Math Tournament Guide

also available at khaacademy.github.io

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Introduction

Here's a bunch of tournament techniques and knowledge you can use. It's a work in progress as of now. I'll try to update it regularly. Also, feel free to skim around, since you might know a lot of it already.

Enjoy! Sweep!

Derivative Tricks

1 x^n

Differentiating x^n

Know these three facts:

- The nth derivative of x^n is $n!$
- The $(n-1)$th derivative of x^n is $n! \cdot x$
- The $(n+1)$th derivative of x^n is 0

Memorizing this is extremely useful for ciphering/mathbowl and will save time on individual tests. Example:

Example 1

What is the 69th derivative of x^{70}?
Answer: $70! \cdot x$

Example 2

What is the 70th derivative of x^{70}?
Answer: 70!

Example 3

What is the 71th derivative of x^{70}?
Answer: 0

Example 4

What is the 72th derivative of x^{70}?
Answer: 0

Example 5

What is the 89th derivative of x^{70}?
Answer: 0

Notice that for any n greater than $n+1$, the derivative of x^n will also be 0. Keep that in mind.

Practice Problem 1

What is the 30th derivative of x^{31}?

Practice Problem 2

What is the 10th derivative of x^{11}?

Practice Problem 3

What is the 6th derivative of x^5?

Practice Problem 4

What is the 77th derivative of x^{77}?

What about something like 39th derivative of x^{70}
There is an easy trick to this too!

Differentiating x^n less than $(n-1)$ times

Suppose α is a number less than $(n-1)$:

$$\text{The } \alpha\text{th derivative of } x^n \text{ is } n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot (n-\alpha+1) \cdot x^{(n-\alpha)}$$

This may look intimidating at first, but looking at a few examples will make it obvious how simple it is:

Example 1

What is the 50th derivative of x^{70}?
$70 \cdot 69 \cdot 68 \cdot \ldots \cdot (70-50+1) \cdot x^{70-50}$
$= 70 \cdot 69 \cdot 68 \cdot \ldots \cdot 21 \cdot x^{20}$

Example 2

What is the 20th derivative of x^{50}?
$50 \cdot 49 \cdot 48 \cdot \ldots \cdot 31 \cdot x^{30}$

Example 3

What is the 24th derivative of x^{78}?
$78 \cdot 77 \cdot 76 \cdot \ldots \cdot 55 \cdot x^{54}$

Example 4

What is the 19th derivative of x^{44}?
$44 \cdot 43 \cdot 42 \cdot \ldots \cdot 26 \cdot x^{25}$

Example 5

What is the 5th derivative of x^{10}?
$10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot x^5$

Try some out yourself. Finding the pattern is key here. Practice with smaller derivatives if you are having trouble with this.

Practice Problem 1

What is the 30th derivative of x^{42}?

Practice Problem 2

What is the 10th derivative of x^{57}?

Practice Problem 3

What is the 6th derivative of x^{12}?

Practice Problem 4

What is the 77th derivative of x^{99}?

2 $\sin$ & $\cos$ cycles

Differentiating $\sin x$ n times

Divide n by 4, and find the remainder, r. If:

$$r = 0, f^n = \sin x$$
$$r = 1, f^n = \cos x$$
$$r = 2, f^n = -\sin x$$
$$r = 3, f^n = -\cos x$$

Example 1

What is the 64th derivative of $\sin x$?
64/4 leaves remainder 0, so it is $\sin x$

Example 2

What is the 33th derivative of $\sin x$?
33/4 leaves remainder 1, so it is $\cos x$

Example 3

What is the 74th derivative of $\sin x$?
74/4 leaves remainder 2, so it is $-\sin x$

Example 4

What is the 19th derivative of $\sin x$?
19/4 leaves remainder 3, so it is $-\cos x$

Example 5

What is the 25th derivative of $\sin x$?
25/4 leaves remainder 1, so it is $\cos x$

For $\cos x$, it's the same as $\sin x$, but one forward. So, what I usually do is, if asked to do 6th derivative of $\cos x$, I just do 7th derivative of $\sin x$ instead. General case: The nth derivative of $\cos x$ is the $(n+1)$th derivative of $\sin x$. If you would rather memorize the ones for $\cos x$, go ahead and derive them yourself!

Everything About Limits

Here are some limit techniques:

1 Limits by Direct Substitution

Plug in the x-value.
Example:

$$\lim_{x \to 3} (23x) = 69$$

2 Limits with Fractions

Plug x-value, if indeterminate ($\frac{0}{0}, \frac{\infty}{\infty}, \infty - \infty, 0^0, 1^\infty, \infty^0$), use one of following:

L'Hopital's Rule: Differentiate top and bottom. (No quotient rule).
Example:

$$\lim_{x \to 0} \left(\frac{\sin x}{x} \right) \Rightarrow \lim_{x \to 0} \left(\frac{\cos x}{1} \right) \Rightarrow 1$$

Factoring: Find the common factor.
Example:

$$\lim_{x \to 0} \left(\frac{e^x - 1}{e^{2x} - 1} \right) \Rightarrow \lim_{x \to 0} \left(\frac{\cancel{(e^x - 1)}}{\cancel{(e^x - 1)}(e^x + 1)} \right) \Rightarrow \frac{1}{2}$$

Rationalization: You multiply by the conjugate.
Example:

$$\lim_{x \to -1} \left(\frac{x+1}{\sqrt{x+5} - 2} \right) \Rightarrow \lim_{x \to -1} \left(\frac{x+1}{\sqrt{x+5} - 2} \right) \left(\frac{\sqrt{x+5} + 2}{\sqrt{x+5} + 2} \right) \Rightarrow \lim_{x \to -1} \left(\frac{\cancel{(x+1)} \cdot (\sqrt{x+5} + 2)}{\cancel{x+1}} \right) \Rightarrow 4$$

3 Limits Involving Number e

Limit Definition of Derivative

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n}=e$$

$$\lim_{n\to 0}(1+n)^{\frac{1}{n}}=e$$

$$\lim_{x\to\infty}\left(1+\frac{r}{x}\right)^{tx}=e^{rt}$$

Example 1

$$\lim_{x\to\infty}\left(1+\frac{1}{2x}\right)^{3x}=e^{3/2}$$

4 Limits at Infinity

Use dominance (Note: $\sqrt{x^2}=-x$ if approaching $-\infty$) :

Example 1

$$\lim_{n\to\infty}\frac{x^6+18x^3}{2x^2+48x^8}\Rightarrow\lim_{n\to\infty}\frac{x^6}{48x^8}=0$$

Example 2

$$\lim_{n\to-\infty}x-\sqrt{x^2+4}\Rightarrow\lim_{n\to-\infty}x-\sqrt{x^2+4x+4}\Rightarrow$$

$$\lim_{n\to-\infty}x-\sqrt{(x+2)^2}\Rightarrow\lim_{n\to-\infty}x-(-(x+2))\Rightarrow\lim_{n\to-\infty}2x-2\Rightarrow DNE$$

Practice Problem 1

$$\lim_{x\to-\infty}\left(\sqrt{x^2-\frac{4}{3}x+2}+x\right)$$

Answer: 8/3

5 Trigonometric Limits

Know that $\sin x$ and $\cos x$ oscillate between 1 and -1. This is important when taking dominance into account. If a limit ends in oscillation, it is DNE. If the trig fraction results in 0/0 L'hopital's is possible.

$\sin x/x$

$$\lim_{n\to 0} \frac{\sin x}{x} = 1$$

1 - $\cos x/x$

$$\lim_{n\to 0} \frac{1-\cos x}{x} = 0$$

Example 1

$$\lim_{n\to 0} \frac{\sin x}{2x} = \frac{1}{2}$$

Example 2

$$\lim_{n\to 0} \frac{\sin 5x}{x} = 5$$

Example 3

$$\lim_{n\to 0} \frac{\sin \frac{2x}{7}}{x} = \frac{2}{7}$$

Example 4

$$\lim_{n\to \infty} \frac{\sin 2x}{x} = 0$$

Example 5

$$\lim_{n\to \infty} \frac{x}{\sin x} = DNE$$

Example 6

$$\lim_{n\to 0} \frac{\sin 2x}{\sin 5x} = \frac{2}{5}$$

Extra 1

$$\text{Evaluate: } \lim_{x \to 0} tanx = 0$$

Extra 2

$$\lim_{x \to \infty} arctan(x) = \frac{\pi}{2}$$

6 Limit Definition of a Derivative (and the Alternate)

Limit Definition of Derivative

$$\text{Regular} \Rightarrow \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = f'(x)$$

$$\text{Alternate} \Rightarrow \lim_{x \to \alpha} \frac{f(x) - f(\alpha)}{x - \alpha} = f'(\alpha)$$

$$\lim_{x \to \alpha} f(x) \text{ means that } \lim_{x \to \alpha^+} f(x) \text{ and } \lim_{x \to \alpha^-} f(x)$$

Example 1

$$\lim_{h \to 0} \frac{f'(x+h) - f'(x)}{h} = f''(x)$$

Example 2

$$\lim_{x \to \alpha} \frac{\sin(x) - \sin(\alpha)}{x - \alpha} = \cos \alpha$$

Example 3

Evaluate $\lim_{x \to 5} \frac{x+5}{x-5}$.
Solution:

$$\text{DNE, since } \lim_{x \to 5^+} \frac{x+5}{x-5} = \infty \text{ and } \lim_{x \to 5^-} \frac{x+5}{x-5} = -\infty \text{ are not equal}$$

Example 4 (Really Good Problem)

Find α such that $\lim_{x\to 0}\frac{tan(\alpha x+\pi/4)-1}{x}=4$
Solution:

$$\lim_{x\to 0}\frac{tan(\alpha x+\pi/4)-1}{x}\Rightarrow \lim_{x\to 0}\alpha\cdot\frac{tan(\alpha x+\pi/4)-1}{\alpha x}\Rightarrow \alpha f'(\pi/4)\Rightarrow \alpha\sec^2(\pi/4)\Rightarrow \alpha\cdot 2$$

Therefore, $\alpha\cdot 2=4$, so $\alpha=2$

Example 5 (Another Really Good Problem)

Find $\lim_{x\to 2}\frac{f(x)-f(2)}{\sqrt{x}-\sqrt{2}}$
Solution:

$$\lim_{x\to 2}\frac{f(x)-f(2)}{\sqrt{x}-\sqrt{2}}\cdot\frac{\sqrt{x}+\sqrt{2}}{\sqrt{x}+\sqrt{2}}\Rightarrow \lim_{x\to 2}\frac{f(x)-f(2)}{x-2}\cdot(\sqrt{x}+\sqrt{2})\Rightarrow f'(2)\cdot(\sqrt{2}+\sqrt{2})$$

$$=2\sqrt{2}f'(2)$$

Note: for limit definition of integral, go to Various Forgettable Topics section

7 Change of Limit

If all nothing else works, this is your last technique to rely on. The trigger to use this is if you see a fraction, typically $\frac{1}{x}$.
Know that:

$$\text{If } u=\frac{1}{x}, \lim_{x\to\infty}=\lim_{u\to 0}$$

Example 1

$$\lim_{x\to\infty}x\sin(\frac{1}{x})\Rightarrow \lim_{u\to 0}\frac{\sin u}{u}=1$$

Ciphering & Mathbowl Tricks

Gotta go fast. Use these tricks when applicable and if you're comfortable with them. Here:

Solving answer while saying it - ex: $f(x) = x\cos x$, find $f'(x)$
you can raise your hand immediately and solve it without writing it down. (mathbowl trick)

Skipping derivatives - ex: $f(x) = 3x^2 + 2x + 1$, find $f''(x)$
don't bother with first derivative, go straight to second (use the x^n derivative trick)
you can also use this for inflection point questions

Circle - ex: Evaluate $\int_{-1}^{1} \sqrt{1 - x^2}dx$
If you see $\int \sqrt{k - x^2}$, it is the same as finding the area of a circle (remember circle is $x^2+y^2 = r^2$ and rewritten is $y = \sqrt{r^2 - x^2}$). find radius and plug into area formula (make sure to look at bounds, it could be a half/quarter of the circle)

Shorthand notation - write less, use y' instead of $\frac{dy}{dx}$. save like 1-2 secs. maybe even make your own for some (like maybe t(x) instead of tan(x))

Various Forgettable Topics

Linear Approximation (Differentials)

Practice: http://tutorial.math.lamar.edu/Problems/CalcI/Differentials.aspx
Ex: Compute the differential for $y = x^2$ as x changes from 1 to 1.01
Answer: $\frac{dy}{dx} = 2x \Rightarrow dy = 2xdx \Rightarrow dy = 2(1)(.01) \Rightarrow dy = 0.02$
You just compute derivative, but multiply by dx at the end. Plug in "integer" value of x (i.e. x = 1) for x, and Δx for dx.

Inverse Functions $(f^{-1}(x))$

Practice: http://tutorial.math.lamar.edu/Problems/CalcI/InverseFunctions.aspx
Find the inverse function by swapping x and y.
If it is not easy to differentiate, do the following:
Let $g(x) = f^{-1}(x)$ and $g(f(x)) = x$. Differentiate and solve.

Average Value vs. Average Rate of Change

Practice (Value): http://tutorial.math.lamar.edu/Problems/CalcI/AvgFcnValue.aspx
Practice (ROC): https://www.khanacademy.org/math/algebra/algebra-functions/average-rate-of-change-word-problems/a/average-rate-of-change-review
Average Value is $\frac{\int_a^b f(x)dx}{b-a}$. Average ROC is $\frac{f(b)-f(a)}{b-a}$.

Cross-sections

Practice: https://www.khanacademy.org/math/ap-calculus-ab/ab-applications-of-integration-new/ab-8-7/e/volumes-with-square-and-rectangle-cross-sections
Volume with cross-sections: $\int_a^b A(x)dx$, where $A(x)$ is the area of cross-section

Inverse Trigonometric Functions

Practice (Diff.): https://www.math.ucdavis.edu/~kouba/CalcOneDIRECTORY/invtrigderivdirectory/InvTrigDeriv.html
Practice (Int.): https://www.intmath.com/methods-integration/6-integration-inverse-trigonometric-forms.php
You can derive with triangle or implicit differentiation, or memorize them.

Logarithmic/Exponential Differentiation

Practice (Logs): https://www.intmath.com/differentiation-transcendental/5-derivative-logarithm.php
Practice (Expo): https://www.math.ucdavis.edu/~kouba/CalcOneDIRECTORY/logdiffdirectory/LogDiff.html
All you really have to know is derivative of $\log_\alpha x$ is $\frac{1}{x \cdot ln\alpha}$ and derivative of α^x is $\alpha^x \cdot \ln \alpha$
This can be easily derived from knowing $\log_\alpha x = \frac{lnx}{ln\alpha}$ and $\alpha^x = e^{\ln \alpha \cdot x}$
Quick Examples:
Differentiating $\log_{\sin x} x$ is the same as differentiating $\frac{\ln x}{\ln \sin x}$, which you know how to do.
Differentiating $x^{\sin x}$ is the same as differentiating $e^{\ln x \cdot \sin x}$, which you know how to do.

Limit Definition of Integral
Practice (Probs #9-13): https://www.math.ucdavis.edu/~kouba/CalcTwoDIRECTORY/defintdirectory/DefInt.html
Basically, $\lim_{n\to\infty}\sum_{i=1}^{\infty} f(c_i)\Delta x_i = \int_a^b f(x)dx$, where $\Delta x_i = \frac{b-a}{n}$ and $c_i = a + \frac{b-a}{n} \cdot i$
Quick Example:

$$\lim_{n\to\infty}\sum_{i=1}^{\infty}(2 + \frac{4i}{n})^2 \cdot \frac{4}{n} = \int_2^6 x^2 dx$$

Hard Example:

$$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{i}{i^2+n^2} = \lim_{n\to\infty}\sum_{i=1}^{n}\frac{\frac{i^2}{n^2}}{\frac{i^2}{n^2}+\frac{n^2}{n^2}} = \lim_{n\to\infty}\sum_{i=1}^{n}\frac{\frac{i}{n}}{(\frac{i}{n})^2+1}\cdot\frac{1}{n} = \int_0^1 \frac{x}{x^2+1}dx$$

Shell Method
Practice: https://www.khanacademy.org/math/old-ap-calculus-ab/ab-applications-definite-integrals/ab-shell-method/a/stacy-scaling-3
Difficult to explain through text alone. I recommend looking at Khan Academy's video on shell method. This is a good alternative to ring/washer method sometimes when working with difficult solids of revolution.

Obscure Topics from Calculus

The chances these topics will show up on locals is little to none. This is just for if you're extra.

Euler's Method

Practice: https://www.khanacademy.org/math/ap-calculus-bc/bc-differential-equations-new/bc-7-5/e/euler-s-method
You make a chart with x, y, and dy/dx on the top. It's difficult to explain with just text. Watching the Khan Academy video and doing some problems is the easiest way to learn it.

Propagated Error

Practice (lol): https://www.youtube.com/watch?v=R7t9Iv_lTc8
The same thing as differentials, except you plug in the error value into dx (or whatever variable you are with respect to).
Example: There is a square with side s = 12 inches, with a possible error of 1/64 inches. Calculate the propagated error in computing the area of the square.
$A = s^2$ (area of square)
$\frac{dA}{ds} = 2s$
$dA = 2sds$
$dA = 2 \cdot 12 \cdot \pm 1/64 \Rightarrow \pm\frac{3}{8}\text{in.}^2$

Marginal Cost

Practice: http://tutorial.math.lamar.edu/Problems/CalcI/BusinessApps.aspx
Marginal cost is a joke. It's basically find the derivative at x = some constant.
Example: A company's production cost when producing x toys is represented by $C(x) = x^2$. What is the marginal cost when x=2?
Answer: Evaluate $C'(2) = 4$

Epsilon-Delta Definition of Limit

Practice: https://brilliant.org/wiki/epsilon-delta-definition-of-a-limit/
This one is easy, but hard to explain. Solving some problems is usually the best way to learn this. ACTM (from the resources file) has quite a few of these.