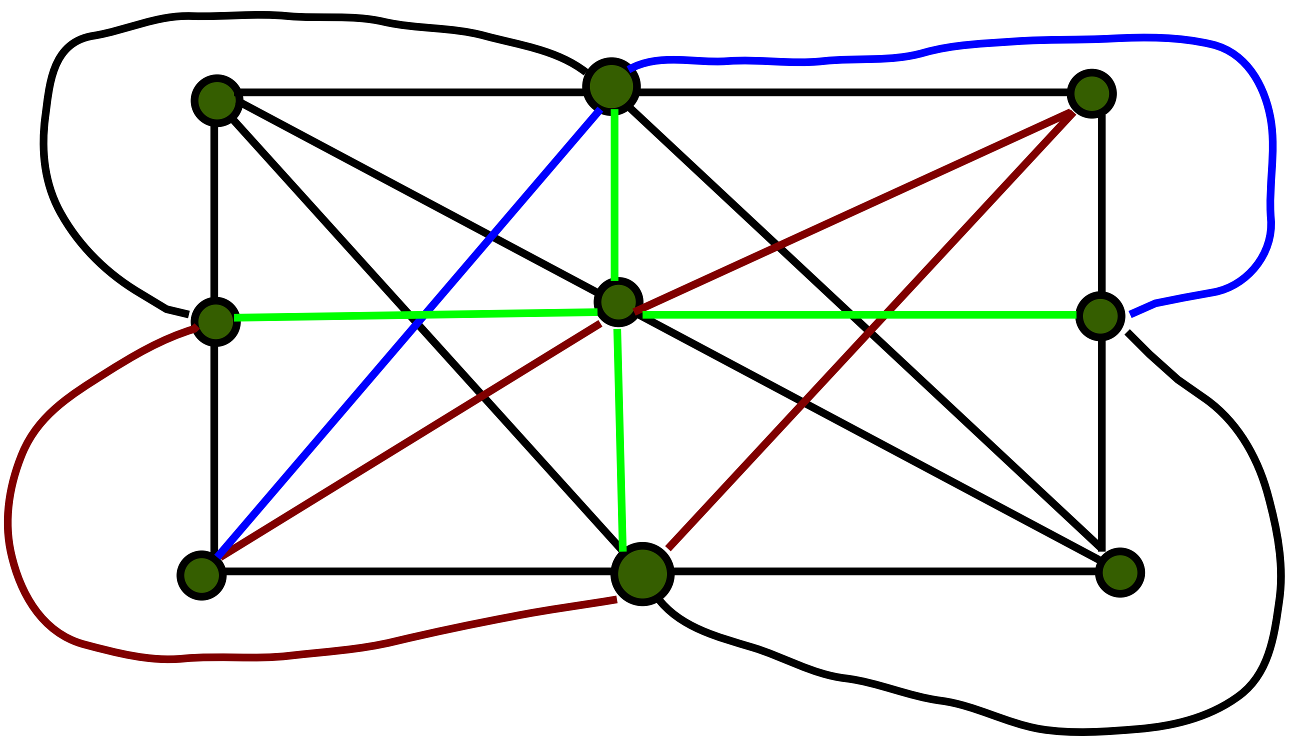
Difference Set Construction

Young's Geometry

1. There exists at least one line.
2. Every line of the geometry has exactly 3 points on it.
3. Not all points of the geometry are on the same line.
4. For two distinct points, there exists exactly one line on both of them.

5'. If a point does not lie on a given line, then there exists exactly one line on that point that does not intersect the given line.

Young's Geometry



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