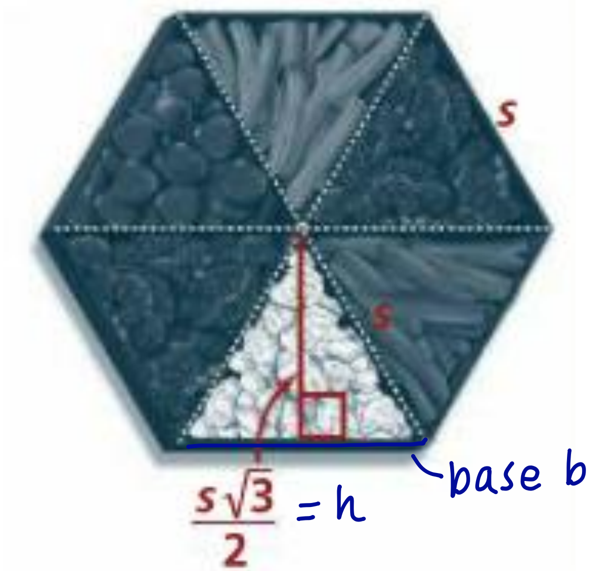
Unit 6A ~ Lesson 1:
Special Right Triangles and the Unit Circle

Objectives:

- I can label and identify angles on the unit circle in both degrees and radians.
- I can find the exact value of sine, cosine or tangent of angles on the unit circle.

A hexagonal tray of vegetables has an area of 450 square centimeters. What is the length of each side of the hexagon?

each side is "s" cm. Long



•What is the area of the triangle in terms of the side length?

$$A = \frac{1}{2} b \cdot h = \frac{1}{2}(s)\left(\frac{\sqrt{3}}{2}s\right) = \frac{\sqrt{3}}{4}s^2$$

•How can you use the diagram at the right to find the formula for the area of the hexagon?

There are 6 triangles so Area of the hexagon = 6 times the area of the triangle.

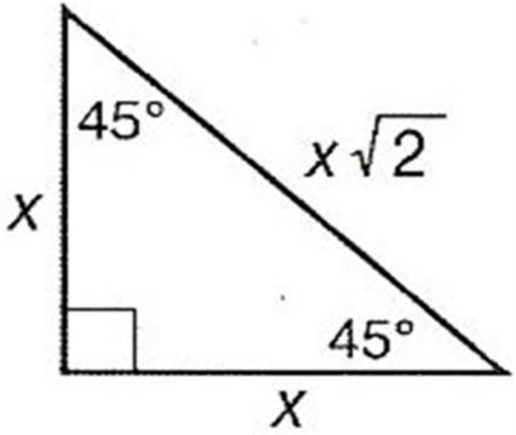
$$\text{Area} = 6\left(\frac{\sqrt{3}}{4}s^2\right) = \frac{3\sqrt{3}}{2}s^2$$

$$2 \cdot 450 = \frac{3\sqrt{3}}{2}s^2 \cdot 2 \rightarrow \frac{900}{3\sqrt{3}} = \frac{3\sqrt{3}s^2}{3\sqrt{3}} \rightarrow \sqrt{s^2} = \sqrt{\frac{300}{\sqrt{3}}} \rightarrow s = \sqrt{\frac{300}{\sqrt{3}}} = \boxed{13.16 \text{ cm}}$$

The previous problem is an example of a "Special Right Triangle". These are triangles that always have the same proportions. Our two special right triangles are the 45-45-90 triangle, and the 30-60-90 triangle.

Let's start with the 45-45-90 triangle:

If the base angles are congruent, then the legs are congruent. Given congruent sides, our triangle will always have the same ratio of sides.



How did we figure out the side lengths?

By using the Pythagorean Theorem.

What if we know the length of the hypotenuse and we want to know the length of the legs?

If the hypotenuse is $\sqrt{2}$ times Longer than the sides, divide the hypotenuse by $\sqrt{2}$ to find the side measures.

Say we have a 45-45-90 triangle with a hypotenuse length of 1. Find the lengths of the legs.

side $= x$

hypotenuse $= x\sqrt{2}$

$$\frac{1}{\sqrt{2}} = \frac{x\sqrt{2}}{\sqrt{2}}$$

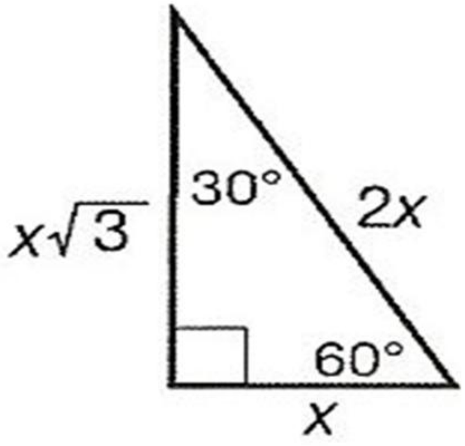
$$x = \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}}$$ ← rationalize the denominator!

$$x = \frac{\sqrt{2}}{\sqrt{4}}$$

$$\boxed{x = \frac{\sqrt{2}}{2}}$$

The sides will be equal to the $\frac{\sqrt{2}}{2}$.

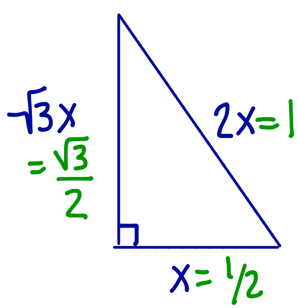
Let's look at the 30-60-90 triangle now. Think back to the vegetable tray problem. We solved that problem by using the Pythagorean Theorem on ½ of an equilateral triangle. That gave us a 30-60-90 triangle. Just like with the 45-45-90 triangle, we always have the same ratio of side lengths. If we know the length of the shortest side, we can find the length of the other sides.



What if we know the length of the hypotenuse, and we want to know the lengths of the long side and of the short side?

We can divide the hypotenuse by 2 to find the short side, then multiply that answer by $\sqrt{3}$ to find the long side.

Say we have a 30-60-90 triangle with a hypotenuse length of 1. Find the lengths of the legs.



$$\frac{2x}{2} = \frac{1}{2}$$

$$\left. x = \frac{1}{2} \right\}$$ short side

$$\text{Long side} = \sqrt{3} \cdot \text{shortside}$$

$$\text{Long side} = \sqrt{3} \cdot 1/2 = \frac{\sqrt{3}}{2}$$

The Unit Circle is a circle with a radius of 1, that is centered at the origin on the coordinate plane.

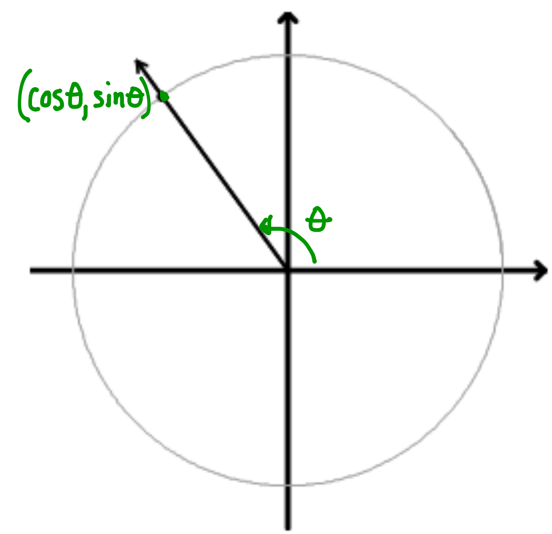
Angles in Standard position always begin on the positive x-axis. Positive angles go in a counter-clockwise direction from the x-axis, and negative angles go in a clockwise direction from the x-axis.

One of the ways we use the unit circle is for finding the exact measure of trigonometric values for common angle measures (both in degrees and in radians). The rays that form the sides of angles intersect our unit circle in specific coordinate points, and these points give us the value of the cosine of the angle, and the sine of the angle.

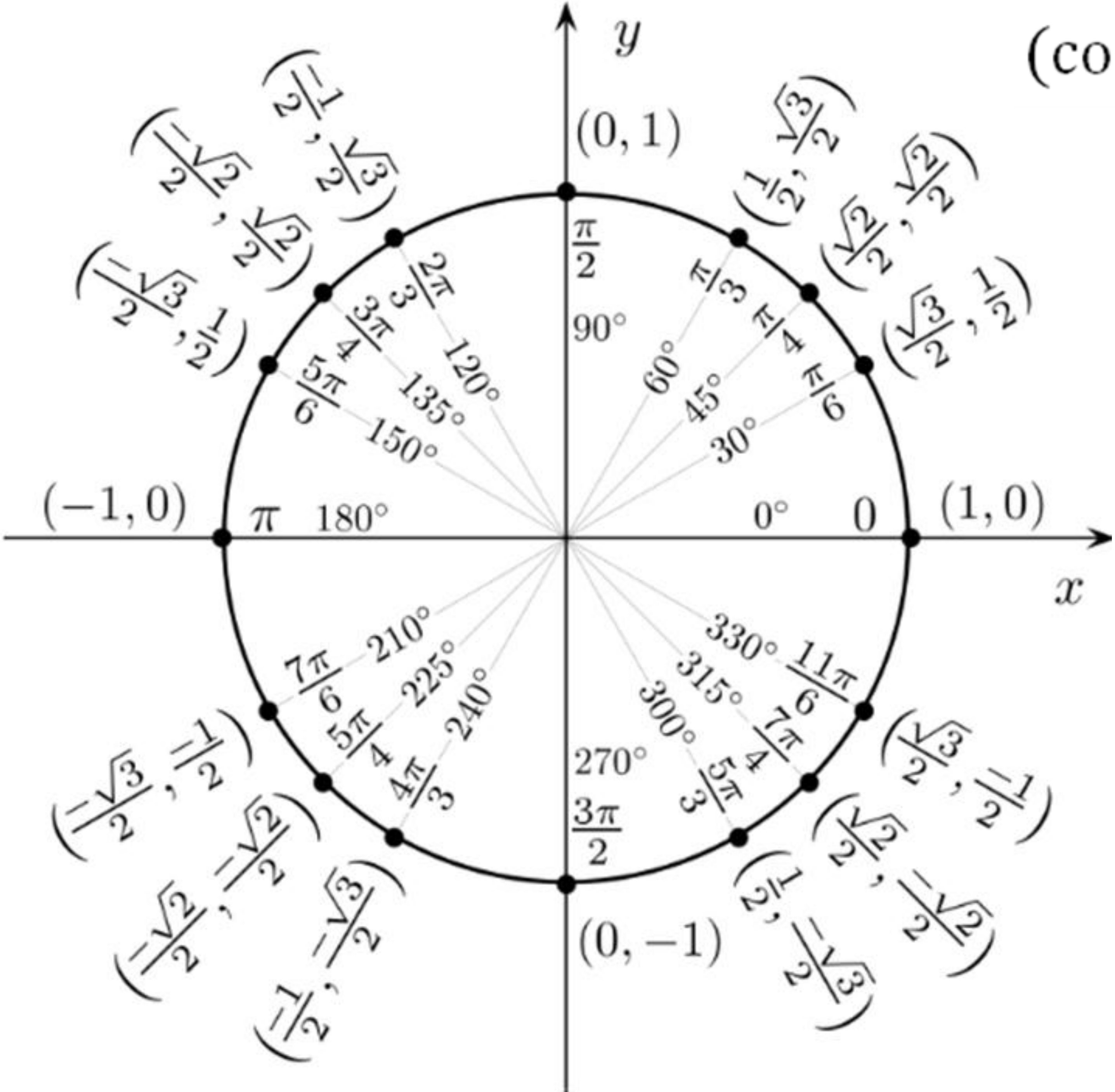
When an angle is in standard position, you can use points on the unit circle to find the trigonometric ratios for the angle.

<u>**Points (x, y) on the unit circle**</u>:

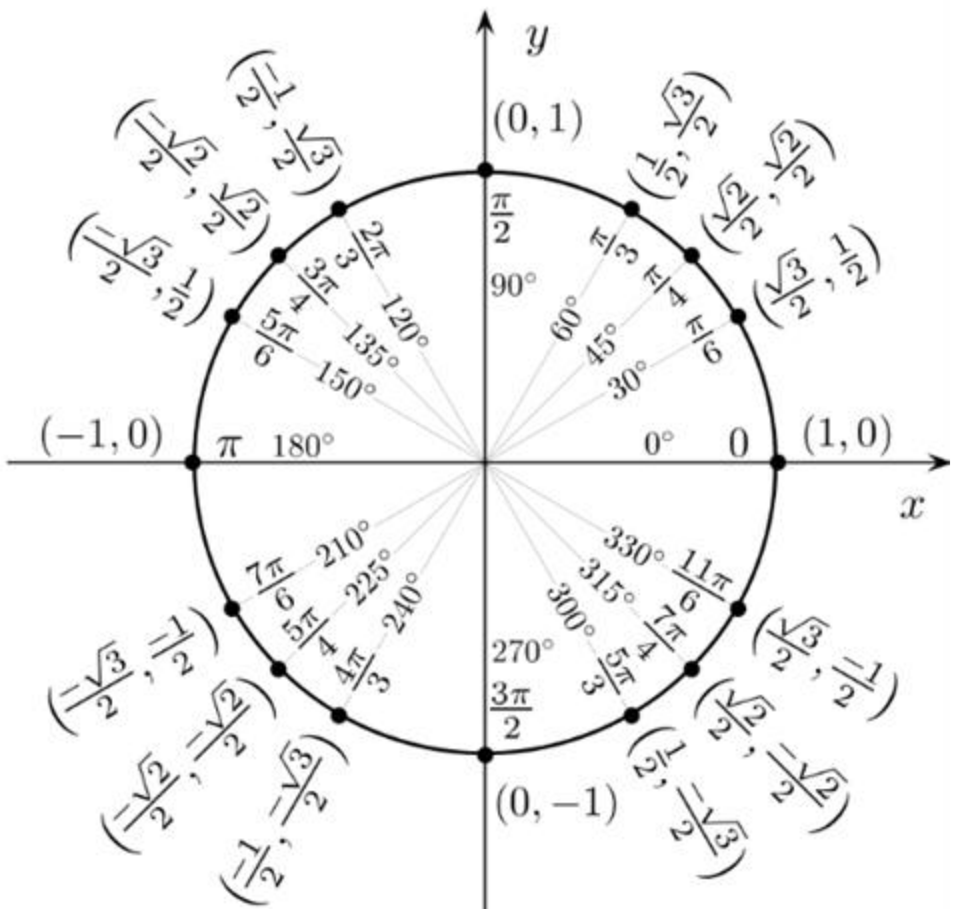
$\sin\theta = y \quad \cos\theta = x \quad \tan\theta = \dfrac{y}{x}$



$(\cos\theta, \sin\theta)$



$$(\cos\theta, \sin\theta) \qquad \tan(\theta) = \frac{y}{x}$$



$$\tan\left(\frac{\pi}{6}\right) = \frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}} = \left(\frac{1}{2}\right)\left(\frac{2}{\sqrt{3}}\right) = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

$$\tan\left(\frac{\pi}{4}\right) = \frac{\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}} = \left(\frac{\sqrt{2}}{2}\right)\left(\frac{2}{\sqrt{2}}\right) = 1$$

$$\tan\left(\frac{\pi}{3}\right) = \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} = \left(\frac{\sqrt{3}}{2}\right)\left(\frac{2}{1}\right) = \sqrt{3}$$

Examples: Find the exact value of each trigonometric function using the unit circle diagram.

A. $\cos 45° = \boxed{\dfrac{\sqrt{2}}{2}}$

x-value at 45°

B. $\sin \dfrac{2\pi}{3} = \boxed{\dfrac{\sqrt{3}}{2}}$

y-value at $\frac{2\pi}{3}$

C. $\tan 240° = \boxed{\sqrt{3}}$

$= \dfrac{\sin 240°}{\cos 240°} = \dfrac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} = \dfrac{\sqrt{3}}{\cancel{2}} \cdot \dfrac{\cancel{2}}{1} = \sqrt{3}$

D. $\sin \dfrac{\pi}{2} = \boxed{1}$

y-value

E. $\cos \dfrac{\pi}{2} = \boxed{0}$

x-value

F. $\tan \dfrac{\pi}{2} = \dfrac{\sin \pi/2}{\cos \pi/2} = \dfrac{1}{0} = \boxed{\text{undefined}}$

Examples: Find the exact value of each trigonometric function using the unit circle diagram.

G. $\tan 315° = \dfrac{\sqrt{2}/2}{\sqrt{2}/2} = \boxed{1}$

H. $\sin(-30°) = \boxed{\dfrac{-1}{2}}$

$= \sin(330°) =$

I. $\cos \dfrac{11\pi}{4} = \cos\left(\dfrac{3\pi}{4}\right) = \boxed{\dfrac{-\sqrt{2}}{2}}$

Examples: Find each angle, theta. There may be more than one answer.

J. $\sin\theta = -\frac{\sqrt{2}}{2}$ $\theta = 225°$ and $315°$ or $\frac{5\pi}{4}$ and $\frac{7\pi}{4}$ radians

K. $\sin\theta = -1$ $\theta = 270°$ or $\frac{3\pi}{2}$ radians

L. $\cos\theta = \frac{\sqrt{3}}{2}$ $\theta = 30°$ and $330°$ or $\frac{\pi}{6}$ and $\frac{11\pi}{6}$

M. $\tan\theta = \frac{\sqrt{3}}{3}$ $\theta = 30°$ and $210°$ or $\frac{\pi}{6}$ and $\frac{7\pi}{6}$

Homework:

Memorize the patterns on the Unit Circle. Remember – If you know how to construct the first quadrant, all of them follow from there.