The boy's guide to pricing and hedging

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There is an unfortunate strain of pedantry running through the teaching of quantitative finance, one involving an excess of abstraction, formality, rigour and axiomatisation that makes the subject unnecessarily daunting and difficult. Over the years I've seen a few too many fresh graduates who, on being asked why one believes one can obtain a credible value for an option, reply that it's because of Girsanov's theorem.

I'd like to think most of the useful aspects of quantitative finance are relatively simple, and so, here is my rather abbreviated poor man's guide to the field.

Price and value

You must first distinguish between price and value. Price is what you pay to acquire a security; value is what it is worth. The price is fair when it is equal to the value.

But what is the value? How do you estimate it? Judging value, in even the simplest way, involves the construction of a model or theory.

The one and only commandment of quantitative finance

According to legend, Hillel, a famous sage, was asked to recite the essence of God's laws while standing on one leg. "Do not do unto others as you would not have them do unto you," he is supposed to have said. "All the rest is commentary. Go and learn."

You can summarise the essence of quantitative finance on one leg too: "If you want to know the value of a security, use the price of another security that's as similar to it as possible. All the rest is modelling. Go and build."

The wonderful thing about this law, when compared with almost everything else in economics, is that it dispenses with utility functions, those unobservable hidden variables whose ghostly presences permeate economic theory. But don't think you can escape all human perceptions by using this law; the models of quantitative finance inevitably involve expectations and estimates of future behaviour, and those estimates and expectations are people's estimates.

Financial economists refer to their essential principle as the law of one price, or the principle of no risk-free arbitrage, which states that: 'Any two securities with identical future payouts, no matter how the future turns out, should have identical current prices.'

The law of one price is not a law of nature. It's a general reflection on the practices of human beings, who, when they have enough time and information, will grab a bargain when they see one. This law usually holds in the long run, in well-oiled markets with enough savvy participants, but there are always short-term or even longer-term exceptions that persist.

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Valuation by replication

How do you use the law of one price to determine value? If you want to estimate the value of a target security, the law of one price tells you to find some other replicating portfolio, a collection of more liquid securities that, collectively, has the same future payouts as the target, no matter how the future turns out. The target's value is then simply the price of the replicating portfolio.

Where do models enter? It takes a model to show that the target and the replicating portfolio have identical future payouts under all circumstances. To demonstrate the identity of payouts, you must specify what you mean by 'all circumstances' for each security, and find a strategy for creating a replicating portfolio that, in each future scenario, will have identical payouts to those of the target.

Most of the mathematical complexity in finance involves the description of the range of future behaviour of each security's price.

There are two kinds of replication, static and dynamic. A static replicating portfolio is a collection of securities that, once defined and never altered, reproduces the payout of a target security under all future scenarios. Static replication is the simplest and most comprehensive method of valuation, but is feasible only in the rare cases when the target security closely resembles the liquid securities available.

For more complex non-linear securities, such as stock options, static replicating portfolios are unavailable, but sometimes you can find a dynamic replicating portfolio, a mixture of liquid securities whose proportions, continually adjusted by trading out of one security and into another, will replicate the payout of the target. Static or dynamic, the initial price of the replicating portfolio is the estimated value of the target.

Models are only models, toy-like descriptions of idealised worlds. Simple models envisage a simple future; more sophisticated models incorporate a more complex set of future scenarios that can better approximate actual markets. But no mathematical model will capture the intricacies of human psychology. If you listen to the models' siren song for too long, you may end up on the rocks or in the whirlpool.

We now proceed to apply replication

and the law of one price to various securities of interest.

Modelling (relatively) risk-free bonds

How do you value the promise of a future payment? The simplest answer is: find an institution whose liquid bonds have an estimated risk of default similar to your debtor's and use that institution's term structure of interest rates to discount the payouts of your target bond.

Why use interest rates? Because they allow you to compare investments. Quoting bond prices in terms of rates or yields is a sort of model too, albeit a simple one that projects bonds of different coupons and maturities on to a one-dimensional yield scale that's more enlightening than mere dollar price. The most easily embraceable models convert inexpressive prices into a more eloquent one- or two-dimensional scale that makes comparison more intuitive.

Complex bonds, mortgages or swaptions, for example, whose payouts are contingent on interest rates or other market parameters, are best modelled using dynamically replicating portfolios, as described later.

The risk of stocks

A stock's most germane characteristic is the uncertainty of its return. About the most elementary model of uncertainty is the risk involved in flipping a coin. Figure 1 illustrates a similarly simple model – the binary tree – for the evolution of the price of a stock with return volatility σ over a small instant of time Δt. On an up-move the price increases by the percentage:

$$\mu\Delta t + \sigma\sqrt{\Delta t}$$

while on an equally probable down-move it increases by only:

$$\mu\Delta t - \sigma\sqrt{\Delta t}$$

The volatility σ is the measure of the stock's risk.

According to the law of one price, the risk-free rate of return r must lie in the zone between the up and down returns. If both the up and down returns were greater than the risk-free return, you could create a portfolio long \$100 of stock and short \$100 of a risk-free bond with zero price and a paradoxically positive payout under all future scenarios. Any model with such possibilities is in trouble before it leaves the ground.

This apparently naive 'either-up-or-down' model captures much of the inherent risk of owning a stock and many other risky securities. Repeated over and over again for small time steps, it mimics the more-or-less continuous motion of prices in a reasonable though imperfect way, much as movies produce the illusion of life by changing images at the rate of 24 frames a second.

Risk reduction by adding risk-free bonds implies 'more risk, more return'

What future expected reward justifies a particular present risk? This is the paramount question of life and finance. If you know the future volaility of a stock σ, what rate of return μ should you expect?

The law of one price tells you how to value securities you can replicate, but some payouts simply cannot be replicated. Certain risk factors are intrinsic, unavoidable, the thing in itself. We need to extend the law of one price – same payout, same return – to demand that the same unavoidable risk should lead to the same expected return, or, more precisely, that risk factors with equal risk should have equal expected return. We can use this principle to determine the rational relationship between risk and return.

The key point is that, by adding a risk-free short-term investment in a bond to any risky position, one can reduce the magnitude of the risk and return in a predictable manner while preserving the risk's character.

Figure 2 shows the binary tree for a risk-free bond. It is degenerate: whether the stock moves up or down, the bond produces a guaranteed return $r\Delta t$. By adding a risk-free bond of this type with zero volatility to a stock of volatility σ and return μ, you can commensurately reduce both the risk and return of your investment. The binary tree for a 50:50 mix of stock and cash suffers half the volatility σ and produces half the expected excess return $(\mu - r)$ over the risk-free rate. The principle of equal-unavoidable-risk-equal-expected-return then dictates that any security with half the risk produces half the excess return, or, even more generally, that excess return is proportional to risk, so that:

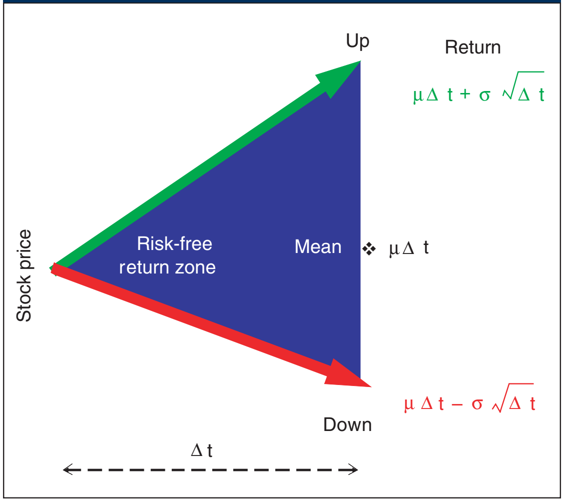
$$\mu - r = \lambda\sigma$$

where λ is the Sharpe ratio.

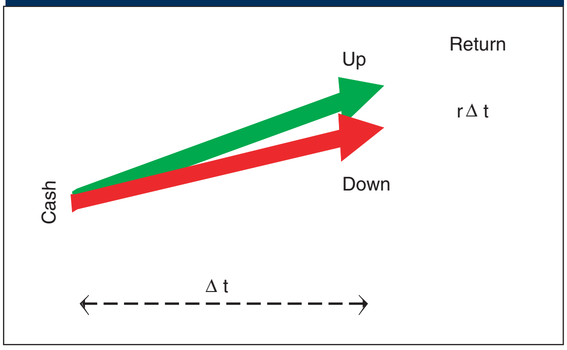
Risk reduction by diversification implies the Sharpe ratio is zero

We have shown that since you can reduce risk by keeping part of your money in risk-free bonds, it follows that excess return is proportional to risk. But you can also diminish risk by diversifying. If you can accumulate a portfolio of so many uncorrelated unavoidable risks that their

1. Evolution of a stock's price



2. Evolution of a risk-free bond's price



risks cancel, so that the portfolio's net volatility is zero, then, by the law of one price, since it is risk-free, it must produce the risk-free rate of return. In that case, each stock in the portfolio must earn the risk-free rate too, so that $\mu = r$ and the Sharpe ratio λ must be zero.

If all risk factors can be cancelled by diversification, investors should expect only the risk-free return on any single stock.

Risk reduction by hedging implies only factor risk is rewarded

Stock markets aren't truly amenable to the total diversification we assumed above. That's because large financial markets are more than a collection of individual uncorrelated risk factors. Experienced investors are always trying to detect patterns in the universe of stock returns. They perceive stocks as belonging to groups that have in common their sensitivity to a particular asset, factor or set of factors.

Let ρ be the correlation of the returns

3. A stock is straight, an option curved

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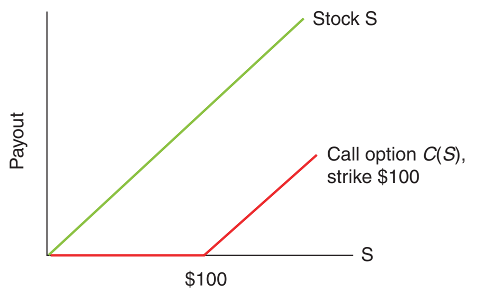
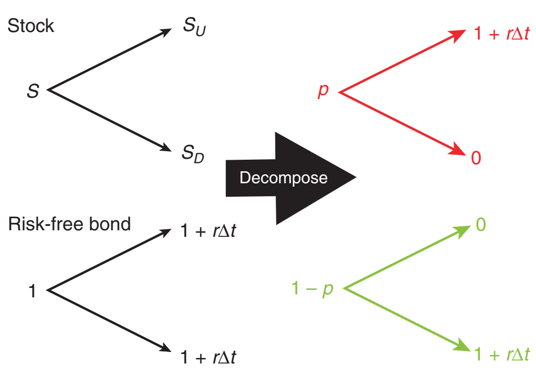
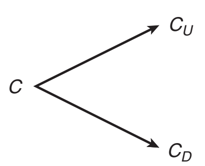


Binary price trees for a stock S and a $1 investment in a risk-free bond. The stock and bond can be decomposed into a security p that pays out only in the up state and a security $1 - p$ that pays out only in the down state



You can replicate an option's non-linear payout over each instant by suitable investments in the elemental securities p and $1 - p$



$$(1 + r\Delta t)C = (C_U) \times p + (C_D) \times (1 - p)$$

between a stock S with volatility σ and some tradable risky asset M with volatility σ_M. Because of the correlation between each stock's return and that of M, you can hedge the M-related risk of any stock. If you have $100 invested in a stock S, you can short β times as many dollars of the factor M against it, where $\beta = \rho(\sigma/\sigma_M)$ is the number of percentage points that stock S is expected to rise when M rises by 1%. This factor-hedged portfolio, consisting of a $100 long position in the stock combined with β times as much of a short position in M, will carry no net exposure to the price of M, because any increase in the price of the stock will, on average, be cancelled by a correlated decrease in value of the short position in M.

The net expected return on this factor-hedged M-neutral portfolio is proportional to the return of the stock S less β times the return of M, namely $\mu - \beta\mu_M$. Assuming there are no other factors influencing the stock S, this residual risk is unavoidable.

If you can diversify over a large enough M-neutral portfolio of stocks so that their accumulated unavoidable risk cancels, then this M-neutral portfolio of zero volatility must earn the risk-free rate r. The same must therefore be true of each M-neutral element of the portfolio. This leads to the result that:

$$(\mu - r) = \beta(\mu_M - r)$$

This is the result of the capital asset pricing model or arbitrage pricing theory: in a world of rational investors, the excess return you can expect from buying a stock is its β times the expected return of its hedgeable factor. Put differently, you can only expect to be rewarded for the unavoidable factor risk of each stock, since all other risk can be eliminated by diversification.

The rationality of the investing world is still hotly debated. In finance – a social science overlaid with a useful veneer of quantitative analysis – the exact truth is hard to determine.

The choice-of-currency trick

It seems obvious that the value of a security in dollars should be independent of the currency (dollars, yen, shares of IBM, etc) you choose for modelling its evolution. A little thought often suggests a natural choice of currency that can greatly simplify thinking about a problem. Just as stock analysts find it convenient to measure a stock's price in units of projected earnings rather than dollars, thereby making stock comparison easier, so financial modellers can sometimes cleverly choose a more meaningful currency than the dollar when valuing a complex security. Convertible bonds, for example, which involve an option to exchange a bond for stock, can sometimes be fruitfully modelled by choosing a bond itself as the natural valuation currency. Financial mathematicians call this trick 'the choice of numeraire'.

Derivatives are not independent securities

A derivative is a contract whose value is determined by the non-linear relationship between its payout and the value of some other, simpler security called its underlyer. The most interesting characteristic of a derivative is the curvature of its payout $C(S)$, as illustrated for a simple call option in figure 3.

If you know the price of a simple stock, whose payout is linear, what is the value of the non-linear option?

In the binary tree model, there are only two possible states (up and down) for the stock or the option after a time Δt. These states are spanned by two independent securities, the stock itself and a risk-free bond. You can decompose the stock and bond into a basis of two more elemental securities – p and $1 - p$ – each respectively paying out in only one of the final states, as shown in figure 3. By creating a portfolio that is a suitable mixture of these two securities, you can instantaneously replicate the payout of any non-linear function $C(S)$ over the next instant of time Δt, no matter into which state the stock evolves.

The value of the option is therefore the price of the mixture of stock and bond that replicates it. By choosing Δt to be infinitesimally small, you can repeat this replication strategy instant after instant, so that, if the final payout $C(S)$ is known, its dynamic replication at all earlier times is determined. The value of the replicating portfolio depends on the size of the stock price movements in the stock's binary tree – that is, on the stock's volatility σ. If you know the future volatility, you can synthesise an option out of stock and bonds.

The same strategy of dynamic replication can be extended to more complex and realistic models of stock evolution, as well as to the replication of derivatives on all sorts of other underliers, as long as there are enough independent underliers to span all future states.

Conclusion

Axiomatisation is helpful when axioms hold true to a high degree of accuracy. But, as Paul Wilmott aptly expressed it, "every financial axiom ... ever seen is demonstrably wrong". Therefore, quantitative finance practitioners need to know where best to spend their effort in the grittily messy world they inhabit. In preparing for that world it's best to start with the concrete and then proceed to the abstract. ■

[1] *Wilmott, Paul, in the prologue to* Derivatives, *John Wiley & Sons, 1998*