CS135 Spring 2015 logic guide, version 0

Some laws of propositional logic

Binding power: $\wedge, \vee$ bind more tightly than $\rightarrow$, less tightly than $\neg$. For example, $P \wedge \neg Q \rightarrow P$ means $(P \wedge (\neg Q)) \rightarrow P$.

$\neg(\neg p)$	$\equiv$	p	double negation
$p \wedge p$	$\equiv$	p	idempotent laws
$p \vee p$	$\equiv$	p	
$p \wedge T$	$\equiv$	p	identity elements
$p \vee F$	$\equiv$	p	
$p \wedge F$	$\equiv$	F	zero elements ('domination laws')
$p \vee T$	$\equiv$	T	
$p \wedge q$	$\equiv$	$q \wedge p$	commutativity
$p \vee q$	$\equiv$	$q \vee p$	
$p \vee \neg p$	$\equiv$	T	negation laws (excluded middle and contradiction)
$p \wedge \neg p$	$\equiv$	F	
$p \wedge (q \wedge z)$	$\equiv$	$(p \wedge q) \wedge z$	associativity
$p \vee (q \vee z)$	$\equiv$	$(p \vee q) \vee z$	
$p \vee (p \wedge q)$	$\equiv$	p	absorption
$p \wedge (p \vee q)$	$\equiv$	p	
$p \vee (q \wedge z)$	$\equiv$	$(p \vee q) \wedge (p \vee z)$	distributive laws
$p \wedge (q \vee z)$	$\equiv$	$(p \wedge q) \vee (p \wedge z)$	
$\neg(p \wedge q)$	$\equiv$	$(\neg p) \vee (\neg q)$	De Morgan's laws
$\neg(p \vee q)$	$\equiv$	$(\neg p) \wedge (\neg q)$	
$p \rightarrow q$	$\equiv$	$\neg p \vee q$	definition of $\rightarrow$
$p \leftrightarrow q$	$\equiv$	$(p \rightarrow q) \wedge (q \rightarrow p)$	definition of $\leftrightarrow$

Connection between laws and tautologies:
$p \equiv q$ is a valid law if, and only if, $p \leftrightarrow q$ is a tautology.

Inference rules for propositional logic

Rule	Corresponding tautology	Name of rule
$\dfrac{p \qquad p \rightarrow q}{q}$	$(p \wedge (p \rightarrow q)) \rightarrow q$	Modus ponens
$\dfrac{p \qquad p \leftrightarrow q}{q}$	$(p \wedge (p \leftrightarrow q)) \rightarrow q$	Equivalence rule
$\dfrac{p \qquad q}{p \wedge q}$		Conjunction rule
$\dfrac{p \rightarrow q \qquad q \rightarrow r}{p \rightarrow r}$	(p→q)∧(q→r)→(p→r)	Hypothetical syllogism
$\dfrac{\begin{array}{c}[p]\\ \vdots \\ q\end{array}}{p \rightarrow q}$		Discharge hypothesis
$\dfrac{\forall x\ P(x)}{P(c)}$	$(\forall x\ P(x)) \rightarrow P(c)$	Universal instantiation (c is any expression)
$\dfrac{P(x) \qquad (x \text{ is arbitrary})}{\forall x\ P(x)}$		Universal Generalization
$\dfrac{P(c)}{\exists x\ P(x)}$	$P(c) \rightarrow \exists x\ P(x)$	Existential Generalization
$\dfrac{P(x) \qquad x = e}{P(e)}$		Substitution of equals
$\dfrac{P(0) \qquad \forall n.\ (n \neq 0 \rightarrow (P(n-1) \rightarrow P(n))}{\forall n \in \mathbf{N}\ P(n)))}$		Induction (on naturals)
$\dfrac{P('()) \qquad \forall lst.\ (lst \neq' () \rightarrow (P((\texttt{cdr}\ lst)) \rightarrow P(lst)))}{\forall lst\ P(lst)}$		Induction (on lists)

For Universal Generalization "x is arbitrary" means the proof of $P(x)$ makes no assumptions about x. By comparison, for Existential Generalization you choose an expression c and prove $P(c)$.

When doing a proof in step-by-step style, these rules justify that one line follows from previous lines. For example, here is a proof of $P \to Q \lor P$.

1.	P	assumption
2.	$P \lor Q$	from 1 by Addition rule
3.	$P \lor Q \equiv Q \lor P$	commutativity law for $\lor$ (and Univ. Instantiation!)
4.	$Q \lor P$	from 2 and 3 by Equivalence rule
5.	$P \to Q \lor P$	from 1 and 4 by Discharge hypothesis

A complete proof shouldn't have any assumptions that were not discharged.

When doing a proof in equational style, we are implicitly using one the following rule:

$$\frac{P \equiv Q \qquad Q \equiv R}{P \equiv R}$$

and similarly for equality (=) for numeric expressions etc.