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Elements of econometrics

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Undergraduate study in
Economics, Management, Finance and the Social Sciences

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LSE

THE LONDON SCHOOL OF ECONOMICS AND POLITICAL SCIENCE

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Preface

0.1 Introduction

0.2 What is econometrics, and why study it?

Econometrics is the application of statistical methods to the quantification and critical assessment of hypothetical economic relationships using data. It is with the aid of econometrics that we discriminate between competing economic theories and put numerical clothing onto the successful ones. Econometric analysis may be motivated by a simple desire to improve our understanding of how the economy works, at either the microeconomic or macroeconomic level, but more often it is undertaken with a specific objective in mind. In the private sector, the financial benefits that accrue from a sophisticated understanding of relevant markets and an ability to predict change may be the driving factor. In the public sector, the impetus may come from an awareness that evidence-based policy initiatives are likely to have the greatest impact.

It is now generally recognised that nearly all professional economists, not just those actually working with data, should have a basic understanding of econometrics. There are two major benefits. One is that it facilitates communication between econometricians and the users of their work. The other is the development of the ability to obtain a perspective on econometric work and to make a critical evaluation of it. Econometric work is more robust in some contexts than in others. Experience with the practice of econometrics and a knowledge of the potential problems that can arise are essential for developing an instinct for judging how much confidence should be placed on the findings of a particular study.

Such is the importance of econometrics that, in common with intermediate macroeconomics and microeconomics, an introductory course forms part of the core of any serious undergraduate degree in economics and is a prerequisite for admission to a serious Master's level course in economics or finance.

0.3 Aims

The aim of **EC2020 Elements of econometrics** is to give you an opportunity to develop an understanding of econometrics to a standard that will equip you to understand and evaluate most applied analysis of cross-sectional data and to be able to undertake such analysis yourself. The restriction to cross-sectional data (data raised at one moment in time, often through a survey of households, individuals, or enterprises) should be emphasised because the analysis of time series data (observations on a set of variables over a period of time) is much more complex. Chapters 11 to 13 of the textbook, *Introduction to econometrics*, and this subject guide are devoted to the analysis of time series data, but, beyond very simple applications, the objectives are confined to giving you an understanding of the problems involved and making you aware of the need for a Master's level course if you intend to work with such data.

Specifically the aims of the course are to:

- develop an understanding of the use of regression analysis and related techniques for quantifying economic relationships and testing economic theories
- equip you to read and evaluate empirical papers in professional journals
- provide you with practical experience of using mainstream regression programmes to fit economic models.

0.4 Learning outcomes

By the end of this course, and having completed the Essential reading and activities, you should be able to:

- describe and apply the classical regression model and its application to cross-sectional data
- describe and apply the:
  - Gauss–Markov conditions and other assumptions required in the application of the classical regression model
  - reasons for expecting violations of these assumptions in certain circumstances
  - tests for violations
  - potential remedial measures, including, where appropriate, the use of instrumental variables
- recognise and apply the advantages of logit, probit and similar models over regression analysis when fitting binary choice models
- competently use regression, logit and probit analysis to quantify economic relationships using standard regression programmes (Stata and EViews) in simple applications
- describe and explain the principles underlying the use of maximum likelihood estimation
- apply regression analysis to fit time-series models using stationary time series, with awareness of some of the econometric problems specific to time-series applications (for example, autocorrelation) and remedial measures
- recognise the difficulties that arise in the application of regression analysis to nonstationary time series, know how to test for unit roots, and know what is meant by cointegration.

0.5 How to make use of the textbook

The only reading required for this course is my textbook:

> C. Dougherty, *Introduction to econometrics* (Oxford: Oxford University Press, 2016) fifth edition [ISBN 9780199676828].

The syllabus is the same as that for EC220 Introduction to econometrics, the corresponding internal course at the London School of Economics. The textbook has been written to cover it with very little added and nothing subtracted.

When writing a textbook, there is a temptation to include a large amount of non-core material that may potentially be of use or interest to students. There is much to be said for this, since it allows the same textbook to be used to some extent for reference as well as a vehicle for a taught course. However, my textbook is stripped down to nearly the bare minimum for two reasons. First, the core material provides quite enough content for an introductory year-long course and I think that students should initially concentrate on gaining a good understanding of it. Second, if the textbook is focused narrowly on the syllabus, students can read through it as a continuous narrative without a need for directional guidance. Obviously, this is particularly important for those who are studying the subject on their own, as is the case for most of those enrolled on **EC2020 Elements of econometrics**.

An examination syllabus is provided as an appendix to this subject guide, but its function is mostly to indicate the expected depth of understanding of each topic, rather than the selection of the topics themselves.

0.6 How to make use of this subject guide

The function of this subject guide differs from that of other subject guides you may be using. Unlike those for other courses, this subject guide acts as a supplementary resource, with the textbook as the main resource. Each chapter forms an extension to a corresponding chapter in the textbook with the same title. You *must* have a copy of the textbook to be able to study this course. The textbook will give you the information you need to carry out the activities and achieve the learning outcomes in the subject guide.

The main purpose of the subject guide is to provide you with opportunities to gain experience with econometrics through practice with exercises. Each chapter of the subject guide falls into two parts. The first part begins with an overview of the corresponding chapter in the textbook. Then there is a checklist of learning outcomes anticipated as a result of studying the chapter in the textbook, doing the exercises in the subject guide, and making use of the corresponding resources on the website. Finally, in some of the chapters, comes a section headed 'Further material'. This consists of new topics that may be included in the next edition of the textbook. The second part of each chapter consists of additional exercises, followed by answers to the starred exercises in the text and answers to the additional exercises.

You should organise your studies in the following way:

- first read this introductory chapter

- read the Overview section from the Review chapter of the subject guide
- read the Review chapter of the textbook and do the starred exercises
- refer to the subject guide for answers to the starred exercises in the text and for additional exercises
- check that you have covered all the items in the learning outcomes section in the subject guide.

You should repeat this process for each of the numbered chapters. Note that the subject guide chapters have the same titles as the chapters in the text. In those chapters where there is a 'Further material' section in the subject guide, this should be read after reading the chapter in the textbook.

0.7 How to make use of the website

You should make full use of the resources available at the Online Resource Centre maintained by the publisher, Oxford University Press (OUP): www.oup.com/uk/orc/bin/9780199567089. Here you will find PowerPoint slideshows that provide a graphical treatment of the topics covered in the textbook, data sets for practical work and statistical tables.

0.7.1 Slideshows

In principle you will be able to acquire mastery of the subject by studying the contents of the textbook with the support of this subject guide and doing the exercises conscientiously. However, I strongly recommend that you do study all the slideshows as well. Some do not add much to the material in the textbook, and these you can skim through quickly. Some, however, provide a much more graphical treatment than is possible with print and they should improve your understanding. Some present and discuss regression results and other hands-on material that could not be included in the text for lack of space, and they likewise should be helpful.

0.7.2 Data sets

To use the data sets, you must have access to a proper statistics application with facilities for regression analysis, such as Stata or EViews. The student versions of such applications are adequate for doing all, or almost all, the exercises and of course are much cheaper than the professional ones. Product and pricing information can be obtained from the applications' websites, the URL usually being the name of the application sandwiched between 'www.' and '.com'.

If you do not have access to a commercial econometrics application, you should use gretl. This is a sophisticated application almost as powerful as the commercial ones, and it is free. See the gretl manual on the OUP website for further information.

Whatever you do, do not be tempted to try to get by with the regression engines built into some spreadsheet applications, such as Microsoft Excel. They are not remotely adequate for your needs.

There are three major data sets on the website. The most important one, for the purposes of this subject guide, is the Consumer Expenditure Survey (*CES*) data set. You will find on the website versions in the formats used by Stata, EViews and gretl. If you are using some other application, you should download the text version (comma-delimited ASCII) and import it. Answers to all of the exercises are provided in the relevant chapters of this subject guide.

The exercises for the *CES* data set cover Chapters 1–10 of the text. For Chapters 11–13, you should use the Demand Functions data set, another major data set, to do the additional exercises in the corresponding chapters of this subject guide. Again you should download the data set in the appropriate format. For these exercises, also, answers are provided.

The third major data set on the website is the Educational Attainment and Earnings Function data set, which provides practical work for the first 10 chapters of the text and Chapter 14. No answers are provided, but many parallel examples will be found in the text.

0.8 Online study resources

In addition to the subject guide and the Essential reading, it is crucial that you take advantage of the study resources that are available online for this course, including the VLE and the Online Library.

You can access the VLE, the Online Library and your University of London email account via the Student Portal at: http://my.londoninternational.ac.uk

You should have received your login details for the Student Portal with your official offer, which was emailed to the address that you gave on your application form. You have probably already logged into the Student Portal in order to register! As soon as you registered, you will automatically have been granted access to the VLE, Online Library and your fully functional University of London email account.

If you forget your login details at any point, please email uolia.support@london.ac.uk quoting your student number.

0.8.1 The VLE

The VLE, which complements this subject guide, has been designed to enhance your learning experience, providing additional support and a sense of community. It forms an important part of your study experience with the University of London and you should access it regularly.

The VLE provides a range of resources for EMFSS courses:

- **Electronic study materials:** All of the printed materials which you receive from the University of London are available to download, to give you flexibility in how and where you study.
- **Discussion forums:** An open space for you to discuss interests and seek support

from your peers, working collaboratively to solve problems and discuss subject material. Some forums are moderated by an LSE academic.

- **Videos:** Recorded academic introductions to many subjects; interviews and debates with academics who have designed the courses and teach similar ones at LSE.
- **Recorded lectures:** For a few subjects, where appropriate, various teaching sessions of the course have been recorded and made available online via the VLE.
- **Audio-visual tutorials and solutions:** For some of the first year and larger later courses such as Introduction to Economics, Statistics, Mathematics and Principles of Banking and Accounting, audio-visual tutorials are available to help you work through key concepts and to show the standard expected in examinations.
- **Self-testing activities:** Allowing you to test your own understanding of subject material.
- **Study skills:** Expert advice on getting started with your studies, preparing for examinations and developing your digital literacy skills.

Note: Students registered for Laws courses also receive access to the dedicated Laws VLE.

Some of these resources are available for certain courses only, but we are expanding our provision all the time and you should check the VLE regularly for updates.

0.8.2 Making use of the Online Library

The Online Library (http://onlinelibrary.london.ac.uk) contains a huge array of journal articles and other resources to help you read widely and extensively.

To access the majority of resources via the Online Library you will either need to use your University of London Student Portal login details, or you will be required to register and use an Athens login.

The easiest way to locate relevant content and journal articles in the Online Library is to use the **Summon** search engine.

If you are having trouble finding an article listed in a reading list, try removing any punctuation from the title, such as single quotation marks, question marks and colons.

For further advice, please use the online help pages (http://onlinelibrary.london.ac.uk/resources/summon) or contact the Online Library team: onlinelibrary@shl.london.ac.uk

0.9 Prerequisite for studying this subject

The prerequisite for studying this subject is a solid background in mathematics and elementary statistical theory. The mathematics requirement is a basic understanding of multivariate differential calculus. With regard to statistics, you must have a clear understanding of what is meant by the sampling distribution of an estimator, and of the principles of statistical inference and hypothesis testing. This is absolutely essential. I find that most problems that students have with introductory econometrics are not econometric problems at all but problems with statistics, or rather, a lack of understanding of statistics. There are no short cuts. If you do not have this background knowledge, you should put your study of econometrics on hold and study statistics first. Otherwise there will be core parts of the econometrics syllabus that you do not begin to understand.

In addition, it would be helpful if you have some knowledge of economics. However, although the examples and exercises relate to economics, most of them are so straightforward that a previous study of economics is not a requirement.

0.10 Application of linear algebra to econometrics

At the end of this subject guide you will find a primer on the application of linear algebra (matrix algebra) to econometrics. It is *not* part of the syllabus for the examination, and studying it is unlikely to confer any advantage for the examination. It is provided for the benefit of those students who intend to take a further course in econometrics, especially at the Master's level. The present course is ambitious, by undergraduate standards, in terms of its coverage of concepts and, above all, its focus on the development of an intuitive understanding. For its purposes, it has been quite sufficient and appropriate to work with uncomplicated regression models, typically with no more than two explanatory variables.

However, when you progress to the next level, it is necessary to generalise the theory to cover multiple regression models with many explanatory variables, and linear algebra is ideal for this purpose. The primer does not attempt to teach it. There are many excellent texts and there is no point in duplicating them. The primer assumes that such basic study has already been undertaken, probably taking about 20 to 50 hours, depending on the individual. It is intended to show how the econometric theory in the text can be handled with this more advanced mathematical approach, thus serving as preparation for the higher-level course.

0.11 The examination

Important: the information and advice given here are based on the examination structure used at the time this subject guide was written. Please note that subject guides may be used for several years. Because of this we strongly advise you to always check both the current Programme regulations for relevant information about the examination, and the VLE where you should be advised of any forthcoming changes. You should also carefully check the rubric/instructions on the paper you actually sit and follow those instructions.

Candidates should answer eight out of 10 questions in three hours: all of the questions in Section A (8 marks each) and three questions from Section B (20 marks each).

A calculator may be used when answering questions on this paper and it must comply in all respects with the specification given with your Admission Notice.

Remember, it is important to check the VLE for:

- up-to-date information on examination and assessment arrangements for this course
- where available, past examination papers and *Examiners' commentaries* for the course which give advice on how each question might best be answered.

Review: Random variables and sampling theory

0.12 Overview

The textbook and this subject guide assume that you have previously studied basic statistical theory and have a sound understanding of the following topics:

- descriptive statistics (mean, median, quartile, variance, etc.)
- random variables and probability
- expectations and expected value rules
- population variance, covariance, and correlation
- sampling theory and estimation
- unbiasedness and efficiency
- loss functions and mean square error
- normal distribution
- hypothesis testing, including:
  - t tests
  - Type I and Type II error
  - the significance level and power of a t test
  - one-sided versus two-sided t tests
- confidence intervals
- convergence in probability, consistency, and plim rules
- convergence in distribution and central limit theorems.

There are many excellent textbooks that offer a first course in statistics. The Review chapter of my textbook is not a substitute. It has the much more limited objective of providing an opportunity for revising some key statistical concepts and results that will be used time and time again in the course. They are central to econometric analysis and if you have not encountered them before, you should postpone your study of econometrics and study statistics first.

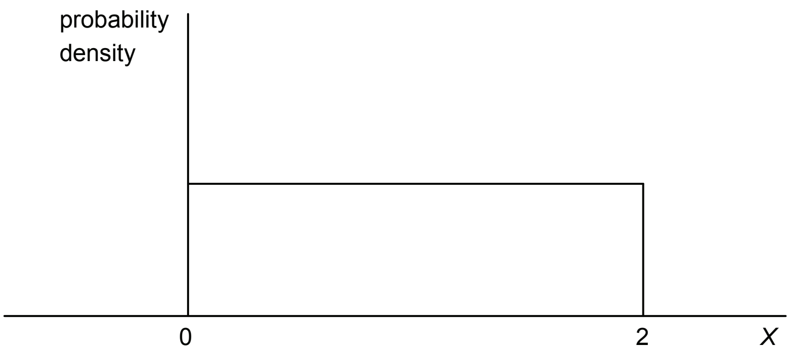
0.13 Learning outcomes

After working through the corresponding chapter in the textbook, studying the corresponding slideshows, and doing the starred exercises in the textbook and the additional exercises in this subject guide, you should be able to explain what is meant by all of the items listed in the Overview. You should also be able to explain why they are important. The concepts of efficiency, consistency, and power are often misunderstood by students taking an introductory econometrics course, so make sure that you aware of their precise meanings.

0.14 Additional exercises

[**Note:** Each chapter has a set of additional exercises. The answers to them are provided at the end of the chapter after the answers to the starred exercises in the text.]

AR.1 A random variable X has a continuous uniform distribution from 0 to 2. Define its probability density function.



AR.2 Find the expected value of X in Exercise AR.1, using the expression given in Box R.1 in the text.

AR.3 Derive $E(X^2)$ for X defined in Exercise AR.1, using the expression given in Box R.1.

AR.4 Derive the population variance and the standard deviation of X as defined in Exercise AR.1, using the expression given in Box R.1.

AR.5 Using equation (R.9), find the variance of the random variable X defined in Exercise AR.1 and show that the answer is the same as that obtained in Exercise AR.4. (Note: You have already calculated $E(X)$ in Exercise AR.2 and $E(X^2)$ in Exercise AR.3.)

AR.6 In Table R.6, μ_0 and μ_1 were three standard deviations apart. Construct a similar table for the case where they are two standard deviations apart.

AR.7 Suppose that a random variable X has a normal distribution with unknown mean μ and variance σ^2. To simplify the analysis, we shall assume that σ^2 is known. Given a sample of observations, an estimator of μ is the sample mean, $\overline{X}$. An investigator wishes to test $H_0 : \mu = 0$ and believes that the true value cannot be negative. The appropriate alternative hypothesis is therefore $H_1 : \mu > 0$ and the investigator decides to perform a one-sided test. However, the investigator is mistaken because μ could in fact be negative. What are the consequences of erroneously performing a one-sided test when a two-sided test would have been appropriate?

AR.8 Suppose that a random variable X has a normal distribution with mean μ and variance σ^2. Given a sample of n independent observations, it can be shown that:

$$\widehat{\sigma}^2 = \frac{1}{n-1}\sum\left(X_i - \overline{X}\right)^2$$

is an unbiased estimator of σ^2. Is $\sqrt{\widehat{\sigma}^2}$ either an unbiased or a consistent estimator of σ?

0.15 Answers to the starred exercises in the textbook

R.2 A random variable X is defined to be the larger of the two values when two dice are thrown, or the value if the values are the same. Find the probability distribution for X.

Answer:

The table shows the 36 possible outcomes. The probability distribution is derived by counting the number of times each outcome occurs and dividing by 36. The probabilities have been written as fractions, but they could equally well have been written as decimals.

red green	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	2	3	4	5	6
3	3	3	3	4	5	6
4	4	4	4	4	5	6
5	5	5	5	5	5	6
6	6	6	6	6	6	6

Value of X	1	2	3	4	5	6
Frequency	1	3	5	7	9	11
Probability	1/36	3/36	5/36	7/36	9/36	11/36

R.4 Find the expected value of X in Exercise R.2.

Answer:

The table is based on Table R.2 in the text. It is a good idea to guess the outcome before doing the arithmetic. In this case, since the higher numbers have the largest probabilities, the expected value should clearly lie between 4 and 5. If the calculated value does not conform with the guess, it is possible that this is because the guess was poor. However, it may be because there is an error in the arithmetic, and this is one way of catching such errors.

X	p	Xp
1	1/36	1/36
2	3/36	6/36
3	5/36	15/36
4	7/36	28/36
5	9/36	45/36
6	11/36	66/36
Total		$161/36 = 4.4722$

R.7 Calculate $E(X^2)$ for X defined in Exercise R.2.

Answer:

The table is based on Table R.3 in the text. Given that the largest values of X^2 have the highest probabilities, it is reasonable to suppose that the answer lies somewhere in the range 15–30. The actual figure is 21.97.

X	X^2	p	X^2p
1	1	1/36	1/36
2	4	3/36	12/36
3	9	5/36	45/36
4	16	7/36	112/36
5	25	9/36	225/36
6	36	11/36	396/36
Total			$791/36 = 21.9722$

R.10 Calculate the population variance and the standard deviation of X as defined in Exercise R.2, using the definition given by equation (R.8).

Answer:

The table is based on Table R.4 in the textbook. In this case it is not easy to make a guess. The population variance is 1.97, and the standard deviation, its square root, is 1.40. Note that four decimal places have been used in the working, even though the estimate is reported to only two. This is to eliminate the possibility of the estimate being affected by rounding error.

X	p	$X-\mu_X$	$(X-\mu_X)^2$	$(X-\mu_X)^2p$
1	1/36	−3.4722	12.0563	0.3349
2	3/36	−2.4722	6.1119	0.5093
3	5/36	−1.4722	2.1674	0.3010
4	7/36	−0.4722	0.2230	0.0434
5	9/36	0.5278	0.2785	0.0696
6	11/36	1.5278	2.3341	0.7132
Total				1.9715

R.12 Using equation (R.9), find the variance of the random variable X defined in Exercise R.2 and show that the answer is the same as that obtained in Exercise R.10. (Note: You have already calculated μ_X in Exercise R.4 and $E(X^2)$ in Exercise R.7.)

Answer:

$E(X^2)$ is 21.9722 (Exercise R.7). $E(X)$ is 4.4722 (Exercise R.4), so μ_X^2 is 20.0006. Thus the variance is $21.9722 - 20.0006 = 1.9716$. The last-digit discrepancy between this figure and that in Exercise R.10 is due to rounding error.

R.14 Suppose a variable Y is an exact linear function of X:

$$Y = \lambda + \mu X$$

where λ and μ are constants, and suppose that Z is a third variable. Show that $\rho_{XZ} = \rho_{YZ}$

Answer:

We start by noting that $\left(Y_i - \overline{Y}\right) = \mu\left(X_i - \overline{X}\right)$. Then:

$$\begin{aligned}
\rho_{YZ} &= \frac{E\left[\left(Y_i-\overline{Y}\right)\left(Z_i-\overline{Z}\right)\right]}{\sqrt{E\left[\left(Y_i-\overline{Y}\right)^2\right]E\left[\left(Z_i-\overline{Z}\right)^2\right]}} \\
&= \frac{E\left[\mu\left(X_i-\overline{X}\right)\left(Z_i-\overline{Z}\right)\right]}{\sqrt{E\left[\mu^2\left(X_i-\overline{X}\right)^2\right]E\left[\mu^2\left(Z_i-\overline{Z}\right)^2\right]}} \\
&= \frac{\mu E\left[\left(X_i-\overline{X}\right)\left(Z_i-\overline{Z}\right)\right]}{\sqrt{\mu^2 E\left[\left(X_i-\overline{X}\right)^2\right]E\left[\left(Z_i-\overline{Z}\right)^2\right]}} \\
&= \rho_{XZ}.
\end{aligned}$$

R.16 Show that, when you have n observations, the condition that the generalised estimator $(\lambda_1 X_1 + \cdots + \lambda_n X_n)$ should be an unbiased estimator of μ_X is $\lambda_1 + \cdots + \lambda_n = 1$.

Answer:

$$\begin{aligned} E(Z) &= E(\lambda_1 X_1 + \cdots + \lambda_n X_n) \\ &= E(\lambda_1 X_1) + \cdots + E(\lambda_n X_n) \\ &= \lambda_1 E(X_1) + \cdots + \lambda_n E(X_n) \\ &= \lambda_1 \mu_X + \cdots + \lambda_n \mu_X \\ &= (\lambda_1 + \cdots + \lambda_n)\mu_X. \end{aligned}$$

Thus $E(Z) = \mu_X$ requires $\lambda_1 + \cdots + \lambda_n = 1$.

R.19 In general, the variance of the distribution of an estimator decreases when the sample size is increased. Is it correct to describe the estimator as becoming more efficient?

Answer:

No, it is incorrect. When the sample size increases, the variance of the estimator decreases, and as a consequence it is more likely to give accurate results. Because it is improving in this important sense, it is very tempting to describe the estimator as becoming more efficient. But it is the wrong use of the term. Efficiency is a comparative concept that is used when you are comparing two or more alternative estimators, all of them being applied to the same data set with the same sample size. The estimator with the smallest variance is said to be the most efficient. You cannot use efficiency as suggested in the question because you are comparing the variances of the *same* estimator with *different* sample sizes.

R.21 Suppose that you have observations on three variables X, Y, and Z, and suppose that Y is an exact linear function of Z:

$$Y = \lambda + \mu Z$$

where λ and μ are constants. Show that $\widehat{\rho}_{XZ} = \widehat{\rho}_{XY}$. (This is the counterpart of Exercise R.14.)

Answer:

We start by noting that $\left(Y_i - \overline{Y}\right) = \mu\left(Z_i - \overline{Z}\right)$. Then:

$$\begin{aligned} \widehat{\rho}_{XY} &= \frac{\sum\left(X_i - \overline{X}\right)\left(Y_i - \overline{Y}\right)}{\sqrt{\sum\left(X_i - \overline{X}\right)^2 \sum\left(Y_i - \overline{Y}\right)^2}} \\ &= \frac{\sum\left(X_i - \overline{X}\right)\mu\left(Z_i - \overline{Z}\right)}{\sqrt{\sum\left(X_i - \overline{X}\right)^2 \sum \mu^2\left(Z_i - \overline{Z}\right)^2}} \\ &= \frac{\sum\left(X_i - \overline{X}\right)\left(Z_i - \overline{Z}\right)}{\sqrt{\sum\left(X_i - \overline{X}\right)^2 \sum\left(Z_i - \overline{Z}\right)^2}} \\ &= \widehat{\rho}_{XZ} \end{aligned}$$

R.26 Show that, in Figures R.18 and R.22, the probabilities of a Type II error are 0.15 in the case of a 5 per cent significance test and 0.34 in the case of a 1 per cent test. Note that the distance between μ_0 and μ_1 is three standard deviations. Hence the right-hand 5 per cent rejection region begins 1.96 standard deviations to the right of μ_0. This means that it is located 1.04 standard deviations to the left of μ_1. Similarly, for a 1 per cent test, the right-hand rejection region starts 2.58 standard deviations to the right of μ_0, which is 0.42 standard deviations to the left of μ_1.

Answer:

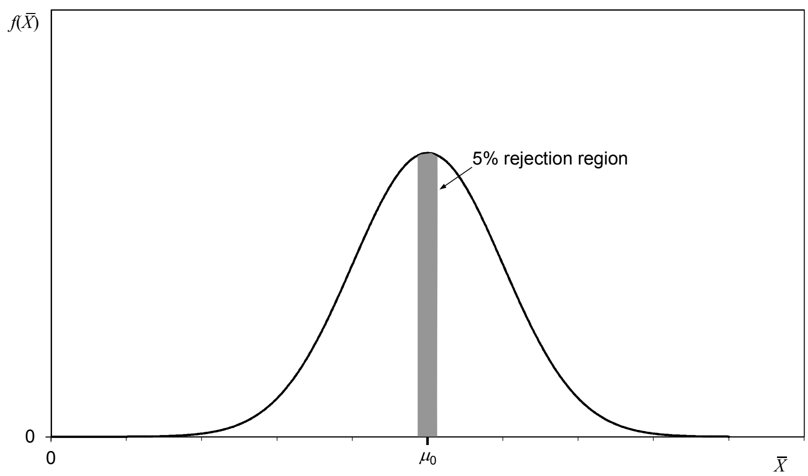
For the 5 per cent test, the rejection region starts $3 - 1.96 = 1.04$ standard deviations below μ_1, given that the distance between μ_1 and μ_0 is 3 standard deviations. See Figure R.18. According to the standard normal distribution table, the cumulative probability of a random variable lying 1.04 standard deviations (or less) above the mean is 0.8508. This implies that the probability of it lying 1.04 standard deviations below the mean is 0.1492. For the 1 per cent test, the rejection region starts $3 - 2.58 = 0.42$ standard deviations below the mean. See Figure R.22. The cumulative probability for 0.42 in the standard normal distribution table is 0.6628, so the probability of a Type II error is 0.3372.

R.27 Explain why the difference in the power of a 5 per cent test and a 1 per cent test becomes small when the distance between μ_0 and μ_1 becomes large.

Answer:

The powers of both tests tend to one as the distance between μ_0 and μ_1 becomes large. The difference in their powers must therefore tend to zero.

R.28 A random variable X has unknown population mean μ. A researcher has a sample of observations with sample mean $\overline{X}$. He wishes to test the null hypothesis $H_0 : \mu = \mu_0$. The figure shows the potential distribution of $\overline{X}$ conditional on H_0 being true. It may be assumed that the distribution is known to have variance equal to one.



The researcher decides to implement an unorthodox (and unwise) decision rule. He decides to reject H_0 if $\overline{X}$ lies in the central 5 per cent of the distribution (the tinted area in the figure).

(a) Explain why his test is a 5 per cent significance test.

(b) Explain in intuitive terms why his test is unwise.

(c) Explain in technical terms why his test is unwise.

Answer:

The following discussion assumes that you are performing a 5 per cent significance test, but it applies to any significance level.

If the null hypothesis is true, it does not matter how you define the 5 per cent rejection region. By construction, the risk of making a Type I error will be 5 per cent. Issues relating to Type II errors are irrelevant when the null hypothesis is true.

The reason that the central part of the conditional distribution is not used as a rejection region is that it leads to problems when the null hypothesis is false. The probability of not rejecting H_0 when it is false will be lower. To use the obvious technical term, the power of the test will be lower.

The figure shows the power functions for the test using the conventional upper and lower 2.5 per cent tails and the test using the central region. The horizontal axis is the difference between the true value and the hypothetical value μ_0 in terms of standard deviations. The vertical axis is the power of the test. The first figure has been drawn for the case where the true value is greater than the hypothetical value. The second figure is for the case where the true value is lower than the hypothetical value. It is the same, but reflected horizontally.

The greater the difference between the true value and the hypothetical mean, the more likely is it that the sample mean will lie in the right tail of the distribution conditional on H_0 being true, and so the more likely is it that the null hypothesis will be rejected by the conventional test. The figure shows that the power of the test approaches one asymptotically. However, if the central region of the distribution is used as the rejection region, the probability of the sample mean lying in it will *diminish* as the difference between the true and hypothetical values increases, and the power of the test approaches zero asymptotically. This is an extreme example of a very bad test procedure.

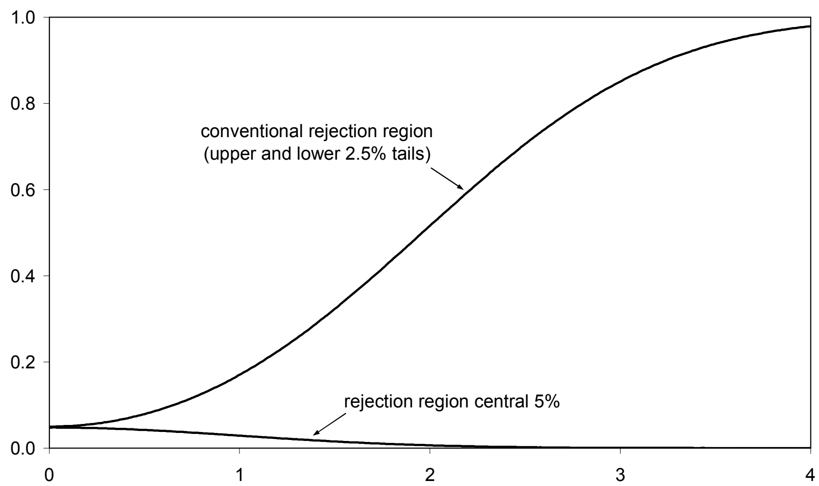


Figure 1: Power functions of a conventional 5 per cent test and one using the central region (true value $> \mu_0$).

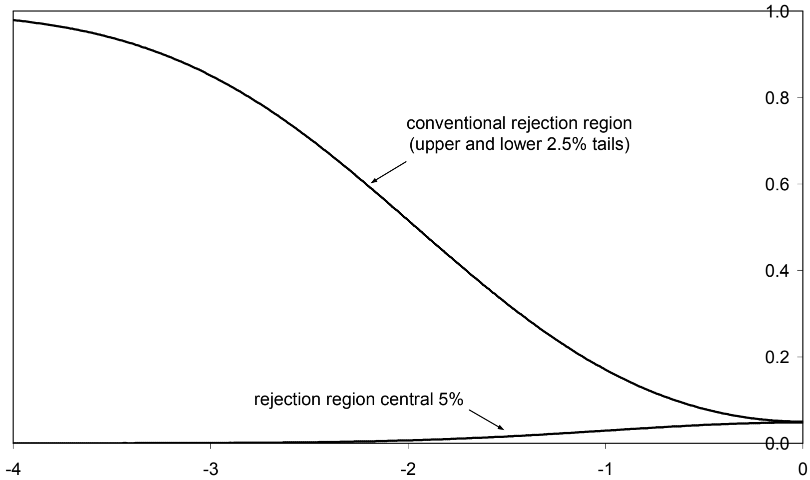


Figure 2: Power functions of a conventional 5 per cent test and one using the central region (true value $< \mu_0$).

R.29 A researcher is evaluating whether an increase in the minimum hourly wage has had an effect on employment in the manufacturing industry in the following three months. Taking a sample of 25 firms, what should she conclude if:

(a) the mean decrease in employment is 9 per cent, and the standard error of the mean is 5 per cent

(b) the mean decrease is 12 per cent, and the standard error is 5 per cent

(c) the mean decrease is 20 per cent, and the standard error is 5 per cent

(d) there is a mean *increase* of 11 per cent, and the standard error is 5 per cent?

Answer:

There are 24 degrees of freedom, and hence the critical values of t at the 5 per cent, 1 per cent, and 0.1 per cent levels are 2.06, 2.80, and 3.75, respectively.

(a) The t statistic is -1.80. Fail to reject H_0 at the 5 per cent level.

(b) $t = -2.40$. Reject H_0 at the 5 per cent level but not the 1 per cent level.

(c) $t = -4.00$. Reject H_0 at the 1 per cent level. Better, reject at the 0.1 per cent level.

(d) $t = 2.20$. This would be a surprising outcome, but if one is performing a two-sided test, then reject H_0 at the 5 per cent level but not the 1 per cent level.

R3.33 Demonstrate that the 95 per cent confidence interval defined by equation (R.89) has a 95 per cent probability of capturing μ_0 if H_0 is true.

Answer:

If H_0 is true, there is 95 per cent probability that:

$$\left| \frac{\overline{X} - \mu_0}{\text{s.e.}(\overline{X})} \right| < t_{\text{crit}}.$$

Hence there is 95 per cent probability that $|\overline{X} - \mu_0| \leq t_{\text{crit}} \times \text{s.e.}(\overline{X})$. Hence there is 95 per cent probability that (a) $\overline{X} - \mu_0 < t_{\text{crit}} \times \text{s.e.}(\overline{X})$ and (b) $\mu_0 - \overline{X} < t_{\text{crit}} \times \text{s.e.}(\overline{X})$.

(a) can be rewritten $\overline{X} - t_{\text{crit}} \times \text{s.e.}(\overline{X}) < \mu_0$, giving the lower limit of the confidence interval.

(b) can be rewritten $\overline{X} - \mu_0 > -t_{\text{crit}} \times \text{s.e.}(\overline{X})$ and hence $\overline{X} + t_{\text{crit}} \times \text{s.e.}(\overline{X}) > \mu_0$, giving the upper limit of the confidence interval.

Hence there is 95 per cent probability that μ_0 will lie in the confidence interval.

R.34 In Exercise R.29, a researcher was evaluating whether an increase in the minimum hourly wage has had an effect on employment in the manufacturing industry. Explain whether she might have been justified in performing one-sided tests in cases (a) – (d), and determine whether her conclusions would have been different.

Answer:

First, there should be a discussion of whether the effect of an increase in the minimum wage could have a positive effect on employment. If it is decided that it cannot, we can use a one-sided test and the critical values of t at the 5 per cent, 1 per cent, and 0.1 per cent levels become 1.71, 2.49, and 3.47, respectively.

1. The t statistic is -1.80. We can now reject H_0 at the 5 per cent level.
2. $t = -2.40$. No change, but much closer to rejecting at the 1 per cent level.
3. $t = -4.00$. No change. Reject at the 1 per cent level (and 0.1 per cent level).
4. $t = 2.20$. Here there is a problem because the coefficient has the unexpected sign. In principle we should stick to our guns and fail to reject H_0. However, we should consider two further possibilities. One is that the justification for a one-sided test is incorrect (not very likely in this case). The other is that the model is misspecified in some way and the misspecification is responsible for the unexpected sign. For example, the coefficient might be distorted by omitted variable bias, to be discussed in Chapter 6.

R.37 A random variable X has population mean μ_X and population variance σ_X^2. A sample of n observations $\{X_1, \ldots, X_n\}$ is generated. Using the plim rules, demonstrate that, subject to a certain condition that should be stated:

$$\operatorname{plim}\left(\frac{1}{\overline{X}}\right) = \frac{1}{\mu_X}.$$

Answer:

$\operatorname{plim} \overline{X} = \mu_X$ by the weak law of large numbers. Provided that $\mu_X \neq 0$, we are entitled to use the plim quotient rule, so:

$$\operatorname{plim}\left(\frac{1}{\overline{X}}\right) = \frac{\operatorname{plim} 1}{\operatorname{plim} \overline{X}} = \frac{1}{\mu_X}.$$

R.39 A random variable X has unknown population mean μ_X and population variance σ_X^2. A sample of n observations $\{X_1, \ldots, X_n\}$ is generated. Show that:

$$Z = \frac{1}{2}X_1 + \frac{1}{4}X_2 + \frac{1}{8}X_3 + \cdots + \frac{1}{2^{n-1}} + \frac{1}{2^{n-1}}X_n$$

is an unbiased estimator of μ_X. Show that the variance of Z does not tend to zero as n tends to infinity and that therefore Z is an inconsistent estimator, despite being unbiased.

Answer:

The weights sum to unity, so the estimator is unbiased. However, its variance is:

$$\sigma_Z^2 = \left(\frac{1}{4} + \frac{1}{16} + \cdots + \frac{1}{4^{n-1}} + \frac{1}{4^{n-1}}\right)\sigma_X^2.$$

This tends to $\sigma_X^2/3$ as n becomes large, not zero, so the estimator is inconsistent.

Note: the sum of a geometric progression is given by:

$$1 + a + a^2 + \cdots + a^n = \frac{1 - a^{n+1}}{1 - a}.$$

Hence:

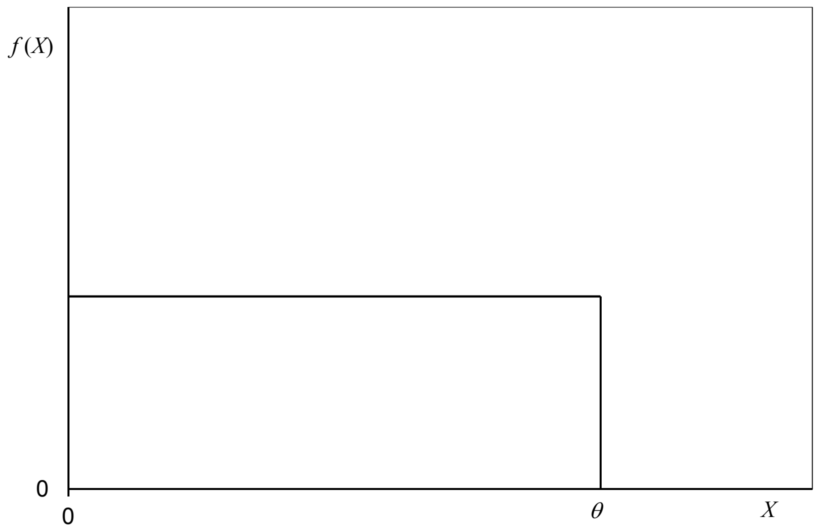
$$\begin{aligned}
\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^{n-1}} + \frac{1}{2^{n-1}} &= \frac{1}{2}\left(1 + \frac{1}{2} + \cdots + \frac{1}{2^{n-2}}\right) + \frac{1}{2^{n-1}} \\
&= \frac{1}{2} \times \frac{1 - \left(\frac{1}{2}\right)^{n-1}}{1 - \frac{1}{2}} + \frac{1}{2^{n-1}} \\
&= 1 - \frac{1}{2^{n-1}} + \frac{1}{2^{n-1}} = 1
\end{aligned}$$

and:

$$\begin{aligned}
\frac{1}{4} + \frac{1}{16} + \cdots + \frac{1}{4^{n-1}} + \frac{1}{4^{n-1}} &= \frac{1}{4}\left(1 + \frac{1}{4} + \cdots + \frac{1}{4^{n-2}}\right) + \frac{1}{4^{n-1}} \\
&= \frac{1}{4} \times \frac{1 - \left(\frac{1}{4}\right)^{n-1}}{1 - \frac{1}{4}} + \frac{1}{4^{n-1}} \\
&= \frac{1}{3}\left(1 - \left(\frac{1}{4}\right)^{n-1}\right) + \frac{1}{4^{n-1}} \to \frac{1}{3}
\end{aligned}$$

as n becomes large.

R.41 A random variable X has a continuous uniform distribution over the interval from 0 to θ, where θ is an unknown parameter.



The following three estimators are used to estimate θ, given a sample of n observations on X:

(a) twice the sample mean

(b) the largest value of X in the sample

(c) the sum of the largest and smallest values of X in the sample.

Explain verbally whether or not each estimator is (1) unbiased, and (2) consistent.

Answer:

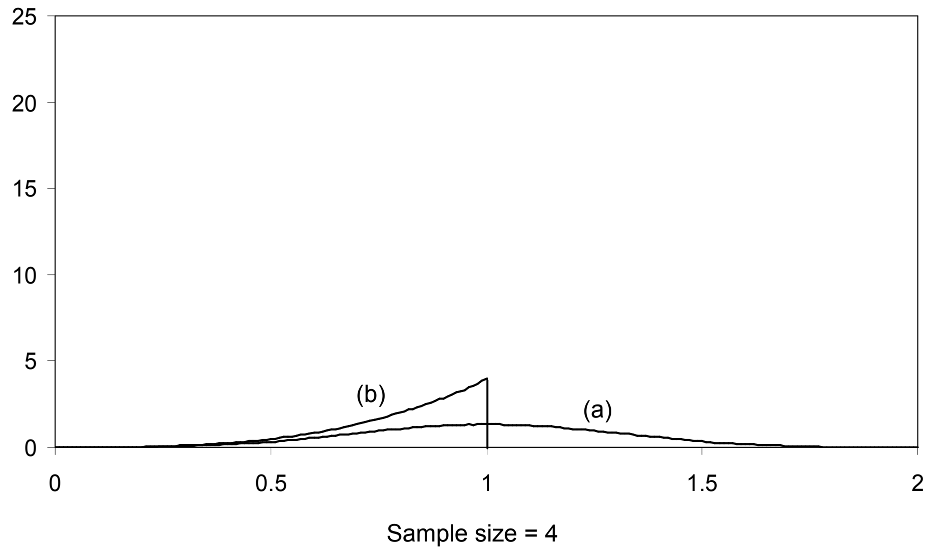
(a) It is evident that $E(X) = E(\overline{X}) = \theta/2$. Hence $2\overline{X}$ is an unbiased estimator of θ. The variance of $\overline{X}$ is σ_X^2/n. The variance of $2\overline{X}$ is therefore $4\sigma_{\overline{X}}^2/n$. This will tend to zero as n tends to infinity. Thus the distribution of $2\overline{X}$ will collapse to a spike at θ and the estimator is consistent.

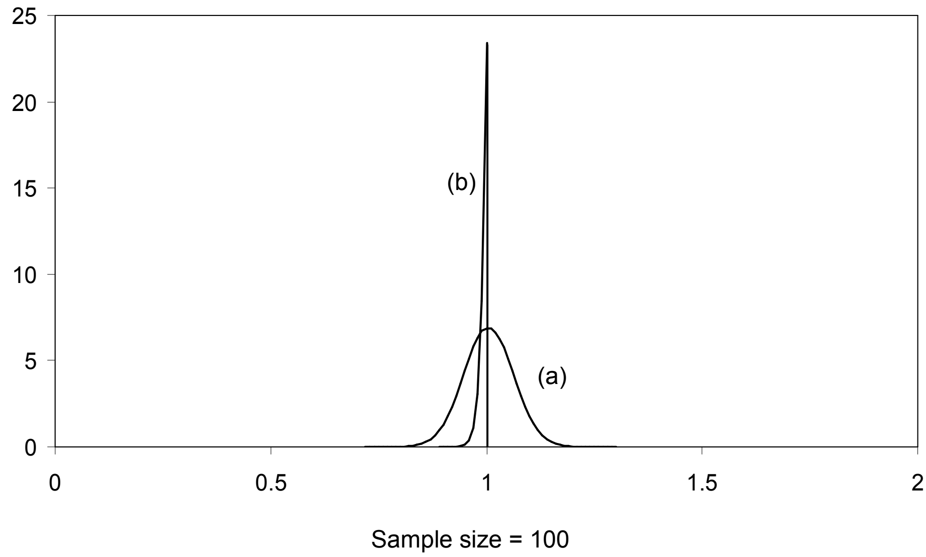
(b) The estimator will be biased downwards since the highest value of X in the sample will always be less than θ. However, as n increases, the distribution of the estimator will be increasingly concentrated in a narrow range just below θ. To put it formally, the probability of the highest value being more than ϵ below θ will be $\left(1 - \frac{\epsilon}{\theta}\right)^n$ and this will tend to zero, no matter how small ϵ is, as n tends to infinity. The estimator is therefore consistent. It can in fact be shown that the expected value of the estimator is $\frac{n}{n+1}\theta$ and this tends to θ as n becomes large.

(c) The estimator will be unbiased. Call the maximum value of X in the sample X_{max} and the minimum value X_{min}. Given the symmetry of the distribution of X, the distributions of X_{max} and X_{min} will be identical, except that that of X_{min} will be to the right of 0 and that of X_{max} will be to the left of θ. Hence, for any n, $E(X_{\text{min}}) - 0 = \theta - E(X_{\text{max}})$ and the expected value of their sum is equal to θ. The estimator will be consistent for the same reason as explained in (b).

The first figure shows the distributions of the estimators (a) and (b) for 1,000,000 samples with only four observations in each sample, with $\theta = 1$. The second figure shows the distributions when the number of observations in each sample is equal to

100. The table gives the means and variances of the distributions as computed from the results of the simulations. If the mean square error is used to compare the estimators, which should be preferred for sample size 4? For sample size 100?





	Sample size 4		Sample size 100	
	(a)	(b)	(a)	(b)
Mean	1.0000	0.8001	1.0000	0.9901
Variance	0.0833	0.0267	0.0033	0.0001
Estimated bias	0.0000	−0.1999	0.0000	−0.0099
Estimated mean square error	0.0833	0.0667	0.0033	0.0002

It can be shown (Larsen and Marx, *An Introduction to Mathematical Statistics and Its Applications*, p.382, that estimator (b) is biased downwards by an amount $\theta/(n+1)$ and that its variance is:

$$\frac{n\theta^2}{(n+1)^2(n+2)}$$

while estimator (a) has variance $\theta^2/3n$. How large does n have to be for (b) to be preferred to (a) using the mean square error criterion?

The crushing superiority of (b) over (a) may come as a surprise, so accustomed are we to finding that the sample mean in the best estimator of a parameter. The underlying reason in this case is that we are estimating a boundary parameter, which, as its name implies, defines the limit of a distribution. In such a case the optimal properties of the sample mean are no longer guaranteed and it may be eclipsed by a score statistic such as the largest observation in the sample. Note that the standard deviation of the sample mean is inversely proportional to $\sqrt{n}$, while that of (b) is inversely proportional to n (disregarding the differences between n, $n+1$, and $n+2$). (b) therefore approaches its limiting (asymptotically unbiased) value much faster than (a) and is said to be superconsistent. We will encounter superconsistent estimators again when we come to cointegration in Chapter 13. Note that if we multiply (b) by $(n+1)/n$, it is unbiased for finite samples as well as superconsistent.

0.16 Answers to the additional exercises

AR.1 The total area under the function over the interval [0, 2] must be equal to 1. Since the length of the rectangle is 2, its height must be 0.5. Hence $f(X) = 0.5$ for $0 \leq X \leq 2$, and $f(X) = 0$ for $X < 0$ and $X > 2$.

AR.2 Obviously, since the distribution is uniform, the expected value of X is 1. However we will derive this formally.

$$E(X) = \int_0^2 Xf(X)\,\mathrm{d}X = \int_0^2 0.5X\,\mathrm{d}X = \left[\frac{X^2}{4}\right]_0^2 = \left[\frac{2^2}{4}\right] - \left[\frac{0^2}{4}\right] = 1.$$

AR.3 The expected value of X^2 is given by:

$$E(X^2) = \int_0^2 X^2 f(X)\,\mathrm{d}X = \int_0^2 0.5X^2\,\mathrm{d}X = \left[\frac{X^3}{6}\right]_0^2 = \left[\frac{2^3}{6}\right] - \left[\frac{0^3}{6}\right] = 1.3333.$$

AR.4 The variance of X is given by:

$$\begin{aligned} E\left([X-\mu_X]^2\right) = \int_0^2 [X-\mu_X]^2 f(X)\,\mathrm{d}X &= \int_0^2 0.5[X-1]^2\,\mathrm{d}X \\ &= \int_0^2 (0.5X^2 - X + 0.5)\,\mathrm{d}X \\ &= \left[\frac{X^3}{6} - \frac{X^2}{2} + \frac{X}{2}\right]_0^2 \\ &= \left[\frac{8}{6} - 2 + 1\right] - [0] = 0.3333. \end{aligned}$$

The standard deviation is equal to the square root, 0.5774.

AR.5 From Exercise AR.3, $E(X^2) = 1.3333$. From Exercise AR.2, the square of $E(X)$ is 1. Hence the variance is 0.3333, as in Exercise AR.4.

AR.6 Table R.6 is reproduced for reference:

Table R.6 Trade-off between Type I and Type II errors, one-sided and two-sided tests

	Probability of Type II error if $\mu = \mu_1$	
	One-sided test	Two-sided test
5 per cent significance test	0.09	0.15
2.5 per cent significance test	0.15	(not investigated)
1 per cent significance test	0.25	0.34

Note: The distance between μ_1 and μ_0 in this example was 3 standard deviations.

Two-sided tests

Under the (false) $H_0 : \mu = \mu_0$, the right rejection region for a two-sided 5 per cent significance test starts 1.96 standard deviations above μ_0, which is 0.04 standard deviations below μ_1. A Type II error therefore occurs if $\overline{X}$ is more than 0.04 standard deviations to the left of μ_1. Under $H_1 : \mu = \mu_1$, the probability is 0.48.

Under H_0, the right rejection region for a two-sided 1 per cent significance test starts 2.58 standard deviations above μ_0, which is 0.58 standard deviations *above* μ_1. A Type II error therefore occurs if $\overline{X}$ is less than 0.58 standard deviations to the right of μ_1. Under $H_1 : \mu = \mu_1$, the probability is 0.72.

One-sided tests

Under $H_0 : \mu = \mu_0$, the right rejection region for a one-sided 5 per cent significance test starts 1.65 standard deviations above μ_0, which is 0.35 standard deviations below μ_1. A Type II error therefore occurs if $\overline{X}$ is more than 0.35 standard deviations to the left of μ_1. Under $H_1 : \mu = \mu_1$, the probability is 0.36.

Under H_0, the right rejection region for a one-sided 1 per cent significance test starts 2.33 standard deviations above μ_0, which is 0.33 standard deviations *above* μ_1. A Type II error therefore occurs if $\overline{X}$ is less than 0.33 standard deviations to the right of μ_1. Under $H_1 : \mu = \mu_1$, the probability is 0.63.

Hence the table is:

Trade-off between Type I and Type II errors, one-sided and two-sided tests

	Probability of Type II error if $\mu = \mu_1$	
	One-sided test	Two-sided test
5 per cent significance test	0.36	0.48
1 per cent significance test	0.63	0.72

AR.7 We will assume for sake of argument that the investigator is performing a 5 per cent significance test, but the conclusions apply to all significance levels.

If the true value is 0, the null hypothesis is true. The risk of a Type I error is, by construction, 5 per cent for both one-sided and two-sided tests. Issues relating to Type II error do not arise because the null hypothesis is true.

If the true value is positive, the investigator is lucky and makes the gain associated with a one-sided test. Namely, the power of the test is uniformly higher than that for a two-sided test for all positive values of μ. The power functions for one-sided and two-sided tests are shown in the first figure below.

If the true value is negative, the power functions are as shown in the second figure. That for the two-sided test is the same as that in the first figure, but reflected horizontally. The larger (negatively) is the true value of μ, the greater will be the probability of rejecting H_0 and the power approaches 1 asymptotically. However, with a one-sided test, the power function will decrease from its already very low value. The power is not automatically zero for true values that are negative because even for these it is possible that a sample might have a mean that lies in the right tail of the distribution under the null hypothesis. But the probability rapidly falls to zero as the (negative) size of μ grows.

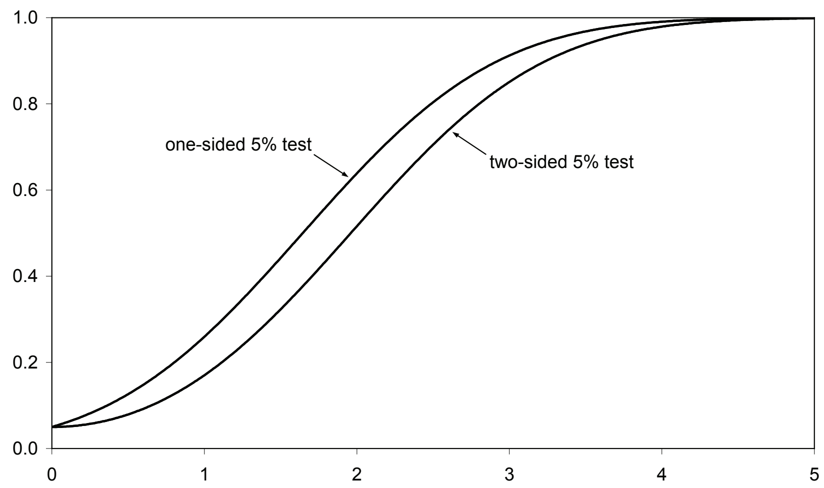


Figure 3: Power functions of one-sided and two-sided 5 per cent tests (true value > 0).

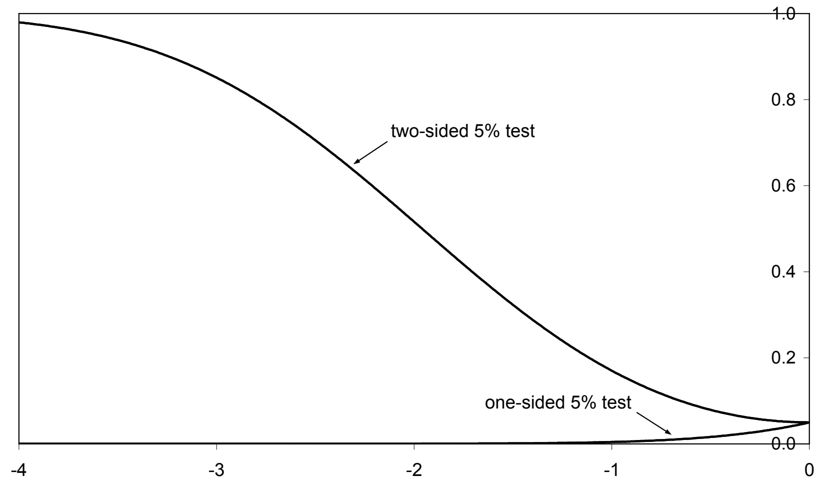


Figure 4: Power functions of one-sided and two-sided 5 per cent tests (true value < 0).

AR.8 We will refute the unbiasedness proposition by considering the more general case where Z^2 is an unbiased estimator of θ^2. We know that:

$$E\left[(Z-\theta)^2\right] = E(Z^2) - 2\theta E(Z) + \theta^2 = 2\theta^2 - 2\theta E(Z).$$

Hence:

$$E(Z) = \theta - \frac{1}{2\theta} E\left[(Z-\theta)^2\right].$$

Z is therefore a biased estimator of θ except for the special case where Z is equal to θ for all samples, that is, in the trivial case where there is no sampling error.

Nevertheless, since a function of a consistent estimator will, under quite general conditions, be a consistent estimator of the function of the parameter, $\sqrt{\widehat{\sigma}^2}$ will be a consistent estimator of σ.

Chapter 1
Simple regression analysis

1.1 Overview

This chapter introduces the least squares criterion of goodness of fit and demonstrates, first through examples and then in the general case, how it may be used to develop expressions for the coefficients that quantify the relationship when a dependent variable is assumed to be determined by one explanatory variable. The chapter continues by showing how the coefficients should be interpreted when the variables are measured in natural units, and it concludes by introducing R^2, a second criterion of goodness of fit, and showing how it is related to the least squares criterion and the correlation between the fitted and actual values of the dependent variable.

1.2 Learning outcomes

After working through the corresponding chapter in the text, studying the corresponding slideshows, and doing the starred exercises in the text and the additional exercises in this subject guide, you should be able to explain what is meant by:

- dependent variable
- explanatory variable (independent variable, regressor)
- parameter of a regression model
- the nonstochastic component of a true relationship
- the disturbance term
- the least squares criterion of goodness of fit
- ordinary least squares (OLS)
- the regression line
- fitted model
- fitted values (of the dependent variable)
- residuals
- total sum of squares, explained sum of squares, residual sum of squares
- R^2.

In addition, you should be able to explain the difference between:

- the nonstochastic component of a true relationship and a fitted regression line, and
- the values of the disturbance term and the residuals.

1.3 Additional exercises

A1.1 The output below gives the result of regressing *FDHO*, annual household expenditure on food consumed at home, on *EXP*, total annual household expenditure, both measured in dollars, using the Consumer Expenditure Survey data set. Give an interpretation of the coefficients.

```
. reg FDHO EXP if FDHO>0

      Source |       SS       df       MS              Number of obs =    6334
-------------+------------------------------           F(  1,  6332) = 3431.01
       Model |   972602566     1   972602566           Prob > F      =  0.0000
    Residual |  1.7950e+09  6332  283474.003           R-squared     =  0.3514
-------------+------------------------------           Adj R-squared =  0.3513
       Total |  2.7676e+09  6333   437006.15           Root MSE      =  532.42

------------------------------------------------------------------------------
        FDHO |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
         EXP |   .0627099   .0010706    58.57   0.000     .0606112    .0648086
       _cons |   369.4418   10.65718    34.67   0.000     348.5501    390.3334
------------------------------------------------------------------------------
```

A1.2 Download the *CES* data set from the website (see Appendix B of the text), perform a regression parallel to that in Exercise A1.1 for your category of expenditure, and provide an interpretation of the regression coefficients.

A1.3 The output shows the result of regressing the weight of the respondent, in pounds, in 2011 on the weight in 2004, using *EAWE* Data Set 22. Provide an interpretation of the coefficients. Summary statistics for the data are also provided.

```
. reg WEIGHT11 WEIGHT04

      Source |       SS       df       MS              Number of obs =     500
-------------+------------------------------           F(  1,   498) = 1207.55
       Model |  769248.875     1  769248.875           Prob > F      =  0.0000
    Residual |  317241.693   498  637.031513           R-squared     =  0.7080
-------------+------------------------------           Adj R-squared =  0.7074
       Total |  1086490.57   499  2177.33581           Root MSE      =  25.239
------------------------------------------------------------------------------
    WEIGHT11 |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
    WEIGHT04 |   .9739736   .0280281    34.75   0.000     .9189056    1.029042
       _cons |   17.42232   4.888091     3.56   0.000     7.818493    27.02614
------------------------------------------------------------------------------
```

```
. sum WEIGHT04 WEIGHT11

    Variable |       Obs        Mean    Std. Dev.       Min        Max
-------------+--------------------------------------------------------
    WEIGHT04 |       500     169.686     40.31215        95        330
    WEIGHT11 |       500     182.692     46.66193        95        370
```

A1.4 The output shows the result of regressing the hourly earnings of the respondent, in dollars, in 2011 on height in 2004, measured in inches, using *EAWE* Data Set 22. Provide an interpretation of the coefficients, comment on the plausibility of the interpretation, and attempt to give an explanation.

```
. reg EARNINGS HEIGHT

      Source |       SS       df       MS              Number of obs =     500
-------------+------------------------------           F(  1,   498) =    9.23
       Model |  1393.77592     1  1393.77592           Prob > F      =  0.0025
    Residual |  75171.3726   498  150.946531           R-squared     =  0.0182
-------------+------------------------------           Adj R-squared =  0.0162
       Total |  76565.1485   499  153.437171           Root MSE      =  12.286

------------------------------------------------------------------------------
    EARNINGS |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
      HEIGHT |   .4087231   .1345068     3.04   0.003     .1444523    .6729938
       _cons |  -9.26923   9.125089    -1.02   0.310    -27.19765    8.659188
------------------------------------------------------------------------------
```

A1.5 A researcher has data for 50 countries on N, the average number of newspapers purchased per adult in one year, and G, GDP per capita, measured in US \$, and fits the following regression (RSS = residual sum of squares):

$$\widehat{N} = 25.0 + 0.020G \qquad R^2 = 0.06,\ RSS = 4{,}000.0$$

The researcher realises that GDP has been underestimated by \$100 in every country and that N should have been regressed on G^*, where $G^* = G + 100$. Explain, with mathematical proofs, how the following components of the output would have differed:

- the coefficient of GDP
- the intercept
- RSS
- R^2.

A1.6 A researcher with the same model and data as in Exercise A1.5 believes that GDP in each country has been underestimated by 50 per cent and that N should have been regressed on G^*, where $G^* = 2G$. Explain, with mathematical proofs, how the following components of the output would have differed:

- the coefficient of GDP
- the intercept
- RSS
- R^2.

A1.7 Some practitioners of econometrics advocate 'standardising' each variable in a regression by subtracting its sample mean and dividing by its sample standard deviation. Thus, if the original regression specification is:

$$Y_i = \beta_1 + \beta_2 X_i + u_i$$

the revised specification is:

$$Y_i^* = \beta_1^* + \beta_2^* X_i^* + v_i$$

where:

$$Y_i^* = \frac{Y_i - \overline{Y}}{\widehat{\sigma}_Y} \qquad \text{and} \qquad X_i^* = \frac{X_i - \overline{X}}{\widehat{\sigma}_X}$$

$\overline{Y}$ and $\overline{X}$ are the sample means of Y and X, $\widehat{\sigma}_Y$ and $\widehat{\sigma}_X$ are the estimators of the standard deviations of Y and X, defined as the square roots of the estimated variances:

$$\widehat{\sigma}_Y^2 = \frac{1}{n-1}\sum_{i=1}^{n}(Y_i - \overline{Y})^2 \qquad \text{and} \qquad \widehat{\sigma}_X^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \overline{X})^2$$

and n is the number of observations in the sample. We will write the fitted models for the two specifications as:

$$\widehat{Y}_i = \widehat{\beta}_1 + \widehat{\beta}_2 X_i$$

and:

$$\widehat{Y}_i^* = \widehat{\beta}_1^* + \widehat{\beta}_2^* X_i^*.$$

Taking account of the definitions of Y^* and X^*, show that $\widehat{\beta}_1^* = 0$ and that $\widehat{\beta}_2^* = \frac{\widehat{\sigma}_X}{\widehat{\sigma}_Y}\widehat{\beta}_2$. Provide an interpretation of $\widehat{\beta}_2^*$.

A1.8 For the model described in Exercise A1.7, suppose that Y^* is regressed on X^* without an intercept:

$$\widehat{Y}_i^* = \widehat{\beta}_2^{**} X_i^*.$$

Determine how $\widehat{\beta}_2^{**}$ is related to $\widehat{\beta}_2^*$.

A1.9 A variable Y_i is generated as:

$$Y_i = \beta_1 + u_i \tag{1.1}$$

where β_1 is a fixed parameter and u_i is a disturbance term that is independently and identically distributed with expected value 0 and population variance σ_u^2. The least squares estimator of β_1 is $\overline{Y}$, the sample mean of Y. Give a mathematical demonstration that the value of R^2 in such a regression is zero.

1.4 Answers to the starred exercises in the textbook

1.9 The output shows the result of regressing the weight of the respondent in 2004, measured in pounds, on his or her height, measured in inches, using *EAWE* Data Set 21. Provide an interpretation of the coefficients.

```
. reg WEIGHT04 HEIGHT

      Source |       SS       df       MS              Number of obs =     500
-------------+------------------------------           F(  1,   498) =  176.74
       Model |     211309     1      211309            Prob > F      =  0.0000
    Residual |  595389.95   498  1195.56215            R-squared     =  0.2619
-------------+------------------------------           Adj R-squared =  0.2605
       Total |  806698.98   499  1616.63116            Root MSE      =  34.577

------------------------------------------------------------------------------
    WEIGHT04 |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
      HEIGHT |   5.073711     .381639   13.29   0.000      4.32389    5.823532
       _cons |  -177.1703    25.93501   -6.83   0.000    -228.1258   -126.2147
------------------------------------------------------------------------------
```

Answer:

Literally the regression implies that, for every extra inch of height, an individual tends to weigh an extra 5.1 pounds. The intercept, which literally suggests that an individual with no height would weigh −177 pounds, has no meaning.

1.11 A researcher has international cross-sectional data on aggregate wages, W, aggregate profits, P, and aggregate income, Y, for a sample of n countries. By definition:

$$Y_i = W_i + P_i.$$

The regressions:

$$\begin{aligned}\widehat{W}_i &= \widehat{\alpha}_1 + \widehat{\alpha}_2 Y_i \\ \widehat{P}_i &= \widehat{\beta}_1 + \widehat{\beta}_2 Y_i\end{aligned}$$

are fitted using OLS regression analysis. Show that the regression coefficients will automatically satisfy the following equations:

$$\begin{aligned}\widehat{\alpha}_2 + \widehat{\beta}_2 &= 1 \\ \widehat{\alpha}_1 + \widehat{\beta}_1 &= 0.\end{aligned}$$

Explain intuitively why this should be so.

Answer:

$$\begin{aligned}\widehat{\alpha}_2 + \widehat{\beta}_2 &= \frac{\sum\left(Y_i - \overline{Y}\right)\left(W_i - \overline{W}\right)}{\sum\left(Y_i - \overline{Y}\right)^2} + \frac{\sum\left(Y_i - \overline{Y}\right)\left(P_i - \overline{P}\right)}{\sum\left(Y_i - \overline{Y}\right)^2} \\ &= \frac{\sum\left(Y_i - \overline{Y}\right)\left(W_i + P_i - \overline{W} - \overline{P}\right)}{\sum\left(Y_i - \overline{Y}\right)^2} \\ &= \frac{\sum\left(Y_i - \overline{Y}\right)\left(Y_i - \overline{Y}\right)}{\sum\left(Y_i - \overline{Y}\right)^2} \\ &= 1\end{aligned}$$

$$\widehat{\alpha}_1 + \widehat{\beta}_1 = \left(\overline{W} - \widehat{\alpha}_2\overline{Y}\right) + \left(\overline{P} - \widehat{\beta}_2\overline{Y}\right) = \left(\overline{W} + \overline{P}\right) - (\widehat{\alpha}_2 + \widehat{\beta}_2)\overline{Y} = \overline{Y} - \overline{Y} = 0.$$

The intuitive explanation is that the regressions break down income into predicted wages and profits and one would expect the sum of the predicted components of income to be equal to its actual level. The sum of the predicted components is $\widehat{W}_i + \widehat{P}_i = (\widehat{\alpha}_1 + \widehat{\alpha}_2 Y_i) + (\widehat{\beta}_1 + \widehat{\beta}_2 Y_i)$, and in general this will be equal to Y_i only if the two conditions are satisfied.

1.13 Suppose that the units of measurement of X are changed so that the new measure, X^*, is related to the original one by $X_i^* = \mu_2 X_i$. Show that the new estimate of the slope coefficient is $\widehat{\beta}_2/\mu_2$, where $\widehat{\beta}_2$ is the slope coefficient in the original regression.

Answer:

$$\begin{aligned}
\widehat{\beta}_2^* &= \frac{\sum\left(X_i^* - \overline{X}^*\right)\left(Y_i - \overline{Y}\right)}{\sum\left(X_i^* - \overline{X}^*\right)^2} \\
&= \frac{\sum\left(\mu_2 X_i - \mu_2\overline{X}\right)\left(Y_i - \overline{Y}\right)}{\sum\left(\mu_2 X_i - \mu_2\overline{X}\right)^2} \\
&= \frac{\mu_2\sum\left(X_i - \overline{X}\right)\left(Y_i - \overline{Y}\right)}{\mu_2^2\sum\left(X_i - \overline{X}\right)^2} \\
&= \frac{\widehat{\beta}_2}{\mu_2}.
\end{aligned}$$

1.14 Demonstrate that if X is demeaned but Y is left in its original units, the intercept in a regression of Y on demeaned X will be equal to $\overline{Y}$.

Answer:

Let $X_i^* = X_i - \overline{X}$ and $\widehat{\beta}_1^*$ and $\widehat{\beta}_2^*$ be the intercept and slope coefficient in a regression of Y on X^*. Note that $\overline{X}^* = 0$. Then:

$$\widehat{\beta}_1^* = \overline{Y} - \widehat{\beta}_2^*\overline{X}^* = \overline{Y}.$$

The slope coefficient is not affected by demeaning:

$$\widehat{\beta}_2^* = \frac{\sum\left(X_i^* - \overline{X}^*\right)\left(Y_i - \overline{Y}\right)}{\sum\left(X_i^* - \overline{X}^*\right)^2} = \frac{\sum\left([X_i - \overline{X}] - 0\right)\left(Y_i - \overline{Y}\right)}{\sum\left([X_i - \overline{X}] - 0\right)^2} = \widehat{\beta}_2.$$

1.15 The regression output shows the result of regressing weight on height using the same sample as in Exercise 1.9, but with weight and height measured in kilos and centimetres: $WMETRIC = 0.454 * WEIGHT04$ and $HMETRIC = 2.54 * HEIGHT$. Confirm that the estimates of the intercept and slope coefficient are as should be expected from the changes in the units of measurement.

```
. gen WTMETRIC = 0.454*WEIGHT04
. gen HMETRIC = 2.54*HEIGHT

. reg WTMETRIC HMETRIC

      Source |       SS       df       MS              Number of obs =     500
-------------+------------------------------           F(  1,   498) =  176.74
       Model |  43554.1641     1  43554.1641           Prob > F      =  0.0000
    Residual |  122719.394   498  246.424486           R-squared     =  0.2619
-------------+------------------------------           Adj R-squared =  0.2605
       Total |  166273.558   499  333.213544           Root MSE      =  15.698

------------------------------------------------------------------------------
     WMETRIC |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
     HMETRIC |   .9068758   .0682142    13.29   0.000     .7728527    1.040899
       _cons |  -80.43529   11.77449    -6.83   0.000    -103.5691   -57.30148
------------------------------------------------------------------------------
```

Answer:

Abbreviate *WEIGHT04* to W, *HEIGHT* to H, *WMETRIC* to WM, and *HMETRIC* to HM. $WM = 0.454W$ and $HM = 2.54H$. The slope coefficient and intercept for the regression in metric units, $\widehat{\beta}_2^M$ and $\widehat{\beta}_1^M$, are then given by:

$$
\begin{aligned}
\widehat{\beta}_2^{\mathrm{M}} &= \frac{\sum\left(HM_i - \overline{HM}\right)\left(WM_i - \overline{WM}\right)}{\sum\left(HM_i - \overline{HM}\right)^2} \\
&= \frac{\sum 2.54\left(H_i - \overline{H}\right) 0.454\left(W_i - \overline{W}\right)}{\sum 2.54^2\left(H_i - \overline{H}\right)^2} \\
&= 0.179\frac{\sum\left(H_i - \overline{H}\right)\left(W_i - \overline{W}\right)}{\sum\left(H_i - \overline{H}\right)^2} \\
&= 0.179\widehat{\beta}_2 \\
&= 0.179 \times 5.074 \\
&= 0.908 \\
\widehat{\beta}_1^M &= \overline{WM} - \widehat{\beta}_2^M \overline{HM} \\
&= 0.454\overline{W} - \left(\frac{0.454}{2.54}\widehat{\beta}_2\right)(2.54\overline{H}) \\
&= 0.454(\overline{W} - \widehat{\beta}_2\overline{H}) \\
&= 0.454\widehat{\beta}_1 \\
&= 0.454 \times -177.2 \\
&= -80.4.
\end{aligned}
$$

The regression output confirms that the calculations are correct (subject to rounding error in the last digit).

1.16 Consider the regression model:

$$Y_i = \beta_1 + \beta_2 X_i + u_i.$$

It implies:

$$\overline{Y} = \beta_1 + \beta_2 \overline{X} + \bar{u}$$

and hence that:

$$Y_i^* = \beta_2 X_i^* + v_i$$

where $Y_i^* = Y_i - \overline{Y}$, $X_i^* = X_i - \overline{X}$ and $v_i = u_i - \bar{u}$.

Demonstrate that a regression of Y^* on X^* using (1.49) will yield the same estimate of the slope coefficient as a regression of Y on X. Note: (1.49) should be used instead of (1.35) because there is no intercept in this model.

Evaluate the outcome if the slope coefficient were estimated using (1.35), despite the fact that there is no intercept in the model.

Determine the estimate of the intercept if Y^* were regressed on X^* with an intercept included in the regression specification.

Answer:

Let $\widehat{\beta}_2^*$ be the slope coefficient in a regression of Y^* on X^* using (1.49). Then:

$$\widehat{\beta}_2^* = \frac{\sum X_i^* Y_i^*}{\sum X_i^{*2}} = \frac{\sum \left(X_i - \overline{X}\right)\left(Y_i - \overline{Y}\right)}{\sum \left(X_i - \overline{X}\right)^2} = \widehat{\beta}_2.$$

Let $\widehat{\beta}_2^{**}$ be the slope coefficient in a regression of Y^* on X^* using (1.35). Note that $\overline{Y}^*$ and $\overline{X}^*$ are both zero. Then:

$$\widehat{\beta}_2^{**} = \frac{\sum \left(X_i^* - \overline{X}^*\right)\left(Y_i^* - \overline{Y}^*\right)}{\sum \left(X_i^* - \overline{X}^*\right)^2} = \frac{\sum X_i^* Y_i^*}{\sum X_i^{*2}} = \widehat{\beta}_2.$$

Let $\widehat{\beta}_1^{**}$ be the intercept in a regression of Y^* on X^* using (1.35). Then:

$$\widehat{\beta}_1^{**} = \overline{Y}^* - \widehat{\beta}_2^{**}\overline{X}^* = 0.$$

1.18 Demonstrate that the fitted values of the dependent variable are uncorrelated with the residuals in a simple regression model. (This result generalises to the multiple regression case.)

Answer:

The numerator of the sample correlation coefficient for $\widehat{Y}$ and $\widehat{u}$ can be decomposed as follows, using the fact that $\overline{\widehat{u}} = 0$:

$$\begin{aligned}
\frac{1}{n}\sum \left(\widehat{Y}_i - \overline{\widehat{Y}}\right)\left(\widehat{u}_i - \overline{\widehat{u}}\right) &= \frac{1}{n}\sum \left([\widehat{\beta}_1 + \widehat{\beta}_2 X_i] - [\widehat{\beta}_1 + \widehat{\beta}_2 \overline{X}]\right)\widehat{u}_i \\
&= \frac{1}{n}\widehat{\beta}_2 \sum \left(X_i - \overline{X}\right)\widehat{u}_i \\
&= 0
\end{aligned}$$

by (1.65). Hence the correlation is zero.

1.23 Demonstrate that, in a regression with an intercept, a regression of Y on X^* must have the same R^2 as a regression of Y on X, where $X^* = \mu_2 X$.

Answer:

Let the fitted regression of Y on X^* be written $\widehat{Y}_i^* = \widehat{\beta}_1^* + \widehat{\beta}_2^* X_i^*$. $\widehat{\beta}_2^* = \widehat{\beta}_2/\mu_2$ (Exercise 1.13).

$$\widehat{\beta}_1^* = \overline{Y} - \widehat{\beta}_2^* \overline{X}^* = \overline{Y} - \frac{\widehat{\beta}_2}{\mu_2}\mu_2 \overline{X} = \widehat{\beta}_1.$$

Hence:

$$\widehat{Y}_i^* = \widehat{\beta}_1 + \frac{\widehat{\beta}_2}{\mu_2}\mu_2 X_i = \widehat{Y}_i.$$

The fitted and actual values of Y are not affected by the transformation and so R^2 is unaffected.

1.25 The output shows the result of regressing weight in 2011 on height, using *EAWE* Data Set 21. In 2011 the respondents were aged 27–31. Explain why R^2 is lower than in the regression reported in Exercise 1.9.

```
. reg WEIGHT11 HEIGHT

      Source |       SS       df       MS              Number of obs =     500
-------------+------------------------------           F(  1,   498) =  139.97
       Model |  236642.736     1  236642.736           Prob > F      =  0.0000
    Residual |  841926.912   498  1690.61629           R-squared     =  0.2194
-------------+------------------------------           Adj R-squared =  0.2178
       Total |  1078569.65   499  2161.46222           Root MSE      =  41.117

------------------------------------------------------------------------------
    WEIGHT11 |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
      HEIGHT |   5.369246   .4538259    11.83   0.000     4.477597    6.260895
       _cons |  -184.7802    30.8406    -5.99   0.000    -245.3739   -124.1865
------------------------------------------------------------------------------
```

Answer:

The explained sum of squares is actually higher than that in Exercise 1.9. The reason for the fall in R^2 is the huge increase in the total sum of squares, no doubt caused by the cumulative effect of variations in eating habits.

1.5 Answers to the additional exercises

A1.1 Expenditure on food consumed at home increases by 6.3 cents for each dollar of total household expenditure. Literally the intercept implies that \$369 would be spent on food consumed at home if total household expenditure were zero. Obviously, such an interpretation does not make sense. If the explanatory variable were income, and household income were zero, positive expenditure on food at home would still be possible if the household received food stamps or other transfers, but here the explanatory variable is total household expenditure.

A1.2 For each category, the regression sample has been restricted to households with non-zero expenditure. All the slope coefficients are highly significant. Housing has the largest coefficient, as one should expect. Surprisingly, it is followed by education. However, most households spent nothing at all on this category. For those that did, it was important.

EXP			
	n	$\widehat{\beta}_2$	R^2
ADM	2,815	0.0235	0.228
CLOT	4,500	0.0316	0.176
DOM	1,661	0.0409	0.134
EDUC	561	0.1202	0.241
ELEC	5,828	0.0131	0.180
FDAW	5,102	0.0527	0.354
FDHO	6,334	0.0627	0.351
FOOT	1,827	0.0058	0.082
FURN	487	0.0522	0.102
GASO	5,710	0.0373	0.278
HEAL	4,802	0.0574	0.174
HOUS	6,223	0.1976	0.469
LIFE	1,253	0.0193	0.101
LOCT	692	0.0068	0.059
MAPP	399	0.0329	0.102
PERS	3,817	0.0069	0.213
READ	2,287	0.0048	0.104
SAPP	1,037	0.0045	0.034
TELE	5,788	0.0160	0.268
TEXT	992	0.0040	0.051
TOB	1,155	0.0165	0.088
TOYS	2,504	0.0145	0.076
TRIP	516	0.0466	0.186

A1.3 The summary data indicate that, on average, the respondents put on 13 pounds over the period 2004–2011. Was this due to the relatively heavy becoming even heavier, or to a general increase in weight? The regression output indicates that weight in 2011 was approximately equal to weight in 2004 plus 17 pounds, so the second explanation appears to be the correct one. Note that this is an instance where the constant term can be given a meaningful interpretation and where it is as of much interest as the slope coefficient. The R^2 indicates that 2004 weight accounts for 71 per cent of the variance in 2011 weight, so other factors are important.

A1.4 The slope coefficient indicates that hourly earnings increase by 41 cents for every extra inch of height. The negative intercept has no possible interpretation. The interpretation of the slope coefficient is obviously highly implausible, so we know that something must be wrong with the model. The explanation is that this is a very poorly specified earnings function and that, in particular, we are failing to control for the sex of the respondent. Later on, in Chapter 5, we will find that males earn more than females, controlling for observable characteristics. Males also tend to be taller. Hence we find an apparent positive association between earnings and height in a simple regression. Note that R^2 is very low.

A1.5 *The coefficient of GDP:* Let the revised measure of GDP be denoted G^*, where $G^* = G + 100$. Since $G_i^* = G_i + 100$ for all i, $\overline{G}^* = \overline{G} + 100$ and so $G_i^* - \overline{G}^* = G_i - \overline{G}$ for all i. Hence the new slope coefficient is:

$$\widehat{\beta}_2^* = \frac{\sum \left(G_i^* - \overline{G}^*\right)\left(N_i - \overline{N}\right)}{\sum \left(G_i^* - \overline{G}^*\right)^2} = \frac{\sum \left(G_i - \overline{G}\right)\left(N_i - \overline{N}\right)}{\sum \left(G_i - \overline{G}\right)^2} = \widehat{\beta}_2.$$

The coefficient is unchanged.

The intercept: The new intercept is:

$$\widehat{\beta}_1^* = \overline{N} - \widehat{\beta}_2^* \overline{G}^* = \overline{N} - \widehat{\beta}_2\left(\overline{G} + 100\right) = \widehat{\beta}_1 - 100\widehat{\beta}_2 = 23.0.$$

RSS: The residual in observation i in the new regression, $\widehat{u}_i^*$, is given by:

$$\widehat{u}_i^* = N_i - \widehat{\beta}_1^* - \widehat{\beta}_2^* G_i^* = N_i - (\widehat{\beta}_1 - 100\widehat{\beta}_2) - \widehat{\beta}_2(G_i + 100) = \widehat{u}_i$$

the residual in the original regression. Hence RSS is unchanged.

R^2:

$$R^2 = 1 - \frac{RSS}{\sum \left(N_i - \overline{N}\right)^2}$$

and is unchanged since RSS and $\sum \left(N_i - \overline{N}\right)^2$ are unchanged.

Note that this makes sense intuitively. R^2 is unit-free and so it is not possible for the overall fit of a relationship to be affected by the units of measurement.

A1.6 *The coefficient of GDP:* Let the revised measure of GDP be denoted G^*, where $G^* = 2G$. Since $G_i^* = 2G_i$ for all i, $\overline{G}^* = 2\overline{G}$ and so $G_i^* - \overline{G}^* = 2\left(G_i - \overline{G}\right)$ for all i. Hence the new slope coefficient is:

$$\begin{aligned}
\widehat{\beta}_2^* &= \frac{\sum \left(G_i^* - \overline{G}^*\right)\left(N_i - \overline{N}\right)}{\sum \left(G_i^* - \overline{G}^*\right)^2} \\
&= \frac{\sum 2\left(G_i - \overline{G}\right)\left(N_i - \overline{N}\right)}{\sum 4\left(G_i - \overline{G}\right)^2} \\
&= \frac{2\sum \left(G_i - \overline{G}\right)\left(N_i - \overline{N}\right)}{4\sum \left(G_i - \overline{G}\right)^2} \\
&= \frac{\widehat{\beta}_2}{2} \\
&= 0.010
\end{aligned}$$

where $\widehat{\beta}_2 = 0.020$ is the slope coefficient in the original regression.

The intercept: The new intercept is:

$$\widehat{\beta}_1^* = \overline{N} - \widehat{\beta}_2^* \overline{G}^* = \overline{N} - \frac{\widehat{\beta}_2}{2} 2\overline{G} = \overline{N} - \widehat{\beta}_2 \overline{G} = \widehat{\beta}_1 = 25.0$$

the original intercept.

RSS: The residual in observation i in the new regression, $\widehat{u}_i^*$, is given by:

$$\widehat{u}_i^* = N_i - \widehat{\beta}_1^* - \widehat{\beta}_2^* G_i^* = N_i - \widehat{\beta}_1 - \frac{\widehat{\beta}_2}{2} 2G_i = \widehat{u}_i$$

the residual in the original regression. Hence *RSS* is unchanged.

R^2:

$$R^2 = 1 - \frac{RSS}{\sum \left(N_i - \overline{N}\right)^2}$$

and is unchanged since *RSS* and $\sum \left(N_i - \overline{N}\right)^2$ are unchanged. As in Exercise A1.6, this makes sense intuitively.

A1.7 By construction, $\overline{Y}^* = \overline{X}^* = 0$. So $\widehat{\beta}_1^* = \overline{Y}^* - \widehat{\beta}_2^* \overline{X}^* = 0$.

$$\begin{aligned}
\widehat{\beta}_2^* &= \frac{\sum \left(X_i^* - \overline{X}^*\right)\left(Y_i^* - \overline{Y}^*\right)}{\sum \left(X_i^* - \overline{X}^*\right)^2} \\
&= \frac{\sum X_i^* Y_i^*}{\sum X_i^{*2}} \\
&= \frac{\sum \left(\frac{X_i - \bar{X}}{\widehat{\sigma}_X}\right)\left(\frac{Y_i - \bar{Y}}{\widehat{\sigma}_Y}\right)}{\sum \left(\frac{X_i - \bar{X}}{\widehat{\sigma}_X}\right)^2} \\
&= \frac{\widehat{\sigma}_X}{\widehat{\sigma}_Y} \frac{\sum \left(X_i - \overline{X}\right)\left(Y_i - \overline{Y}\right)}{\sum \left(X_i - \overline{X}\right)^2} \\
&= \frac{\widehat{\sigma}_X}{\widehat{\sigma}_Y} \widehat{\beta}_2.
\end{aligned}$$

$\widehat{\beta}_2^*$ provides an estimate of the effect on Y, in terms of standard deviations of Y, of a one-standard deviation change in X.

A1.8 We have:

$$\widehat{\beta}_2^{**} = \frac{\sum X_i^* Y_i^*}{\sum X_i^{*2}} = \frac{\sum \left(X_i^* - \overline{X}^*\right)\left(Y_i^* - \overline{Y}^*\right)}{\sum \left(X_i^* - \overline{X}^*\right)^2} = \widehat{\beta}_2^*.$$

A1.9 We have:

$$R^2 = \frac{\sum \left(\widehat{Y}_i - \overline{Y}\right)^2}{\sum \left(Y_i - \overline{Y}\right)^2}$$

and $\widehat{Y}_i = \overline{Y}$ for all i.

Chapter 2
Properties of the regression coefficients and hypothesis testing

2.1 Overview

Chapter 1 introduced least squares regression analysis, a mathematical technique for fitting a relationship given suitable data on the variables involved. It is a fundamental chapter because much of the rest of the text is devoted to extending the least squares approach to handle more complex models, for example models with multiple explanatory variables, nonlinear models, and models with qualitative explanatory variables.

However, the mechanics of fitting regression equations are only part of the story. We are equally concerned with assessing the performance of our regression techniques and with developing an understanding of why they work better in some circumstances than in others. Chapter 2 is the starting point for this objective and is thus equally fundamental. In particular, it shows how two of the three main criteria for assessing the performance of estimators, unbiasedness and efficiency, are applied in the context of a regression model. The third criterion, consistency, will be considered in Chapter 8.

2.2 Learning outcomes

After working through the corresponding chapter in the text, studying the corresponding slideshows, and doing the starred exercises in the text and the additional exercises in this subject guide, you should be able to explain what is meant by:

- cross-sectional, time series, and panel data
- unbiasedness of OLS regression estimators
- variance and standard errors of regression coefficients and how they are determined
- Gauss–Markov theorem and efficiency of OLS regression estimators
- two-sided t tests of hypotheses relating to regression coefficients and one-sided t tests of hypotheses relating to regression coefficients
- F tests of goodness of fit of a regression equation

in the context of a regression model. The chapter is a long one and you should take your time over it because it is essential that you develop a perfect understanding of every detail.

2.3 Further material

Derivation of the expression for the variance of the naïve estimator in Section 2.3.

The variance of the naïve estimator in Section 2.3 and Exercise 2.9 is not of any great interest in itself, but its derivation provides an example of how one obtains expressions for variances of estimators in general.

In Section 2.3 we considered the naïve estimator of the slope coefficient derived by joining the first and last observations in a sample and calculating the slope of that line:

$$\widehat{\beta}_2 = \frac{Y_n - Y_1}{X_n - X_1}.$$

It was demonstrated that the estimator could be decomposed as:

$$\widehat{\beta}_2 = \beta_2 + \frac{u_n - u_1}{X_n - X_1}$$

and hence that $E(\widehat{\beta}_2) = \beta_2$.

The population variance of a random variable X is defined to be $E([X - \mu_X]^2)$ where $\mu_X = E(X)$. Hence the population variance of $\widehat{\beta}_2$ is given by:

$$\sigma^2_{\widehat{\beta}_2} = E\left([\widehat{\beta}_2 - \beta_2]^2\right) = E\left(\left[\left\{\beta_2 + \frac{u_n - u_1}{X_n - X_1}\right\} - \beta_2\right]^2\right) = E\left(\left[\frac{u_n - u_1}{X_n - X_1}\right]^2\right).$$

On the assumption that X is nonstochastic, this can be written as:

$$\sigma^2_{\widehat{\beta}_2} = \left[\frac{1}{X_n - X_1}\right]^2 E\left([u_n - u_1]^2\right).$$

Expanding the quadratic, we have:

$$\begin{aligned}\sigma^2_{\widehat{\beta}_2} &= \left[\frac{1}{X_n - X_1}\right]^2 E\left(u_n^2 + u_1^2 - 2u_n u_1\right) \\ &= \left[\frac{1}{X_n - X_1}\right]^2 \left[E(u_n^2) + E(u_1^2) - 2E(u_n u_1)\right].\end{aligned}$$

Each value of the disturbance term is drawn randomly from a distribution with mean 0 and population variance σ_u^2, so $E(u_n^2)$ and $E(u_1^2)$ are both equal to σ_u^2. u_n and u_1 are drawn independently from the distribution, so $E(u_n u_1) = E(u_n)E(u_1) = 0$. Hence:

$$\sigma^2_{\widehat{\beta}_2} = \frac{2\sigma_u^2}{(X_n - X_1)^2} = \frac{\sigma_u^2}{\frac{1}{2}(X_n - X_1)^2}.$$

Define $A = \frac{1}{2}(X_1 + X_n)$, the average of X_1 and X_n, and $D = X_n - A = A - X_1$. Then:

$$\begin{aligned}
\frac{1}{2}(X_n - X_1)^2 &= \frac{1}{2}(X_n - A + A - X_1)^2 \\
&= \frac{1}{2}\left[(X_n - A)^2 + (A - X_1)^2 + 2(X_n - A)(A - X_1)\right] \\
&= \frac{1}{2}\left[D^2 + D^2 + 2(D)(D)\right] = 2D^2 \\
&= (X_n - A)^2 + (A - X_1)^2 \\
&= (X_n - A)^2 + (X_1 - A)^2 \\
&= (X_n - \overline{X} + \overline{X} - A)^2 + (X_1 - \overline{X} + \overline{X} - A)^2 \\
&= (X_n - \overline{X})^2 + (\overline{X} - A)^2 + 2(X_n - \overline{X})(\overline{X} - A) \\
&\quad +(X_1 - \overline{X})^2 + (\overline{X} - A)^2 + 2(X_1 - \overline{X})(\overline{X} - A) \\
&= (X_1 - \overline{X})^2 + (X_n - \overline{X})^2 + 2(\overline{X} - A)^2 + 2(X_1 + X_n - 2\overline{X})(\overline{X} - A) \\
&= (X_1 - \overline{X})^2 + (X_n - \overline{X})^2 + 2(\overline{X} - A)^2 + 2(2A - 2\overline{X})(\overline{X} - A) \\
&= (X_1 - \overline{X})^2 + (X_n - \overline{X})^2 - 2(\overline{X} - A)^2 \\
&= (X_1 - \overline{X})^2 + (X_n - \overline{X})^2 - 2(A - \overline{X})^2 \\
&= (X_1 - \overline{X})^2 + (X_n - \overline{X})^2 - \frac{1}{2}(X_1 + X_n - 2\overline{X})^2.
\end{aligned}$$

Hence we obtain the expression in Exercise 2.9. There must be a shorter proof.

2.4 Additional exercises

A2.1 A variable Y depends on a nonstochastic variable X with the relationship:

$$Y = \beta_1 + \beta_2 X + u$$

where u is a disturbance term that satisfies the regression model assumptions. Given a sample of n observations, a researcher decides to estimate β_2 using the expression:

$$\widehat{\beta}_2 = \frac{\sum X_i Y_i}{\sum X_i^2}.$$

(This is the OLS estimator of β_2 for the model $Y = \beta_2 X + u$.)

(a) Demonstrate that $\widehat{\beta}_2$ is in general a biased estimator of β_2.

(b) Discuss whether it is possible to determine the sign of the bias.

(c) Demonstrate that $\widehat{\beta}_2$ is unbiased if $\beta_1 = 0$.

(d) Demonstrate that $\widehat{\beta}_2$ is unbiased if $\overline{X} = 0$.

A2.2 A variable Y_i is generated as:

$$Y_i = \beta_1 + u_i$$

where β_1 is a fixed parameter and u_i is a disturbance term that is independently and identically distributed with expected value 0 and population variance σ_u^2. The least squares estimator of β_1 is $\overline{Y}$, the sample mean of Y. However, a researcher believes that Y is a linear function of another variable X and uses ordinary least squares to fit the relationship:

$$\widehat{Y} = \widehat{\beta}_1 + \widehat{\beta}_2 X$$

calculating $\widehat{\beta}_1$ as $\overline{Y} - \widehat{\beta}_2\overline{X}$, where $\overline{X}$ is the sample mean of X. X may be assumed to be a nonstochastic variable. Determine whether the researcher's estimator $\widehat{\beta}_1$ is biased or unbiased, and if biased, determine the direction of the bias.

A2.3 With the model described in Exercise A2.2, standard theory states that the population variance of the researcher's estimator of β_1 is:

$$\sigma_u^2 \left[\frac{1}{n} + \frac{\overline{X}^2}{\sum \left(X_i - \overline{X}\right)^2} \right].$$

In general, this is larger than the population variance of $\overline{Y}$, which is σ_u^2/n. Explain the implications of the difference in the variances.

In the special case where $\overline{X} = 0$, the variances are the same. Give an intuitive explanation.

A2.4 A variable Y depends on a nonstochastic variable X with the relationship:

$$Y = \beta_1 + \beta_2 X + u$$

where u is a disturbance term that satisfies the regression model assumptions. Given a sample of n observations, a researcher decides to estimate β_2 using the expression:

$$\widehat{\beta}_2 = \frac{\sum X_i Y_i}{\sum X_i^2}.$$

It can be shown that the population variance of this estimator is $\sigma_u^2 / \sum X_i^2$.

We saw in Exercise A2.1 that $\widehat{\beta}_2$ is in general a biased estimator of β_2. However, if either $\beta_1 = 0$ or $\overline{X} = 0$, the estimator is unbiased. What can be said in this case about the efficiency of the estimator in these two cases, comparing it with the estimator:

$$\frac{\sum \left(X_i - \overline{X}\right)\left(Y_i - \overline{Y}\right)}{\sum \left(X_i - \overline{X}\right)^2}?$$

Returning to the general case where $\beta_1 \neq 0$ and $\overline{X} \neq 0$, suppose that there is very little variation in X in the sample. Is it possible that $\widehat{\beta}_2$ might be a better estimator than the OLS estimator?

A2.5 Using the output for the regression in Exercise A1.1, reproduced below, perform appropriate statistical tests.

```
. reg FDHO EXP if FDHO>0

      Source |       SS       df       MS              Number of obs =    6334
-------------+------------------------------           F(  1,  6332) = 3431.01
       Model |   972602566     1   972602566           Prob > F      =  0.0000
    Residual |  1.7950e+09  6332  283474.003           R-squared     =  0.3514
-------------+------------------------------           Adj R-squared =  0.3513
       Total |  2.7676e+09  6333   437006.15           Root MSE      =  532.42

------------------------------------------------------------------------------
        FDHO |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
         EXP |   .0627099   .0010706    58.57   0.000     .0606112    .0648086
       _cons |   369.4418   10.65718    34.67   0.000     348.5501    390.3334
------------------------------------------------------------------------------
```

A2.6 Using the output for your regression in Exercise A1.2, perform appropriate statistical tests.

A2.7 Using the output for the regression of weight in 2004 on height in Exercise 1.9, reproduced below, perform appropriate statistical tests.

```
. reg WEIGHT04 HEIGHT

      Source |       SS       df       MS              Number of obs =     500
-------------+------------------------------           F(  1,   498) =  176.74
       Model |     211309     1      211309            Prob > F      =  0.0000
    Residual |  595389.95   498  1195.56215            R-squared     =  0.2619
-------------+------------------------------           Adj R-squared =  0.2605
       Total |  806698.95   499  1616.63116            Root MSE      =  34.577

------------------------------------------------------------------------------
    WEIGHT04 |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
      HEIGHT |   5.073711    .381639    13.29   0.000      4.32389    5.823532
       _cons |  -177.1703   25.93501    -6.83   0.000    -228.1258   -126.2147
------------------------------------------------------------------------------
```

A2.8 Using the output for the regression of earnings on height in Exercise A1.4, reproduced below, perform appropriate statistical tests.

```
. reg EARNINGS HEIGHT

      Source |       SS       df       MS              Number of obs =     500
-------------+------------------------------           F(  1,   498) =    9.23
       Model |  1393.77592     1  1393.77592           Prob > F      =  0.0025
    Residual |  75171.3726   498  150.946531           R-squared     =  0.0182
-------------+------------------------------           Adj R-squared =  0.0162
       Total |  76565.1485   499  153.437171           Root MSE      =  12.286

------------------------------------------------------------------------------
    EARNINGS |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
      HEIGHT |   .4087231   .1345068     3.04   0.003     .1444523    .6729938
       _cons |   -9.26923   9.125089    -1.02   0.310    -27.19765    8.659188
------------------------------------------------------------------------------
```

A2.9 Explain whether it would be justifiable to use a one-sided test on the slope coefficient in the regression of the rate of growth of employment on the rate of growth of GDP in Exercise 2.20.

A2.10 Explain whether it would be justifiable to use a one-sided test on the slope coefficient in the regression of weight on height in Exercise 1.9.

A2.11 With the information given in Exercise A1.5, how would the change in the measurement of GDP affect:

- the standard error of the coefficient of GDP
- the F statistic for the equation?

A2.12 With the information given in Exercise A1.6, how would the change in the measurement of GDP affect:

- the standard error of the coefficient of GDP
- the F statistic for the equation?

A2.13 [This is a continuation of Exercise 1.16 in the text.] A sample of data consists of n observations on two variables, Y and X. The true model is:

$$Y_i = \beta_1 + \beta_2 X_i + u_i$$

where β_1 and β_2 are parameters and u is a disturbance term that satisfies the usual regression model assumptions. In view of the true model:

$$\overline{Y} = \beta_1 + \beta_2 \overline{X} + \bar{u}$$

where $\overline{Y}$, $\overline{X}$, and $\bar{u}$ are the sample means of Y, X, and u. Subtracting the second equation from the first, one obtains:

$$Y_i^* = \beta_2 X_i^* + u_i^*$$

where $Y_i^* = Y_i - \overline{Y}$, $X_i^* = X_i - \overline{X}$ and $u_i^* = u_i - \bar{u}$. Note that, by construction, the sample means of Y^*, X^*, and u^* are all equal to zero.

One researcher fits:

$$\widehat{Y} = \widehat{\beta}_1 + \widehat{\beta}_2 X. \qquad (1)$$

A second researcher fits:

$$\widehat{Y}^* = \widehat{\beta}_1^* + \widehat{\beta}_2^* X^*. \qquad (2)$$

[Note: The second researcher included an intercept in the specification.]

- Comparing regressions (1) and (2), demonstrate that $\widehat{Y}_i^* = \widehat{Y}_i - \overline{Y}$.
- Demonstrate that the residuals in (2) are identical to the residuals in (1).
- Demonstrate that the OLS estimator of the variance of the disturbance term in (2) is equal to that in (1).
- Explain how the standard error of the slope coefficient in (2) is related to that in (1).

- Explain how R^2 in (2) is related to R^2 in (1).
- Explain why, theoretically, the specification (2) of the second researcher is incorrect and he should have fitted:

$$\widehat{Y}^* = \widehat{\beta}_2^* X^* \qquad (3)$$

  not including a constant in his specification.
- If the second researcher had fitted (3) instead of (2), how would this have affected his estimator of β_2? Would dropping the unnecessary intercept lead to a gain in efficiency?

A2.14 For the model described in Exercise A1.7, show that $\widehat{Y}_i^* = (\widehat{Y}_i - \overline{Y})/\widehat{\sigma}_Y$, and thus that $\widehat{u}_i^* = \widehat{u}_i/\widehat{\sigma}_Y$, where $\widehat{Y}_i^*$ and $\widehat{u}_i^*$ are the fitted value of Y_i^* and the residual in the transformed model.

Hence show that:

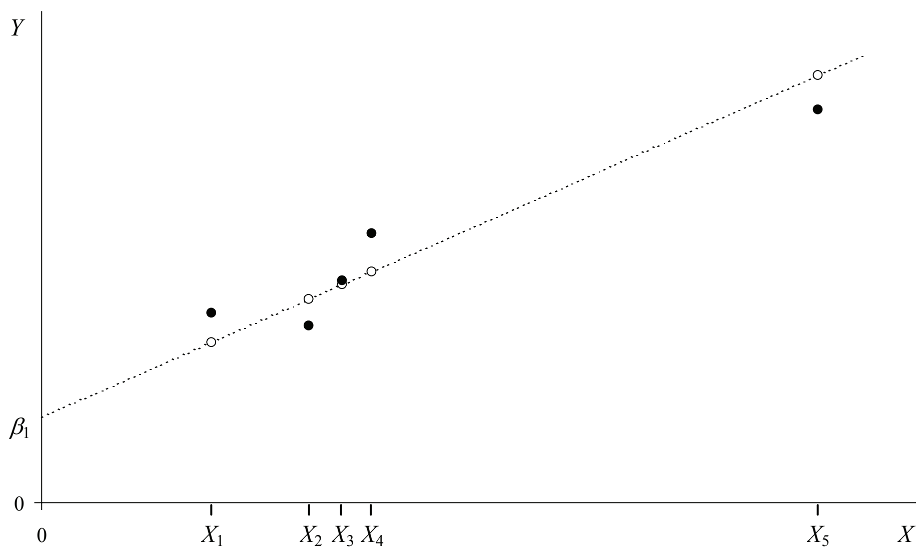
$$\text{s.e.}(\widehat{\beta}_2^*) = \frac{\widehat{\sigma}_X}{\widehat{\sigma}_Y} \times \text{s.e.}(\widehat{\beta}_2).$$

Hence find the relationship between the t statistic for $\widehat{\beta}_2^*$ and the t statistic for $\widehat{\beta}_2$ and the relationship between R^2 for the original specification and R^2 for the revised specification.

A2.15 A variable Y_i is generated as:

$$Y_i = \beta_1 + \beta_2 X_i + u_i \qquad (1)$$

where β_1 and β_2 are fixed parameters and u_i is a disturbance term that satisfies the regression model assumptions. The values of X are fixed and are as shown in the figure. Four of them, X_1 to X_4, are close together. The fifth, X_5, is much larger. The corresponding values that Y would take, if there were no disturbance term, are given by the circles on the line. The presence of the disturbance term in the model causes the actual values of Y in a sample to be different. The solid black circles depict a typical sample of observations.



Discuss the advantages and disadvantages of dropping the observation corresponding to X_5 when regressing Y on X. If you keep the observation in the sample, will this cause the regression estimates to be biased?

2.5 Answers to the starred exercises in the textbook

2.1 Derive the decomposition of $\widehat{\beta}_1$ shown in equation (2.29):

$$\widehat{\beta}_1 = \beta_1 + \sum c_i u_i$$

where $c_i = \frac{1}{n} - a_i\overline{X}$ and a_i is defined in equation (2.23).

Answer:

$$\begin{aligned}
\widehat{\beta}_1 = \overline{Y} - \widehat{\beta}_2\overline{X} &= \left(\beta_1 + \beta_2\overline{X} + \bar{u}\right) - \overline{X}\left(\beta_2 + \sum a_i u_i\right) \\
&= \beta_1 + \frac{1}{n}\sum u_i - \overline{X}\sum a_i u_i \\
&= \beta_1 + \sum c_i u_i.
\end{aligned}$$

2.5 An investigator correctly believes that the relationship between two variables X and Y is given by:

$$Y_i = \beta_1 + \beta_2 X_i + u_i.$$

Given a sample of observations on Y, X, and a third variable Z (which is not a determinant of Y), the investigator estimates β_2 as:

$$\frac{\sum\left(Z_i - \overline{Z}\right)\left(Y_i - \overline{Y}\right)}{\sum\left(Z_i - \overline{Z}\right)\left(X_i - \overline{X}\right)}.$$

Demonstrate that this estimator is unbiased.

Answer:

Noting that $Y_i - \overline{Y} = \beta_2\left(X_i - \overline{X}\right) + u_i - \bar{u}$, we have:

$$\begin{aligned}
\widehat{\beta}_2 &= \frac{\sum\left(Z_i - \overline{Z}\right)\left(Y_i - \overline{Y}\right)}{\sum\left(Z_i - \overline{Z}\right)\left(X_i - \overline{X}\right)} \\
&= \frac{\sum\left(Z_i - \overline{Z}\right)\beta_2\left(X_i - \overline{X}\right) + \sum\left(Z_i - \overline{Z}\right)\left(u_i - \bar{u}\right)}{\sum\left(Z_i - \overline{Z}\right)\left(X_i - \overline{X}\right)} \\
&= \beta_2 + \frac{\sum\left(Z_i - \overline{Z}\right)\left(u_i - \bar{u}\right)}{\sum\left(Z_i - \overline{Z}\right)\left(X_i - \overline{X}\right)}.
\end{aligned}$$

Hence:

$$E(\widehat{\beta}_2) = \beta_2 + \frac{\sum \left(Z_i - \overline{Z}\right) E\left(u_i - \bar{u}\right)}{\sum \left(Z_i - \overline{Z}\right)\left(X_i - \overline{X}\right)} = \beta_2.$$

2.8 Using the decomposition of $\widehat{\beta}_1$ obtained in Exercise 2.1, derive the expression for $\sigma^2_{\widehat{\beta}_1}$ given in equation (2.42).

Answer:

$\widehat{\beta}_1 = \beta_1 + \sum c_i u_i$, where $c_i = \frac{1}{n} - a_i \overline{X}$, and $E(\widehat{\beta}_1) = \beta_1$. Hence:

$$\sigma^2_{\widehat{\beta}_1} = E\left[\left(\sum c_i u_i\right)^2\right] = \sigma_u^2 \sum c_i^2 = \sigma_u^2 \left(n\frac{1}{n^2} - 2\frac{\overline{X}}{n}\sum a_i + \overline{X}^2 \sum a_i^2\right).$$

From Box 2.2, $\sum a_i = 0$ and:

$$\sum a_i^2 = \frac{1}{\sum \left(X_i - \overline{X}\right)^2}.$$

Hence:

$$\sigma^2_{\widehat{\beta}_1} = \sigma_u^2 \left(\frac{1}{n} + \frac{\overline{X}^2}{\sum \left(X_i - \overline{X}\right)^2}\right).$$

2.9 Given the decomposition in Exercise 2.2 of the OLS estimator of β_2 in the model $Y_i = \beta_2 X_i + u_i$, demonstrate that the variance of the slope coefficient is given by:

$$\sigma^2_{\widehat{\beta}_2} = \frac{\sigma_u^2}{\sum X_j^2}.$$

Answer:

$\widehat{\beta}_2 = \beta_2 + \sum_{i=1}^{n} d_i u_i$, where $d_i = X_i / \sum_{j=1}^{n} X_j^2$, and $E(\widehat{\beta}_2) = \beta_2$. Hence:

$$\begin{aligned}
\sigma^2_{\widehat{\beta}_2} = E\left[\left(\sum_{i=1}^{n} d_i u_i\right)^2\right] = \sigma_u^2 \sum_{i=1}^{n} d_i^2 &= \sigma_u^2 \sum_{i=1}^{n} \left(\frac{X_i^2}{\left(\sum_{j=1}^{n} X_j^2\right)^2}\right) \\
&= \frac{\sigma_u^2}{\left(\sum_{j=1}^{n} X_j^2\right)^2} \sum_{i=1}^{n} X_i^2 \\
&= \frac{\sigma_u^2}{\sum_{j=1}^{n} X_j^2}.
\end{aligned}$$

2.12 It can be shown that the variance of the estimator of the slope coefficient in Exercise 2.5:

$$\frac{\sum\left(Z_i-\overline{Z}\right)\left(Y_i-\overline{Y}\right)}{\sum\left(Z_i-\overline{Z}\right)\left(X_i-\overline{X}\right)}$$

is given by:

$$\sigma^2_{\widehat{\beta}_2}=\frac{\sigma^2_u}{\sum\left(X_i-\overline{X}\right)^2}\times\frac{1}{r^2_{XZ}}$$

where r_{XZ} is the correlation between X and Z. What are the implications for the efficiency of the estimator?

Answer:

If Z happens to be an exact linear function of X, the population variance will be the same as that of the OLS estimator. Otherwise $1/r^2_{XZ}$ will be greater than 1, the variance will be larger, and so the estimator will be less efficient.

2.15 Suppose that the true relationship between Y and X is $Y_i=\beta_1+\beta_2X_i+u_i$ and that the fitted model is $\widehat{Y}_i=\widehat{\beta}_1+\widehat{\beta}_2X_i$. In Exercise 1.13 it was shown that if $X_i^*=\mu_2X_i$, and Y is regressed on X^*, the slope coefficient $\widehat{\beta}_2^*=\widehat{\beta}_2/\mu_2$. How will the standard error of $\widehat{\beta}_2^*$ be related to the standard error of $\widehat{\beta}_2$?

Answer:

In Exercise 1.23 it was demonstrated that the fitted values of Y would be the same. This means that the residuals are the same, and hence $\widehat{\sigma}^2_u$, the estimator of the variance of the disturbance term, is the same. The standard error of $\widehat{\beta}_2^*$ is then given by:

$$\begin{aligned}
\text{s.e.}(\widehat{\beta}_2^*) &= \sqrt{\frac{\widehat{\sigma}^2_u}{\sum\left(X_i^*-\overline{X}^*\right)^2}} \\
&= \sqrt{\frac{\widehat{\sigma}^2_u}{\sum\left(\mu_2X_i-\mu_2\overline{X}\right)^2}} \\
&= \sqrt{\frac{\widehat{\sigma}^2_u}{\mu_2^2\sum\left(X_i-\overline{X}\right)^2}} \\
&= \frac{1}{\mu_2}\text{s.e.}(\widehat{\beta}_2).
\end{aligned}$$

2.17 A researcher with a sample of 50 individuals with similar education, but differing amounts of training, hypothesises that hourly earnings, *EARNINGS*, may be related to hours of training, *TRAINING*, according to the relationship:

$$EARNINGS=\beta_1+\beta_2TRAINING+u.$$

He is prepared to test the null hypothesis $H_0 : \beta_2 = 0$ against the alternative hypothesis $H_1 : \beta_2 \neq 0$ at the 5 per cent and 1 per cent levels. What should he report:

(a) if $\widehat{\beta}_2 = 0.30$, s.e.$(\widehat{\beta}_2) = 0.12$?

(b) if $\widehat{\beta}_2 = 0.55$, s.e.$(\widehat{\beta}_2) = 0.12$?

(c) if $\widehat{\beta}_2 = 0.10$, s.e.$(\widehat{\beta}_2) = 0.12$?

(d) if $\widehat{\beta}_2 = -0.27$, s.e.$(\widehat{\beta}_2) = 0.12$?

Answer:

There are 48 degrees of freedom, and hence the critical values of t at the 5 per cent, 1 per cent, and 0.1 per cent levels are 2.01, 2.68, and 3.51, respectively.

(a) The t statistic is 2.50. Reject H_0 at the 5 per cent level but not at the 1 per cent level.

(b) $t = 4.58$. Reject at the 0.1 per cent level.

(c) $t = 0.83$. Fail to reject at the 5 per cent level.

(d) $t = -2.25$. Reject H_0 at the 5 per cent level but not at the 1 per cent level.

2.22 Explain whether it would have been possible to perform one-sided tests instead of two-sided tests in Exercise 2.17. If you think that one-sided tests are justified, perform them and state whether the use of a one-sided test makes any difference.

Answer:

First, there should be a discussion of whether the parameter β_2 in:

$$EARNINGS = \beta_1 + \beta_2 TRAINING + u$$

can be assumed not to be negative. The objective of training is to impart skills. It would be illogical for an individual with greater skills to be paid less on that account, and so we can argue that we can rule out $\beta_2 < 0$. We can then perform a one-sided test. With 48 degrees of freedom, the critical values of t at the 5 per cent, 1 per cent, and 0.1 per cent levels are 1.68, 2.40, and 3.26, respectively.

(a) The t statistic is 2.50. We can now reject H_0 at the 1 per cent level (but not at the 0.1 per cent level).

(b) $t = 4.58$. Not affected by the change. Reject at the 0.1 per cent level.

(c) $t = 0.83$. Not affected by the change. Fail to reject at the 5 per cent level.

(d) $t = -2.25$. Reject H_0 at the 5 per cent level but not at the 1 per cent level. Here there is a problem because the coefficient has an unexpected sign and is large enough to reject H_0 at the 5 per cent level with a two-sided test.

In principle we should ignore this and fail to reject H_0. Admittedly, the likelihood of such a large negative t statistic occurring under H_0 is very small, but it would be smaller still under the alternative hypothesis $H_1 : \beta_2 > 0$.

However, we should consider two further possibilities. One is that the justification for a one-sided test is incorrect. For example, some jobs pay relatively low wages because they offer training that is valued by the employee.

Apprenticeships are the classic example. Alternatively, workers in some low-paid occupations may, for technical reasons, receive a relatively large amount of training. In either case, the correlation between training and earnings might be negative instead of positive.

Another possible reason for a coefficient having an unexpected sign is that the model is misspecified in some way. For example, the coefficient might be distorted by omitted variable bias, to be discussed in Chapter 6.

2.27 Suppose that the true relationship between Y and X is $Y_i = \beta_1 + \beta_2 X_i + u_i$ and that the fitted model is $\widehat{Y}_i = \widehat{\beta}_1 + \widehat{\beta}_2 X_i$. In Exercise 1.13 it was shown that if $X_i^* = \mu_2 X_i$, and Y is regressed on X^*, the slope coefficient $\widehat{\beta}_2^* = \widehat{\beta}_2/\mu_2$. How will the t statistic for $\widehat{\beta}_2^*$ be related to the t statistic for $\widehat{\beta}_2$? (See also Exercise 2.15.)

Answer:

In Exercise 2.15 it was shown that s.e.$(\widehat{\beta}_2^*) = $ s.e.$(\widehat{\beta}_2)/\mu_2$. Hence the t statistic is unaffected by the transformation.

Alternatively, since we saw in Exercise 1.23 that R^2 must be the same, it follows that the F statistic for the equation must be the same. For a simple regression the F statistic is the square of the t statistic on the slope coefficient, so the t statistic must be the same.

2.30 Calculate the 95 per cent confidence interval for β_2 in the price inflation/wage inflation example:

$$\begin{aligned}\widehat{p} &= \underset{(0.05)}{-1.21} + \underset{(0.10)}{0.82}w.\end{aligned}$$

What can you conclude from this calculation?

Answer:

With n equal to 20, there are 18 degrees of freedom and the critical value of t at the 5 per cent level is 2.10. The 95 per cent confidence interval is therefore:

$$0.82 - 0.10 \times 2.10 \leq \beta_2 \leq 0.82 + 0.10 \times 2.10$$

that is:

$$0.61 \leq \beta_2 \leq 1.03.$$

We see that we cannot (quite) reject the null hypothesis $H_0 : \beta_2 = 1$.

2.36 Suppose that the true relationship between Y and X is $Y_i = \beta_1 + \beta_2 X_i + u_i$ and that the fitted model is $\widehat{Y}_i = \widehat{\beta}_1 + \widehat{\beta}_2 X_i$. Suppose that $X_i^* = \mu_2 X_i$, and Y is regressed on X^*. How will the F statistic for this regression be related to the F statistic for the original regression? (See also Exercises 1.23, 2.15, and 2.27.)

Answer:

We saw in Exercise 1.23 that R^2 would be the same, and it follows that F must also be the same.

2.6 Answers to the additional exercises

Note: Each of the exercises below relates to a simple regression. Accordingly, the F test is equivalent to a two-sided t test on the slope coefficient and there is no point in performing both tests. The F statistic is equal to the square of the t statistic and, for any significance level, the critical value of F is equal to the critical value of t. Obviously a one-sided t test, when justified, is preferable to either in that it has greater power for any given significance level.

A2.1 We have:

$$\widehat{\beta}_2 = \frac{\sum X_i Y_i}{\sum X_i^2} = \frac{\sum X_i(\beta_1 + \beta_2 X_i + u_i)}{\sum X_i^2} = \frac{\beta_1 \sum X_i}{\sum X_i^2} + \beta_2 + \frac{\sum X_i u_i}{\sum X_i^2}.$$

Hence:

$$\mathrm{E}(\widehat{\beta}_2) = \frac{\beta_1 \sum X_i}{\sum X_i^2} + \beta_2 + E\left(\frac{\sum X_i u_i}{\sum X_i^2}\right) = \frac{\beta_1 \sum X_i}{\sum X_i^2} + \beta_2 + \frac{\sum X_i\, E(u_i)}{\sum X_i^2}$$

assuming that X is nonstochastic. Since $E(u_i) = 0$, then:

$$\mathrm{E}(\widehat{\beta}_2) = \frac{\beta_1 \sum X_i}{\sum X_i^2} + \beta_2.$$

Thus $\widehat{\beta}_2$ will in general be a biased estimator. The sign of the bias depends on the signs of β_1 and $\sum X_i$. In general, we have no information about either of these. However, if either $\beta_1 = 0$ or $\overline{X} = 0$ (and so $\sum X_i = 0$), the bias term disappears and $\widehat{\beta}_2$ is unbiased after all.

A2.2 First we need to show that $E(\widehat{\beta}_2) = 0$.

$$\widehat{\beta}_2 = \frac{\sum\left(X_i - \overline{X}\right)\left(Y_i - \overline{Y}\right)}{\sum\left(X_i - \overline{X}\right)^2} = \frac{\sum\left(X_i - \overline{X}\right)(\beta_1 + u_i - \beta_1 - \bar{u})}{\sum\left(X_i - \overline{X}\right)^2} = \frac{\sum\left(X_i - \overline{X}\right)(u_i - \bar{u})}{\sum\left(X_i - \overline{X}\right)^2}.$$

Hence, given that we are told that X is nonstochastic:

$$\begin{aligned}
E(\widehat{\beta}_2) &= E\left(\frac{\sum\left(X_i - \overline{X}\right)(u_i - \bar{u})}{\sum\left(X_i - \overline{X}\right)^2}\right) \\
&= \frac{1}{\sum\left(X_i - \overline{X}\right)^2} E\left(\sum\left(X_i - \overline{X}\right)(u_i - \bar{u})\right) \\
&= \frac{1}{\sum\left(X_i - \overline{X}\right)^2} \sum\left(X_i - \overline{X}\right) E\left(u_i - \bar{u}\right) \\
&= 0
\end{aligned}$$

since $E(u) = 0$. Thus:

$$E(\widehat{\beta}_1) = E\left(\overline{Y} - \widehat{\beta}_2\overline{X}\right) = \beta_1 - \overline{X}E(\widehat{\beta}_2) = \beta_1$$

and the estimator is unbiased.

A2.3 In general, the researcher's estimator will have a larger variance than $\overline{Y}$ and therefore will be inefficient. However, if $\overline{X} = 0$, the variances are the same. This is because the estimators are then identical. $\overline{Y} - \widehat{\beta}_2\overline{X}$ reduces to $\overline{Y}$.

A2.4 The variance of the estimator is $\sigma_u^2/\sum X_i^2$ whereas that of the estimator:

$$\frac{\sum(X_i - \overline{X})(Y_i - \overline{Y})}{\sum(X_i - \overline{X})^2}$$

is:

$$\frac{\sigma_u^2}{\sum(X_i - \overline{X})^2} = \frac{\sigma_u^2}{\sum X_i^2 - n\overline{X}^2}.$$

Thus, provided $\overline{X} \neq 0$, $\sigma_u^2/\sum X_i^2$ is more efficient than:

$$\frac{\sum(X_i - \overline{X})(Y_i - \overline{Y})}{\sum(X_i - \overline{X})^2}$$

if $\beta_1 = 0$ because it is unbiased and has a smaller variance. It is the OLS estimator in this case.

If $\overline{X} = 0$, the estimators are equally efficient because the population variance expressions are identical. The reason for this is that the estimators are now identical:

$$\frac{\sum(X_i - \overline{X})(Y_i - \overline{Y})}{\sum(X_i - \overline{X})^2} = \frac{\sum X_i(Y_i - \overline{Y})}{\sum X_i^2} = \frac{\sum X_iY_i}{\sum X_i^2} - \frac{\overline{Y}\sum X_i}{\sum X_i^2} = \frac{\sum X_iY_i}{\sum X_i^2}$$

since $\sum X_i = n\overline{X} = 0$.

Returning to the general case, if there is little variation in X in the sample, $\sum(X_i - \overline{X})^2$ may be small and hence the population variance of $\sum(X_i - \overline{X})(Y_i - \overline{Y})/\sum(X_i - \overline{X})^2$ may be large. Thus using a criterion such as mean square error, $\widehat{\beta}_2$ may be preferable if the bias is small.

A2.5 The t statistic for the coefficient of *EXP* is 58.57, very highly significant. There is little point performing a t test on the intercept, given that it has no plausible meaning. The F statistic is 3431.0, very highly significant. Since this is a simple regression model, the two tests are equivalent.

A2.6 The slope coefficient for every category is significantly different from zero at a very high significance level. (The F test is equivalent to the t test on the slope coefficient.)

EXP						
	n	$\widehat{\beta}_2$	s.e.$(\widehat{\beta}_2)$	t	R^2	F
ADM	2,815	0.0235	0.0008	28.86	0.228	832.8
CLOT	4,500	0.0316	0.0010	30.99	0.176	960.6
DOM	1,661	0.0409	0.0026	16.02	0.134	256.6
EDUC	561	0.1202	0.0090	13.30	0.241	177.0
ELEC	5,828	0.0131	0.0004	35.70	0.180	1274.8
FDAW	5,102	0.0527	0.0010	52.86	0.354	2794.7
FDHO	6,334	0.0627	0.0011	58.57	0.351	3431.0
FOOT	1,827	0.0058	0.0005	12.78	0.082	163.4
FURN	487	0.0522	0.0070	7.44	0.102	55.3
GASO	5,710	0.0373	0.0008	46.89	0.278	2198.5
HEAL	4,802	0.0574	0.0018	31.83	0.174	1013.4
HOUS	6,223	0.1976	0.0027	74.16	0.469	5499.9
LIFE	1,253	0.0193	0.0016	11.86	0.101	140.7
LOCT	692	0.0068	0.0010	6.59	0.059	43.5
MAPP	399	0.0329	0.0049	6.72	0.102	45.1
PERS	3,817	0.0069	0.0002	32.15	0.213	1033.4
READ	2,287	0.0048	0.0003	16.28	0.104	265.1
SAPP	1,037	0.0045	0.0007	6.03	0.034	36.4
TELE	5,788	0.0160	0.0003	46.04	0.268	2119.7
TEXT	992	0.0040	0.0006	7.32	0.051	53.5
TOB	1,155	0.0165	0.0016	10.56	0.088	111.6
TOYS	2,504	0.0145	0.0010	14.34	0.076	205.7
TRIP	516	0.0466	0.0043	10.84	0.186	117.5

A2.7 The t statistic, 13.29, is very highly significant. (The F test is equivalent.)

A2.8 The t statistic for height, 3.04, suggests that the effect of height on earnings is highly significant, despite the very low R^2. In principle the estimate of an extra 41 cents of hourly earnings for every extra inch of height could have been a purely random result of the kind that one obtains with nonsense models. However, the fact that it is apparently highly significant causes us to look for other explanations, the most likely one being that suggested in the answer to Exercise A1.4. Of course, we would not attempt to test the negative constant.

A2.9 One could justify a one-sided test on the slope coefficient in the regression of the rate of growth of employment on the rate of growth of GDP on the grounds that an increase in the rate of growth of GDP is unlikely to cause a decrease in the rate of growth of employment.

A2.10 One could justify a one-sided test on the slope coefficient in the regression of weight on height in Exercise 1.9 on the grounds that an increase in height is unlikely to cause a decrease in weight.

A2.11 *The standard error of the coefficient of GDP.* This is given by:

$$\frac{\sqrt{\widehat{\sigma}_u^{*2}}}{\sqrt{\sum\left(G_i^* - \overline{G}^*\right)^2}}$$

where $\widehat{\sigma}_u^{*2}$, the estimator of the variance of the disturbance term, is $\sum \widehat{u}_i^{*2}/(n-2)$. Since RSS is unchanged, $\widehat{\sigma}_u^{*2} = \widehat{\sigma}_u^2$.

We saw in Exercise A1.6 that $G_i^* - \overline{G}^* = G_i - \overline{G}$ for all i. Hence the new standard error is given by:

$$\frac{\sqrt{\widehat{\sigma}_u^{2}}}{\sqrt{\sum\left(G_i - \overline{G}\right)^2}}$$

and is unchanged.

$$F = \frac{ESS}{RSS/(n-2)}$$

where:

$$ESS = \text{explained sum of squares} = \sum\left(\widehat{Y}_i^* - \overline{\widehat{Y}}^*\right)^2.$$

Since $\widehat{u}_i^* = \widehat{u}_i$, $\widehat{Y}_i^* = \widehat{Y}_i$ and ESS is unchanged. We saw in Exercise A1.6 that RSS is unchanged. Hence F is unchanged.

A2.12 *The standard error of the coefficient of GDP.* This is given by:

$$\frac{\sqrt{\widehat{\sigma}_u^{*2}}}{\sqrt{\sum\left(G_i^* - \overline{G}^*\right)^2}}$$

where $\widehat{\sigma}_u^{*2}$, the estimator of the variance of the disturbance term, is $\sum \widehat{u}_i^{*2}/(n-2)$. We saw in Exercise 1.7 that $\widehat{u}_i^* = \widehat{u}_i$ and so RSS is unchanged. Hence $\widehat{\sigma}_u^{*2} = \widehat{\sigma}_u^2$. Thus the new standard error is given by:

$$\frac{\sqrt{\widehat{\sigma}_u^{2}}}{\sqrt{\sum\left(2G_i - 2\overline{G}\right)^2}} = \frac{1}{2}\frac{\sqrt{\widehat{\sigma}_u^{2}}}{\sqrt{\sum\left(G_i - \overline{G}\right)^2}} = 0.005.$$

$F = ESS/(RSS/(n-2))$ where:

$$ESS = \text{explained sum of squares} = \sum\left(\widehat{Y}_i^* - \overline{\widehat{Y}}^*\right)^2.$$

Since $\widehat{u}_i^* = \widehat{u}_i$, $\widehat{Y}_i^* = \widehat{Y}_i$ and ESS is unchanged. Hence F is unchanged.

A2.13 One way of demonstrating that $\widehat{Y}_i^* = \widehat{Y}_i - \overline{Y}$:

$$\begin{aligned}
\widehat{Y}_i^* &= \widehat{\beta}_1^* + \widehat{\beta}_2^* X_i^* = \widehat{\beta}_2(X_i - \overline{X}) \\
\widehat{Y}_i - \overline{Y} &= (\widehat{\beta}_1 + \widehat{\beta}_2 X_i) - \overline{Y} = \left(\overline{Y} - \widehat{\beta}_2\overline{X}\right) + \widehat{\beta}_2 X_i - \overline{Y} = \widehat{\beta}_2\left(X_i - \overline{X}\right).
\end{aligned}$$

Demonstration that the residuals are the same:

$$\widehat{u}_i^* = Y_i^* - \widehat{Y}_i^* = \left(Y_i - \overline{Y}\right) - \left(\widehat{Y}_i - \overline{Y}\right) = \widehat{u}_i.$$

Demonstration that the OLS estimator of the variance of the disturbance term in (2) is equal to that in (1):

$$\widehat{\sigma}_u^{*2} = \frac{\sum \widehat{u}_i^{*2}}{n-2} = \frac{\sum \widehat{u}_i^2}{n-2} = \widehat{\sigma}_u^2.$$

The standard error of the slope coefficient in (2) is equal to that in (1).

$$\widehat{\sigma}^2_{\widehat{\beta}_2^*} = \frac{\widehat{\sigma}_u^{*2}}{\sum \left(X_i^* - \overline{X}\right)^2} = \frac{\widehat{\sigma}_u^2}{\sum X_i^{*2}} = \frac{\widehat{\sigma}_u^2}{\sum \left(X_i - \overline{X}\right)^2} = \widehat{\sigma}^2_{\widehat{\beta}_2}.$$

Hence the standard errors are the same.

Demonstration that R^2 in (2) is equal to R^2 in (1):

$$R^{2*} = \frac{\sum \left(\widehat{Y}_i^* - \overline{\widehat{Y}^*}\right)^2}{\sum \left(Y_i^* - \overline{Y}^*\right)^2}.$$

$\widehat{Y}_i^* = \widehat{Y}_i - \overline{Y}$ and $\overline{\widehat{Y}} = \overline{Y}$. Hence $\overline{\widehat{Y}^*} = 0$. $\overline{Y^*} = \overline{Y} - \overline{Y} = 0$. Hence:

$$R^{2*} = \frac{\sum \left(\widehat{Y}_i^*\right)^2}{\sum \left(Y_i^*\right)^2} = \frac{\sum \left(\widehat{Y}_i - \overline{Y}\right)^2}{\sum \left(Y_i - \overline{Y}\right)^2} = R^2.$$

The reason that specification (2) of the second researcher is incorrect is that the model does not include an intercept.

If the second researcher had fitted (3) instead of (2), this would not in fact have affected his estimator of β_2. Using (3), the researcher should have estimated β_2 as:

$$\widehat{\beta}_2^* = \frac{\sum X_i^* Y_i^*}{\sum X_i^{*2}}.$$

However, Exercise 1.16 demonstrates that, effectively, he has done exactly this. Hence the estimator will be the same. It follows that dropping the unnecessary intercept would not have led to a gain in efficiency.

A2.14 We have:

$$\widehat{Y}_i^* = \widehat{\beta}_2^* X_i^* = \frac{\widehat{\sigma}_X}{\widehat{\sigma}_Y}\widehat{\beta}_2 \left(\frac{X_i - \overline{X}}{\widehat{\sigma}_X}\right) = \frac{1}{\widehat{\sigma}_Y}\widehat{\beta}_2(X_i - \overline{X})$$

and:

$$\widehat{Y}_i = \widehat{\beta}_1 + \widehat{\beta}_2 X_i = (\overline{Y} - \widehat{\beta}_2\overline{X}) + \widehat{\beta}_2 X_i = \overline{Y} + \widehat{\beta}_2(X_i - \overline{X}).$$

Hence:

$$\widehat{Y}_i^* = \frac{1}{\widehat{\sigma}_Y}(\widehat{Y}_i - \overline{Y}).$$

Also:

$$\widehat{u}_i^* = Y_i^* - \widehat{Y}_i^* = \frac{1}{\widehat{\sigma}_Y}(Y_i - \overline{Y}) - \frac{1}{\widehat{\sigma}_Y}(\widehat{Y}_i - \overline{Y}) = \frac{1}{\widehat{\sigma}_Y}(Y_i - \widehat{Y}_i) = \frac{1}{\widehat{\sigma}_Y}\widehat{u}_i$$

and:

$$\text{s.e.}(\widehat{\beta}_2^*) = \sqrt{\frac{\frac{1}{n-2}\sum \widehat{u}_i^{*2}}{\sum(X_i^* - \overline{X^*})^2}} = \sqrt{\frac{\left(\frac{1}{\widehat{\sigma}_Y}\right)^2 \frac{1}{n-2}\sum \widehat{u}_i^2}{\sum\left(\frac{X_i - \bar{X}}{\widehat{\sigma}_X}\right)^2}} = \frac{\widehat{\sigma}_X}{\widehat{\sigma}_Y} \times \text{s.e.}(\widehat{\beta}_2).$$

Given the expressions for $\widehat{\beta}_2^*$ and s.e.$(\widehat{\beta}_2^*)$, the t statistic for $\widehat{\beta}_2^*$ is the same as that for $\widehat{\beta}_2$. Hence the F statistic will be the same and R^2 will be the same.

A2.15 The inclusion of the fifth observation does not cause the model to be misspecified or the regression model assumptions to be violated, so retaining it in the sample will not give rise to biased estimates. There would be no advantages in dropping it and there would be one major disadvantage. $\sum\left(X_i - \overline{X}\right)^2$ would be greatly reduced and hence the variances of the coefficients would be increased, adversely affecting the precision of the estimates.

This said, in practice one would wish to check whether it is sensible to assume that the model relating Y to X for the other observations really does apply to the observation corresponding to X_5 as well. This question can be answered only by being familiar with the context and having some intuitive understanding of the relationship between Y and X.

Chapter 3
Multiple regression analysis

3.1 Overview

This chapter introduces regression models with more than one explanatory variable. Specific topics are treated with reference to a model with just two explanatory variables, but most of the concepts and results apply straightforwardly to more general models. The chapter begins by showing how the least squares principle is employed to derive the expressions for the regression coefficients and how the coefficients should be interpreted. It continues with a discussion of the precision of the regression coefficients and tests of hypotheses relating to them. Next comes multicollinearity, the problem of discriminating between the effects of individual explanatory variables when they are closely related. The chapter concludes with a discussion of F tests of the joint explanatory power of the explanatory variables or subsets of them, and shows how a t test can be thought of as a marginal F test.

3.2 Learning outcomes

After working through the corresponding chapter in the text, studying the corresponding slideshows, and doing the starred exercises in the text and the additional exercises in this subject guide, you should be able to explain what is meant by:

- the principles behind the derivation of multiple regression coefficients (but you are not expected to learn the expressions for them or to be able to reproduce the mathematical proofs)
- how to interpret the regression coefficients
- the Frisch–Waugh–Lovell graphical representation of the relationship between the dependent variable and one explanatory variable, controlling for the influence of the other explanatory variables
- the properties of the multiple regression coefficients
- what factors determine the population variance of the regression coefficients
- what is meant by multicollinearity
- what measures may be appropriate for alleviating multicollinearity
- what is meant by a linear restriction
- the F test of the joint explanatory power of the explanatory variables

- the F test of the explanatory power of a group of explanatory variables
- why t tests on the slope coefficients are equivalent to marginal F tests.

You should know the expression for the population variance of a slope coefficient in a multiple regression model with two explanatory variables.

3.3 Additional exercises

A3.1 The output shows the result of regressing *FDHO*, expenditure on food consumed at home, on *EXP*, total household expenditure, and *SIZE*, number of persons in the household, using the *CES* data set. Provide an interpretation of the regression coefficients and perform appropriate tests.

```
. reg FDHO EXP SIZE if FDHO>0

      Source |       SS       df       MS              Number of obs =    6334
-------------+------------------------------           F(  2,  6331) = 2257.59
       Model |  1.1521e+09     2   576056293           Prob > F      =  0.0000
    Residual |  1.6154e+09  6331  255164.645           R-squared     =  0.4163
-------------+------------------------------           Adj R-squared =  0.4161
       Total |  2.7676e+09  6333   437006.15           Root MSE      =  505.14

------------------------------------------------------------------------------
        FDHO |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
         EXP |    .056366   .0010435    54.02   0.000     .0543204    .0584116
        SIZE |   115.1636   4.341912    26.52   0.000      106.652    123.6752
       _cons |   130.5997   13.53959     9.65   0.000     104.0575    157.1419
------------------------------------------------------------------------------
```

A3.2 Perform a regression parallel to that in Exercise A3.1 for your *CES* category of expenditure, provide an interpretation of the regression coefficients and perform appropriate tests. Delete observations where expenditure on your category is zero.

A3.3 The output shows the result of regressing *FDHOPC*, expenditure on food consumed at home per capita, on *EXPPC*, total household expenditure per capita, and *SIZE*, number of persons in the household, using the *CES* data set. Provide an interpretation of the regression coefficients and perform appropriate tests.

```
. reg FDHOPC EXPPC SIZE if FDHO>0

      Source |       SS       df       MS             Number of obs =    6334
-------------+------------------------------          F(  2,  6331) = 1572.95
       Model |   202590496     2   101295248          Prob > F      =  0.0000
    Residual |   407705728  6331  64398.3143          R-squared     =  0.3320
-------------+------------------------------          Adj R-squared =  0.3317
       Total |   610296223  6333  96367.6336          Root MSE      =  253.77
```

```
------------------------------------------------------------------------------
      FDHOPC |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
       EXPPC |   .0480294   .0010064    47.72   0.000     .0460564    .0500023
        SIZE |  -26.45917   2.253999   -11.74   0.000    -30.87777   -22.04057
       _cons |   283.2498   8.412603    33.67   0.000     266.7582    299.7413
------------------------------------------------------------------------------
```

A3.4 Perform a regression parallel to that in Exercise A3.3 for your *CES* category of expenditure. Provide an interpretation of the regression coefficients and perform appropriate tests.

A3.5 The output shows the result of regressing *FDHOPC*, expenditure on food consumed at home per capita, on *EXPPC*, total household expenditure per capita, and *SIZEAM*, *SIZEAF*, *SIZEJM*, *SIZEJF*, and *SIZEIN*, numbers of adult males, adult females, junior males, junior females, and infants, respectively, in the household, using the *CES* data set. Provide an interpretation of the regression coefficients and perform appropriate tests.

```
. reg FDHOPC EXPPC SIZEAM SIZEAF SIZEJM SIZEJF SIZEIN if FDHO>0

      Source |       SS       df       MS              Number of obs =    6334
-------------+------------------------------           F(  6,  6327) =  524.59
       Model |  202746894     6   33791149             Prob > F      =  0.0000
    Residual |  407549329  6327  64414.3084            R-squared     =  0.3322
-------------+------------------------------           Adj R-squared =  0.3316
       Total |  610296223  6333  96367.6336            Root MSE      =   253.8

------------------------------------------------------------------------------
      FDHOPC |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
       EXPPC |   .0479717   .0010087    47.56   0.000     .0459943    .0499491
      SIZEAM |  -25.77747   4.757056    -5.42   0.000    -35.10291   -16.45203
      SIZEAF |  -32.38649   5.065782    -6.39   0.000    -42.31714   -22.45584
      SIZEJM |  -20.24693   5.731645    -3.53   0.000     -31.4829   -9.010967
      SIZEJF |  -26.66374   6.122262    -4.36   0.000    -38.66544   -14.66203
      SIZEIN |   -28.6047   11.75666    -2.43   0.015    -51.65174   -5.557656
       _cons |   287.5695   9.280372    30.99   0.000     269.3769    305.7622
------------------------------------------------------------------------------
```

A3.6 Perform a regression parallel to that in Exercise A3.5 for your *CES* category of expenditure. Provide an interpretation of the regression coefficients and perform appropriate tests.

A3.7 A researcher hypothesises that, for a typical enterprise, V, the logarithm of value added per worker, is related to K, the logarithm of capital per worker, and S, the logarithm of the average years of schooling of the workers, the relationship being:

$$V = \beta_1 + \beta_2 K + \beta_3 S + u$$

where u is a disturbance term that satisfies the usual regression model assumptions. She fits the relationship (1) for a sample of 25 manufacturing enterprises, and (2) for a sample of 100 services enterprises. The table provides some data on the samples.

	(1) Manufacturing sample	(2) Services sample
Number of enterprises	25	100
Estimate of variance of u	0.16	0.64
Mean square deviation of K	4.00	16.00
Correlation between K and S	0.60	0.60

The mean square deviation of K is defined as $\frac{1}{n}\sum\left(K_i - \overline{K}\right)^2$, where n is the number of enterprises in the sample and $\overline{K}$ is the average value of K in the sample.

The researcher finds that the standard error of the coefficient of K is 0.050 for the manufacturing sample and 0.025 for the services sample. Explain the difference quantitatively, given the data in the table.

A3.8 A researcher is fitting earnings functions using a sample of data relating to individuals born in the same week in 1958. He decides to relate Y, gross hourly earnings in 2001, to S, years of schooling, and PWE, potential work experience, using the semilogarithmic specification:

$$\log Y = \beta_1 + \beta_2 S + \beta_3 PWE + u$$

where u is a disturbance term assumed to satisfy the regression model assumptions. PWE is defined as age – years of schooling – 5. Since the respondents were all aged 43 in 2001, this becomes:

$$PWE = 43 - S - 5 = 38 - S.$$

The researcher finds that it is impossible to fit the model as specified. Stata output for his regression is reproduced below:

```
. reg LGY S PWE

      Source |       SS       df       MS              Number of obs =    5660
-------------+------------------------------           F(  1,  5658) = 1232.62
       Model |  237.170265     1  237.170265           Prob > F      =  0.0000
    Residual |  1088.66373  5658  .192411405           R-squared     =  0.1789
-------------+------------------------------           Adj R-squared =  0.1787
       Total |    1325.834  5659  .234287682           Root MSE      =  .43865

------------------------------------------------------------------------------
         LGY |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
           S |   .1038011   .0029566    35.11   0.000     .0980051    .1095971
         PWE |  (dropped)
       _cons |   .5000033   .0373785    13.38   0.000     .4267271    .5732795
------------------------------------------------------------------------------
```

Explain why the researcher was unable to fit his specification.

Explain how the coefficient of S might be interpreted.

3.4 Answers to the starred exercises in the textbook

3.5 Explain why the intercept in the regression of $EEARN$ on ES is equal to zero.

Answer:

The intercept is calculated as $\overline{EEARN} - \widehat{\beta}_2\overline{ES}$. However, since the mean of the residuals from an OLS regression is zero, both $\overline{EEARN}$ and $\overline{ES}$ are zero, and hence the intercept is zero.

3.6 Show that, in the general case, the mean of the residuals from a fitted OLS multiple regression is equal to zero, provided that an intercept is included in the specification. Note: This is an extension of one of the useful results in Section 1.5.

Answer:

If the model is:

$$\begin{aligned} Y &= \beta_1 + \beta_2 X_2 + \cdots + \beta_k X_k + u \\ \widehat{\beta}_1 &= \overline{Y} - \widehat{\beta}_2\overline{X}_2 - \cdots - \widehat{\beta}_k\overline{X}_k. \end{aligned}$$

For observation i we have:

$$\widehat{u}_i = Y_i - \widehat{Y}_i = Y_i - \widehat{\beta}_1 - \widehat{\beta}_2 X_{2i} - \cdots - \widehat{\beta}_k X_{ki}.$$

Hence:

$$\begin{aligned} \overline{\widehat{u}} &= \overline{Y} - \widehat{\beta}_1 - \widehat{\beta}_2\overline{X}_2 - \cdots - \widehat{\beta}_k\overline{X}_k \\ &= \overline{Y} - \left[\overline{Y} - \widehat{\beta}_2\overline{X}_2 - \cdots - \widehat{\beta}_k\overline{X}_k\right] - \widehat{\beta}_2\overline{X}_2 - \cdots - \widehat{\beta}_k\overline{X}_k = 0. \end{aligned}$$

3.16 A researcher investigating the determinants of the demand for public transport in a certain city has the following data for 100 residents for the previous calendar year: expenditure on public transport, E, measured in dollars; number of days worked, W; and number of days not worked, NW. By definition NW is equal to $365 - W$. He attempts to fit the following model:

$$E = \beta_1 + \beta_2 W + \beta_3 NW + u.$$

Explain why he is unable to fit this equation. (Give both intuitive and technical explanations.) How might he resolve the problem?

Answer:

There is exact multicollinearity since there is an exact linear relationship between W, NW and the constant term. As a consequence it is not possible to tell whether variations in E are attributable to variations in W or variations in NW, or both. Noting that $NW_i - \overline{NW} = -(W_i - \overline{W})$, we have:

$$\begin{aligned} \widehat{\beta}_2 &= \frac{\sum\left(E_i - \overline{E}\right)\left(W_i - \overline{W}\right)\sum\left(NW_i - \overline{NW}\right)^2 - \sum\left(E_i - \overline{E}\right)\left(NW_i - \overline{NW}\right)\sum\left(W_i - \overline{W}\right)\left(NW_i - \overline{NW}\right)}{\sum\left(W_i - \overline{W}\right)^2\sum\left(NW_i - \overline{NW}\right)^2 - \left(\sum\left(W_i - \overline{W}\right)\left(NW_i - \overline{NW}\right)\right)^2} \\ &= \frac{\sum\left(E_i - \overline{E}\right)\left(W_i - \overline{W}\right)\sum\left(W_i - \overline{W}\right)^2 - \sum\left(E_i - \overline{E}\right)\left(-W_i + \overline{W}\right)\sum\left(W_i - \overline{W}\right)\left(-W_i + \overline{W}\right)}{\sum\left(W_i - \overline{W}\right)^2\sum\left(W_i - \overline{W}\right)^2 - \left(\sum\left(W_i - \overline{W}\right)\left(-W_i + \overline{W}\right)\right)^2} \\ &= \frac{0}{0}. \end{aligned}$$

One way of dealing with the problem would be to drop NW from the regression. The interpretation of $\widehat{\beta}_2$ now is that it is an estimate of the extra expenditure on transport per day worked, compared with expenditure per day not worked.

3.21 The researcher in Exercise 3.16 decides to divide the number of days not worked into the number of days not worked because of illness, I, and the number of days not worked for other reasons, O. The mean value of I in the sample is 2.1 and the mean value of O is 120.2. He fits the regression (standard errors in parentheses):

$$\begin{aligned}\widehat{E} &= \underset{(8.3)}{-9.6} + \underset{(1.98)}{2.10W} + \underset{(1.77)}{0.45O} \qquad R^2 = 0.72\end{aligned}$$

Perform t tests on the regression coefficients and an F test on the goodness of fit of the equation. Explain why the t tests and F test have different outcomes.

Answer:

Although there is not an exact linear relationship between W and O, they must have a very high negative correlation because the mean value of I is so small. Hence one would expect the regression to be subject to multicollinearity, and this is confirmed by the results. The t statistics for the coefficients of W and O are only 1.06 and 0.25, respectively, but the F statistic:

$$F(2,97) = \frac{0.72/2}{(1-0.72)/97} = 124.7$$

is greater than the critical value of F at the 0.1 per cent level, 7.41.

3.5 Answers to the additional exercises

A3.1 The regression indicates that 5.6 cents out of the marginal expenditure dollar is spent on food consumed at home, and that expenditure on this category increases by \$115 for each individual in the household, keeping total expenditure constant. Both of these effects are very highly significant. Just over 40 per cent of the variance in *FDHO* is explained by *EXP* and *SIZE*. The intercept has no plausible interpretation.

A3.2 With the exception of *LOCT*, all of the categories have positive coefficients for *EXP*, with high significance levels, but the *SIZE* effect varies:

- Positive, significant at the 1 per cent level: *FDHO*, *TELE*, *CLOT*, *FOOT*, *GASO*.
- Positive, significant at the 5 per cent level: *LOCT*.
- Negative, significant at the 1 per cent level: *TEXT*, *FEES*, *READ*.
- Negative, significant at the 5 per cent level: *SHEL*, *EDUC*.
- Not significant: *FDAW*, *DOM*, *FURN*, *MAPP*, *SAPP*, *TRIP*, *HEAL*, *ENT*, *TOYS*, *TOB*.

At first sight it may seem surprising that *SIZE* has a significant negative effect for some categories. The reason for this is that an increase in *SIZE* means a reduction in expenditure per capita, if total household expenditure is kept constant, and thus *SIZE* has a (negative) income effect in addition to any direct effect. Effectively poorer, the larger household has to spend more on basics and less on luxuries. To determine the true direct effect, we need to eliminate the income effect, and that is the point of the re-specification of the model in the next exercise.

		EXP		*SIZE*			
	n	$\widehat{\beta}_2$	s.e.$(\widehat{\beta}_2)$	$\widehat{\beta}_3$	s.e.$(\widehat{\beta}_3)$	R^2	F
ADM	2,815	0.0238	0.0008	−8.09	4.19	0.230	418.7
CLOT	4,500	0.0309	0.0010	16.39	4.50	0.178	488.2
DOM	1,661	0.0388	0.0026	52.34	14.06	0.141	136.2
EDUC	561	0.1252	0.0090	−179.23	48.92	0.258	97.2
ELEC	5,828	0.0121	0.0004	18.92	1.57	0.199	725.5
FDAW	5,102	0.0538	0.0010	−20.72	4.47	0.357	1,413.7
FDHO	6,334	0.0564	0.0010	115.16	4.34	0.416	2,257.6
FOOT	1,827	0.0056	0.0005	3.24	2.05	0.083	83.0
FURN	487	0.0541	0.0071	−61.87	35.92	0.108	29.3
GASO	5,710	0.0347	0.0008	50.29	3.40	0.305	1,250.9
HEAL	4,802	0.0580	0.0019	−9.96	8.60	0.175	507.4
HOUS	6,223	0.1997	0.0027	−38.78	11.41	0.470	2,760.4
LIFE	1,253	0.0198	0.0017	−9.01	8.99	0.102	70.9
LOCT	692	0.0062	0.0011	14.61	4.72	0.072	26.8
MAPP	399	0.0309	0.0050	44.48	23.94	0.110	24.4
PERS	3,817	0.0070	0.0002	−2.17	1.03	0.214	519.4
READ	2,287	0.0049	0.0003	−1.06	1.58	0.104	132.7
SAPP	1,037	0.0046	0.0008	−3.12	3.99	0.035	18.5
TELE	5,788	0.0150	0.0004	17.92	1.47	0.287	1,161.2
TEXT	992	0.0041	0.0006	−0.71	2.90	0.051	26.8
TOB	1,155	0.0161	0.0016	6.79	6.24	0.089	56.4
TOYS	2,504	0.0140	0.0010	12.19	4.88	0.078	106.2
TRIP	516	0.0450	0.0045	37.48	31.21	0.188	59.5

A3.3 Another surprise, perhaps. The purpose of this specification is to test whether household size has an effect on expenditure per capita on food consumed at home, controlling for the income effect of variations in household size mentioned in the answer to Exercise A3.2. Expenditure per capita on food consumed at home increases by 4.8 cents out of the marginal dollar of total household expenditure per capita. Now *SIZE* has a very significant negative effect. Expenditure per capita on *FDHO* decreases by \$26 per year for each extra person in the household, suggesting that larger households are more efficient than smaller ones with regard to expenditure on this category, the effect being highly significant. R^2 is lower than in Exercise A3.1, but a comparison is invalidated by the fact that the dependent variable is different.

A3.4 Nearly all of the categories have negative *SIZE* effects, the majority highly significant. One explanation of the negative effects could be economies of scale, but this is not plausible in the case of some. Another might be family composition – larger families having more children. In the case of *DOM*, *SIZE* has a positive effect, significant at the 5 per cent level. Again, this might be attributable to larger families having more children and needing greater expenditure on childcare.

		EXP		*SIZE*			
	n	$\widehat{\beta}_2$	s.e.$(\widehat{\beta}_2)$	$\widehat{\beta}_3$	s.e.$(\widehat{\beta}_3)$	R^2	F
ADM	2,815	0.0244	0.0008	2.56	2.26	0.251	470.4
CLOT	4,500	0.0324	0.0012	−1.07	2.91	0.151	400.8
DOM	1,661	0.0311	0.0025	18.54	7.35	0.086	78.1
EDUC	561	0.1391	0.0108	−31.92	27.57	0.290	113.7
ELEC	5,828	0.0117	0.0004	−17.53	0.89	0.247	953.9
FDAW	5,102	0.0528	0.0011	−13.51	2.53	0.375	1,526.3
FDHO	6,334	0.0480	0.0010	−26.46	2.25	0.332	1,573.0
FOOT	1,827	0.0068	0.0005	−8.13	1.11	0.194	219.5
FURN	487	0.0935	0.0091	3.40	26.82	0.216	66.6
GASO	5,710	0.0308	0.0008	−12.43	1.80	0.255	976.5
HEAL	4,802	0.0597	0.0020	−34.16	4.99	0.197	588.5
HOUS	6,223	0.2127	0.0030	−48.86	6.67	0.501	3,123.3
LIFE	1,253	0.0205	0.0017	−10.33	4.65	0.131	94.4
LOCT	692	0.0062	0.0010	−9.06	2.54	0.098	37.4
MAPP	399	0.0384	0.0051	−15.52	12.32	0.171	41.0
PERS	3,817	0.0071	0.0003	−3.96	0.63	0.228	564.0
READ	2,287	0.0052	0.0003	−3.60	0.84	0.154	208.1
SAPP	1,037	0.0076	0.0010	−6.71	2.61	0.090	51.1
TELE	5,788	0.0139	0.0003	−9.77	0.75	0.307	1,282.6
TEXT	992	0.0041	0.0005	−8.96	1.45	0.138	79.2
TOB	1,155	0.0220	0.0019	−22.68	3.55	0.187	132.1
TOYS	2,504	0.0216	0.0012	−8.86	2.92	0.141	205.7
TRIP	516	0.0361	0.0043	−16.33	16.32	0.150	45.2

A3.5 The coefficients of the *SIZE* variables are fairly similar, suggesting that household composition is not important for this category of expenditure.

A3.6 The regression results for this specification are summarised in the table below. In the case of *SHEL*, the regression indicates that the *SIZE* effect is attributable to *SIZEAM*. To investigate this further, the regression was repeated: (1) restricting the sample to households with at least one adult male, and (2) restricting the sample to households with either no adult male or just 1 adult male. The first regression produces a negative effect for *SIZEAM*, but it is smaller than with the whole sample and not significant. In the second regression the coefficient of *SIZEAM* jumps dramatically, from −\$424 to −\$793, suggesting very strong economies of scale for this particular comparison.

As might be expected, the *SIZE* composition variables on the whole do not appear to have significant effects if the *SIZE* variable does not in Exercise A3.4. The results for TOB are puzzling, in that the apparent economies of scale do not appear to be related to household composition.

Category	*ADM*	*CLOT*	*DOM*	*EDUC*	*ELEC*	*FDAW*	*FDHO*	*FOOT*
EXP	0.0245	0.0309	0.0422	0.1191	0.0120	0.0531	0.0561	0.0056
	(0.0008)	(0.0011)	(0.0026)	(0.0092)	(0.0004)	(0.0010)	(0.0011)	(0.0005)
SIZEAM	−37.17	12.84	−141.47	120.11	23.40	29.36	129.69	2.65
	(9.22)	(10.33)	(32.71)	(107.51)	(3.44)	(9.88)	(9.64)	(4.71)
SIZEAF	−40.47	12.26	−67.26	−58.21	35.73	−45.07	105.17	9.40
	(9.52)	(10.95)	(34.79)	(107.96)	(3.60)	(10.17)	(9.96)	(5.25)
SIZEJM	1.33	17.11	114.68	−413.28	12.53	−24.45	126.94	1.23
	(9.86)	(11.41)	(31.91)	(107.79)	(4.06)	(11.53)	(11.35)	(4.99)
SIZEJF	48.55	29.98	93.82	−287.35	8.93	−26.03	105.01	6.32
	(10.54)	(12.15)	(33.66)	(103.15)	(4.31)	(12.05)	(12.07)	(5.01)
SIZEIN	−34.51	−2.08	441.46	−123.20	−4.05	−61.38	95.90	−16.33
	(22.79)	(22.20)	(59.10)	(289.63)	(8.36)	(23.77)	(23.34)	(11.07)
R^2	0.243	0.179	0.184	0.278	0.204	0.361	0.417	0.086
F	150.1	163.0	62.1	35.6	249.2	480.2	753.6	28.5
n	2,815	4,500	1,661	561	5,828	5,102	6,334	1,827

Category	*FURN*	*GASO*	*HEAL*	*HOUS*	*LIFE*	*LOCT*	*MAPP*	*PERS*
EXP	0.0547	0.0341	0.0579	0.2022	0.0195	0.0061	0.0321	0.0071
	(0.0072)	(0.0008)	(0.0019)	(0.0027)	(0.0017)	(0.0011)	(0.0051)	(0.0002)
SIZEAM	−119.30	90.70	3.01	−175.23	10.54	12.02	2.41	−13.99
	(81.65)	(7.47)	(18.25)	(25.24)	(19.50)	(9.90)	(54.58)	(2.23)
SIZEAF	−55.42	52.23	89.64	−111.39	25.43	19.16	0.75	12.33
	(93.37)	(7.79)	(19.10)	(26.12)	(20.83)	(10.61)	(63.11)	(2.34)
SIZEJM	−27.44	30.83	−62.83	52.32	−23.28	−6.41	131.15	−3.33
	(87.24)	(8.72)	(22.56)	(29.65)	(21.17)	(12.81)	(61.75)	(2.59)
SIZEJF	−15.06	46.24	−57.94	34.65	−15.65	32.97	24.87	−2.10
	(89.23)	(9.27)	(23.96)	(31.58)	(22.98)	(15.85)	(64.61)	(2.71)
SIZEIN	−146.90	−8.90	−109.08	119.91	−116.37	33.48	26.25	−11.30
	(160.29)	(18.02)	(46.46)	(61.40)	(46.00)	(25.82)	(139.98)	(5.32)
R^2	0.110	0.310	0.181	0.475	0.109	0.077	0.116	0.228
F	9.9	427.6	177.0	937.6	25.3	9.6	8.6	187.4
n	487	5,710	4,802	6,223	1,253	692	399	3,817

Category	*READ*	*SAPP*	*TELE*	*TEXT*	*TOB*	*TOYS*	*TRIP*
EXP	0.0049	0.0046	0.0148	0.0040	0.0151	0.0148	0.0448
	(0.0003)	(0.0008)	(0.0004)	(0.0006)	(0.0016)	(0.0010)	(0.0045)
SIZEAM	−6.37	−1.64	29.33	7.42	30.92	−39.66	64.35
	(3.46)	(8.26)	(3.25)	(5.98)	(13.49)	(11.19)	(59.55)
SIZEAF	1.69	8.95	35.59	2.58	22.09	1.30	4.87
	(3.80)	(9.65)	(3.38)	(6.77)	(13.68)	(12.49)	(71.23)
SIZEJM	0.63	−13.21	6.38	−15.90	17.42	42.46	81.61
	(3.93)	(9.73)	(3.78)	(7.51)	(16.52)	(11.30)	(79.96)
SIZEJF	4.73	1.17	12.74	−4.92	−45.12	19.34	102.45
	(4.26)	(10.88)	(4.06)	(7.50)	(16.82)	(11.71)	(91.86)
SIZEIN	−18.98	−19.58	−26.42	19.17	2.92	50.91	−294.14
	(8.56)	(18.58)	(7.82)	(14.13)	(32.83)	(22.49)	(157.82)
R^2	0.108	0.038	0.296	0.059	0.100	0.090	0.197
F	45.8	6.7	404.9	10.4	21.2	41.2	20.8
n	2,287	1,037	5,788	992	368	2,504	516

A3.7 The standard error is given by:

$$\text{s.e.}(\widehat{\beta}_2) = \widehat{\sigma}_u \times \frac{1}{\sqrt{n}} \times \frac{1}{\sqrt{\text{MSD}(K)}} \times \frac{1}{\sqrt{1 - r^2_{K,S}}}.$$

	Data		Factors	
	manufacturing sample	services sample	manufacturing sample	services sample
Number of enterprises	25	100	0.20	0.10
Estimate of variance of u	0.16	0.64	0.40	0.80
Mean square deviation of K	4	16	0.50	0.25
Correlation between K and S	0.6	0.6	1.25	1.25
Standard errors			0.050	0.025

The table shows the four factors for the two sectors. Other things being equal, the larger number of enterprises and the greater MSD of K would separately cause the standard error of $\widehat{\beta}_2$ for the services sample to be half that in the manufacturing sample. However, the larger estimate of the variance of u would, taken in isolation, causes it to be double. The net effect, therefore, is that it is half.

A3.8 Exact multicollinearity. An extra year of schooling implies one fewer year of potential work experience. Thus the coefficient of schooling estimates the proportional increase in earnings associated with an additional year of schooling, taking account of the loss of a year of potential work experience.

Chapter 4
Transformations of variables

4.1 Overview

This chapter shows how least squares regression analysis can be extended to fit nonlinear models. Sometimes an apparently nonlinear model can be linearised by taking logarithms. $Y = \beta_1 X^{\beta_2}$ and $Y = \beta_1 e^{\beta_2 X}$ are examples. Because they can be fitted using linear regression analysis, they have proved very popular in the literature, there usually being little to be gained from using more sophisticated specifications. If you plot earnings on schooling, using the *EAWE* data set, or expenditure on a given category of expenditure on total household expenditure, using the *CES* data set, you will see that there is so much randomness in the data that one nonlinear specification is likely to be just as good as another, and indeed a linear specification may not be obviously inferior. Often the real reason for preferring a nonlinear specification to a linear one is that it makes more sense theoretically. The chapter shows how the least squares principle can be applied when the model cannot be linearised.

4.2 Learning outcomes

After working through the corresponding chapter in the text, studying the corresponding slideshows, and doing the starred exercises in the text and the additional exercises in this subject guide, you should be able to:

- explain the difference between nonlinearity in parameters and nonlinearity in variables
- explain why nonlinearity in parameters is potentially a problem while nonlinearity in variables is not
- define an elasticity
- explain how to interpret an elasticity in simple terms
- perform basic manipulations with logarithms
- interpret the coefficients of semi-logarithmic and logarithmic regressions
- explain why the coefficients of semi-logarithmic and logarithmic regressions should not be interpreted using the method for regressions in natural units described in Chapter 1
- perform a RESET test of functional misspecification

- explain the role of the disturbance term in a nonlinear model
- explain how in principle a nonlinear model that cannot be linearised may be fitted
- perform a transformation for comparing the fits of models with linear and logarithmic dependent variables.

4.3 Further material

Box–Cox tests of functional specification

The theory behind the procedure for discriminating between a linear and a logarithmic specification of the dependent variable is explained in the Appendix to Chapter 10 of the text. However, the exposition there is fairly brief. An expanded version is offered here. It should be skipped on first reading because it makes use of material on maximum likelihood estimation. To keep the mathematics uncluttered, the theory will be described in the context of the simple regression model, where we are choosing between:

$$Y = \beta_1 + \beta_2 X + u$$

and:

$$\log Y = \beta_1 + \beta_2 X + u.$$

It generalises with no substantive changes to the multiple regression model.

The two models are actually special cases of the more general model:

$$Y_\lambda = \frac{Y^\lambda - 1}{\lambda} = \beta_1 + \beta_2 X + u$$

with $\lambda = 1$ yielding the linear model (with an unimportant adjustment to the intercept) and $\lambda = 0$ yielding the logarithmic specification at the limit as λ tends to zero. Assuming that u is iid (independently and identically distributed) $N(0,\, \sigma^2)$, the density function for u_i is:

$$f(u_i) = \frac{1}{\sigma\sqrt{2\pi}} \mathrm{e}^{-u_i^2/2\sigma^2}$$

and hence the density function for $Y_{\lambda i}$ is:

$$f(Y_{\lambda i}) = \frac{1}{\sigma\sqrt{2\pi}} \mathrm{e}^{-(Y_{\lambda i} - \beta_1 - \beta_2 X_i)^2/2\sigma^2}.$$

From this we obtain the density function for Y_i:

$$f(Y_i) = \frac{1}{\sigma\sqrt{2\pi}} \mathrm{e}^{-(Y_{\lambda i} - \beta_1 - \beta_2 X_i)^2/2\sigma^2} \left| \frac{\partial Y_{\lambda i}}{\partial Y_i} \right| = \frac{1}{\sigma\sqrt{2\pi}} \mathrm{e}^{-(Y_{\lambda i} - \beta_1 - \beta_2 X_i)^2/2\sigma^2} Y_i^{\lambda - 1}.$$

The factor $\left| \frac{\partial Y_{\lambda i}}{\partial Y_i} \right|$ is the Jacobian for relating the density function of $Y_{\lambda i}$ to that of Y_i. Hence the likelihood function for the parameters is:

$$L(\beta_1, \beta_2, \sigma, \lambda) = \left(\frac{1}{\sigma\sqrt{2\pi}} \right)^n \prod_{i=1}^{n} \mathrm{e}^{-(Y_{\lambda i} - \beta_1 - \beta_2 X_i)^2/2\sigma^2} \prod_{i=1}^{n} Y_i^{\lambda - 1}$$

and the log-likelihood is:

$$\begin{aligned}\log L(\beta_1, \beta_2, \sigma, \lambda) &= -\frac{n}{2}\log 2\pi\sigma^2 - \sum_{i=1}^{n}\frac{1}{2\sigma^2}(Y_{\lambda i} - \beta_1 - \beta_2 X_i)^2 + \sum_{i=1}^{n}\log Y_i^{\lambda-1} \\ &= -\frac{n}{2}\log 2\pi - n\log\sigma - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(Y_{\lambda i} - \beta_1 - \beta_2 X_i)^2 + (\lambda - 1)\sum_{i=1}^{n}\log Y_i.\end{aligned}$$

From the first-order condition $\partial \log L / \partial \sigma = 0$, we have:

$$-\frac{n}{\sigma} + \frac{1}{\sigma^3}\sum_{i=1}^{n}(Y_{\lambda i} - \beta_1 - \beta_2 X_i)^2 = 0$$

giving:

$$\widehat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}(Y_{\lambda i} - \beta_1 - \beta_2 X_i)^2.$$

Substituting into the log-likelihood function, we obtain the concentrated log-likelihood:

$$\log L(\beta_1, \beta_2, \lambda) = -\frac{n}{2}\log 2\pi - \frac{n}{2}\log\frac{1}{n}\sum_{i=1}^{n}(Y_{\lambda i} - \beta_1 - \beta_2 X_i)^2 - \frac{n}{2} + (\lambda - 1)\sum_{i=1}^{n}\log Y_i.$$

The expression can be simplified (Zarembka, 1968) by working with Y_i^* rather than Y_i, where Y_i^* is Y_i divided by Y_{GM}, the geometric mean of the Y_i in the sample, for:

$$\begin{aligned}\sum_{i=1}^{n}\log Y_i^* &= \sum_{i=1}^{n}\log(Y_i/Y_{\mathrm{GM}}) = \sum_{i=1}^{n}(\log Y_i - \log Y_{\mathrm{GM}}) \\ &= \sum_{i=1}^{n}\log Y_i - n\log Y_{\mathrm{GM}} = \sum_{i=1}^{n}\log Y_i - n\log\left(\prod_{i=1}^{n}Y_i\right)^{1/n} \\ &= \sum_{i=1}^{n}\log Y_i - \log\left(\prod_{i=1}^{n}Y_i\right) = \sum_{i=1}^{n}\log Y_i - \sum_{i=1}^{n}\log Y_i = 0.\end{aligned}$$

With this simplification, the log-likelihood is:

$$\log L(\beta_1, \beta_2, \lambda) = -\frac{n}{2}\left(\log 2\pi + \log\frac{1}{n} + 1\right) - \frac{n}{2}\log\sum_{i=1}^{n}(Y_{\lambda i}^* - \beta_1 - \beta_2 X_i)^2$$

and it will be maximised when β_1, β_2 and λ are chosen so as to minimise $\sum_{i=1}^{n}(Y_{\lambda i}^* - \beta_1 - \beta_2 X_i)^2$, the residual sum of squares from a least squares regression of the scaled, transformed Y on X. One simple procedure is to perform a grid search, scaling and transforming the data on Y for a range of values of λ and choosing the value that leads to the smallest residual sum of squares (Spitzer, 1982).

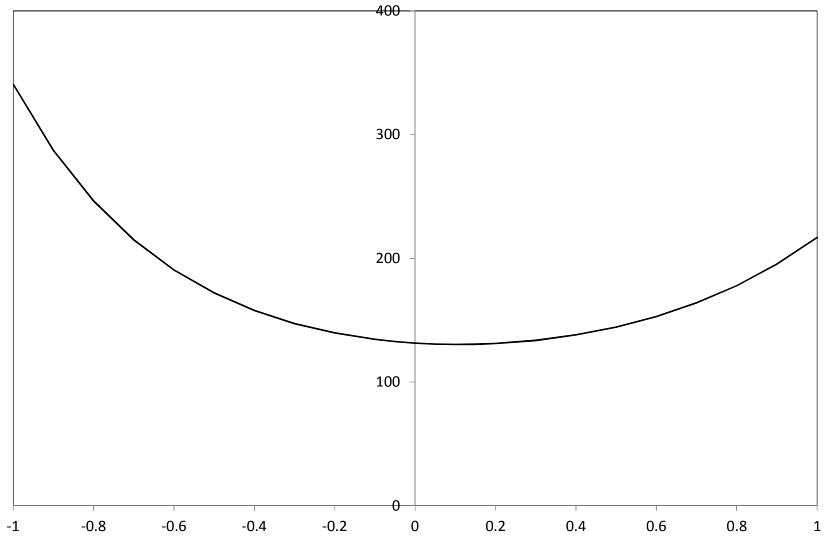
A null hypothesis $\lambda = \lambda_0$ can be tested using a likelihood ratio test in the usual way. Under the null hypothesis, the test statistic $2(\log L_\lambda - \log L_0)$ will have a chi-squared distribution with one degree of freedom, where $\log L_\lambda$ is the unconstrained log-likelihood and L_0 is the constrained one. Note that, in view of the preceding equation:

$$2(\log L_\lambda - \log L_0) = n(\log RSS_0 - \log RSS_\lambda)$$

where RSS_0 and RSS_λ are the residual sums of squares from the constrained and unconstrained regressions with Y^*.

The most obvious tests are $\lambda = 0$ for the logarithmic specification and $\lambda = 1$ for the linear one. Note that it is not possible to test the two hypotheses directly against each other. As with all tests, one can only test whether a hypothesis is incompatible with the sample result. In this case we are testing whether the log-likelihood under the restriction is significantly smaller than the unrestricted log-likelihood. Thus, while it is possible that we may reject the linear but not the logarithmic, or vice versa, it is also possible that we may reject both or fail to reject both.

Example



The figure shows the residual sum of squares for values of λ from -1 to 1 for the wage equation example described in Section 4.2 in the text. The maximum likelihood estimate is 0.10, with $RSS = 130.3$. For the linear and logarithmic specifications, RSS was 217.0 and 131.4, respectively, with likelihood ratio statistics $500(\log 217.0 - \log 130.3) = 255.0$ and $500(\log 131.4 - \log 130.3) = 4.20$.The logarithmic specification is clearly much to be preferred, but even it is rejected at the 5 per cent level, with $\chi^2(1) = 3.84$.

4.4 Additional exercises

A4.1 *Is expenditure on your category per capita related to total expenditure per capita? An alternative model specification.*

Define a new variable $LGCATPC$ as the logarithm of expenditure per capita on your category. Define a new variable $LGEXPPC$ as the logarithm of total household expenditure per capita. Regress $LGCATPC$ on $LGEXPPC$. Provide an interpretation of the coefficients, and perform appropriate statistical tests.

A4.2 *Is expenditure on your category per capita related to household size as well as to total expenditure per capita? An alternative model specification.*

Regress $LGCATPC$ on $LGEXPPC$ and $LGSIZE$. Provide an interpretation of the coefficients, and perform appropriate statistical tests.

A4.3 A researcher is considering two regression specifications:

$$\log Y = \beta_1 + \beta_2 \log X + u \qquad (1)$$

and:

$$\log \frac{Y}{X} = \alpha_1 + \alpha_2 \log X + u \qquad (2)$$

where u is a disturbance term.

Writing $y = \log Y$, $x = \log X$, and $z = \log \frac{Y}{X}$, and using the same sample of n observations, the researcher fits the two specifications using OLS:

$$\widehat{y} = \widehat{\beta}_1 + \widehat{\beta}_2 x \qquad (3)$$

and:

$$\widehat{z} = \widehat{\alpha}_1 + \widehat{\alpha}_2 x. \qquad (4)$$

- Using the expressions for the OLS regression coefficients, demonstrate that $\widehat{\beta}_2 = \widehat{\alpha}_2 + 1$.
- Similarly, using the expressions for the OLS regression coefficients, demonstrate that $\widehat{\beta}_1 = \widehat{\alpha}_1$.
- Hence demonstrate that the relationship between the fitted values of y, the fitted values of z, and the actual values of x, is $\widehat{y}_i - x_i = \widehat{z}_i$.
- Hence show that the residuals for regression (3) are identical to those for (4).
- Hence show that the standard errors of $\widehat{\beta}_2$ and $\widehat{\alpha}_2$ are the same.
- Determine the relationship between the t statistic for $\widehat{\beta}_2$ and the t statistic for $\widehat{\alpha}_2$, and give an intuitive explanation for the relationship.
- Explain whether R^2 would be the same for the two regressions.

A4.4 A researcher has data on a measure of job performance, *SKILL*, and years of work experience, *EXP*, for a sample of individuals in the same occupation. Believing there to be diminishing returns to experience, the researcher proposes the model:

$$SKILL = \beta_1 + \beta_2 \log(EXP) + \beta_3 \log\left(EXP^2\right) + u.$$

Comment on this specification.

A4.5 A researcher hypothesises that a variable Y is determined by a variable X and considers the following four alternative regression specifications, using cross-sectional data:

$$\begin{aligned} Y &= \beta_1 + \beta_2 X + u & (1)\\ \log Y &= \beta_1 + \beta_2 X + u & (2)\\ Y &= \beta_1 + \beta_2 \log X + u & (3)\\ \log Y &= \beta_1 + \beta_2 \log X + u. & (4) \end{aligned}$$

Explain why a direct comparison of R^2, or of *RSS*, in models (1) and (2) is illegitimate. What should be the strategy of the researcher for determining which of the four specifications has the best fit?

A4.6 *Is a logarithmic specification preferable to a linear specification for an expenditure function?*

Use your category of expenditure from the *CES* data set. Define *CATPCST* as *CATPC* scaled by its geometric mean and *LGCATST* as the logarithm of *CATPCST*. Regress *CATPCST* on *EXPPC* and *SIZE* and regress *LGCATST* on *LGEXPPC* and *LGSIZE*. Compare the *RSS* for these equations.

A4.7

```
. reg LGEARN S EXP ASVABC SASVABC

      Source |       SS       df       MS              Number of obs =     500
-------------+------------------------------           F(  4,   495) =   22.68
       Model |  23.6368302     4  5.90920754           Prob > F      =  0.0000
    Residual |   128.96239   495   .26053008           R-squared     =  0.1549
-------------+------------------------------           Adj R-squared =  0.1481
       Total |   152.59922   499   .30581006           Root MSE      =  .51042

------------------------------------------------------------------------------
      LGEARN |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
           S |   .0764243   .0116879     6.54   0.000     .0534603    .0993883
         EXP |   .0400506   .0096479     4.15   0.000     .0210948    .0590065
      ASVABC |  -.2096325   .1406659    -1.49   0.137    -.4860084    .0667434
     SASVABC |   .0188685   .0093393     2.02   0.044     .0005189    .0372181
       _cons |   1.386753   .2109596     6.57   0.000     .9722664     1.80124
------------------------------------------------------------------------------
```

The output above shows the result of regressing the logarithm of hourly earnings on years of schooling, years of work experience, *ASVABC* score, and *SASVABC*, an interactive variable defined as the product of *S* and *ASVABC*, using *EAWE* Data Set 21. The mean values of *S*, *EXP*, and *ASVABC* in the sample were 14.9, 6.4, and 0.27, respectively. Give an interpretation of the regression output.

A4.8 Perform a RESET test of functional misspecification. Using your *EAWE* data set, regress *WEIGHT11* on *HEIGHT*. Save the fitted values as *YHAT* and define *YHATSQ* as its square. Add *YHATSQ* to the regression specification and test its coefficient.

4.5 Answers to the starred exercises in the textbook

4.8 Suppose that the logarithm of Y is regressed on the logarithm of X, the fitted regression being:

$$\log \widehat{Y} = \widehat{\beta}_1 + \widehat{\beta}_2 \log X.$$

Suppose $X^* = \mu X$, where μ is a constant, and suppose that $\log Y$ is regressed on $\log X^*$. Determine how the regression coefficients are related to those of the original regression. Determine also how the t statistic for $\widehat{\beta}_2$ and R^2 for the equation are related to those in the original regression.

Answer:

Nothing of substance is affected since the change amounts only to a fixed constant shift in the measurement of the explanatory variable.

Let the fitted regression be:

$$\log \widehat{Y} = \widehat{\beta}_1^* + \widehat{\beta}_2^* \log X^*.$$

Note that:

$$\begin{aligned}
\log X_i^* - \overline{\log X^*} &= \log \mu X_i - \frac{1}{n}\sum_{j=1}^{n} \log X_j^* \\
&= \log \mu X_i - \frac{1}{n}\sum_{j=1}^{n} \log \mu X_j \\
&= \log \mu + \log X_i - \frac{1}{n}\sum_{j=1}^{n} (\log \mu + \log X_j) \\
&= \log X_i - \frac{1}{n}\sum_{j=1}^{n} \log X_j \\
&= \log X_i - \overline{\log X}.
\end{aligned}$$

Hence $\widehat{\beta}_2^* = \widehat{\beta}_2$. To compute the standard error of $\widehat{\beta}_2^*$, we will also need $\widehat{\beta}_1^*$.

$$\begin{aligned}
\widehat{\beta}_1^* = \overline{\log Y} - \widehat{\beta}_2^* \overline{\log X^*} &= \overline{\log Y} - \widehat{\beta}_2 \frac{1}{n}\sum_{j=1}^{n} (\log \mu + \log X_j) \\
&= \overline{\log Y} - \widehat{\beta}_2 \log \mu - \widehat{\beta}_2 \overline{\log X} \\
&= \widehat{\beta}_1 - \widehat{\beta}_2 \log \mu.
\end{aligned}$$

Thus the residual $\widehat{u}_i^*$ is given by:

$$\widehat{u}_i^* = \log Y_i - \widehat{\beta}_1^* - \widehat{\beta}_2^* \log X_i^* = \log Y_i - (\widehat{\beta}_1 - \widehat{\beta}_2 \log \mu) - \widehat{\beta}_2 (\log X_i + \log \mu) = \widehat{u}_i.$$

Hence the estimator of the variance of the disturbance term is unchanged and so the standard error of $\widehat{\beta}_2^*$ is the same as that for $\widehat{\beta}_2$. As a consequence, the t statistic must be the same. R^2 must also be the same:

$$R^{2*} = 1 - \frac{\sum \widehat{u}_i^{*2}}{\sum \left(\log Y_i - \overline{\log Y}\right)} = 1 - \frac{\sum \widehat{u}_i^2}{\sum \left(\log Y_i - \overline{\log Y}\right)} = R^2.$$

4.11 *RSS* was the same in Tables 4.6 and 4.8. Demonstrate that this was not a coincidence.

Answer:

This is a special case of the transformation in Exercise 4.7.

4.14

```
. gen LGHTSQ = ln(HEIGHTSQ)

. reg LGWT04 LGHEIGHT LGHTSQ


      Source |       SS       df       MS              Number of obs =     500
-------------+------------------------------           F(  1,   498) =  211.28
       Model |  7.90843858     1  7.90843858           Prob > F      =  0.0000
    Residual |  18.6403163   498  .037430354           R-squared     =  0.2979
-------------+------------------------------           Adj R-squared =  0.2965
       Total |  26.5487548   499  .053203918           Root MSE      =  .19347

------------------------------------------------------------------------------
      LGWT04 |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
    LGHEIGHT |  (dropped)
      LGHTSQ |   1.053218   .0724577    14.54   0.000      .9108572   1.195578
       _cons |   -3.78834    .610925    -6.20   0.000     -4.988648  -2.588031
------------------------------------------------------------------------------
```

The output shows the results of regressing, *LGWT04*, the logarithm of *WEIGHT04*, on *LGHEIGHT*, the logarithm of *HEIGHT*, and *LGHTSQ*, the logarithm of the square of *HEIGHT*, using *EAWE* Data Set 21. Explain the regression results, comparing them with those in Exercise 4.2.

Answer:

LGHTSQ = 2 LGHEIGHT, so the specification is subject to exact multicollinearity. In such a situation, Stata drops one of the variables responsible.

4.18

```
. nl (S = {beta1} + {beta2}/({beta3} + SIBLINGS)) if SIBLINGS>0
(obs = 473)

Iteration 0:  residual SS =  3502.041
Iteration 1:  residual SS =  3500.884
....................................
Iteration 14:  residual SS =  3482.794

      Source |       SS       df       MS
-------------+------------------------------         Number of obs =       473
       Model |  132.339291     2  66.1696453         R-squared     =    0.0366
    Residual |   3482.7939   470  7.41019979         Adj R-squared =    0.0325
-------------+------------------------------         Root MSE      =  2.722168
       Total |  3615.13319   472  7.65918049         Res. dev.     =  2286.658

------------------------------------------------------------------------------
           S |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
      /beta1 |   10.45811   5.371492     1.95   0.052    -.0970041    21.01322
      /beta2 |   47.95198   125.3578     0.38   0.702    -198.3791    294.2831
      /beta3 |     8.6994   15.10277     0.58   0.565    -20.97791    38.37671
------------------------------------------------------------------------------
  Parameter beta1 taken as constant term in model & ANOVA table
```

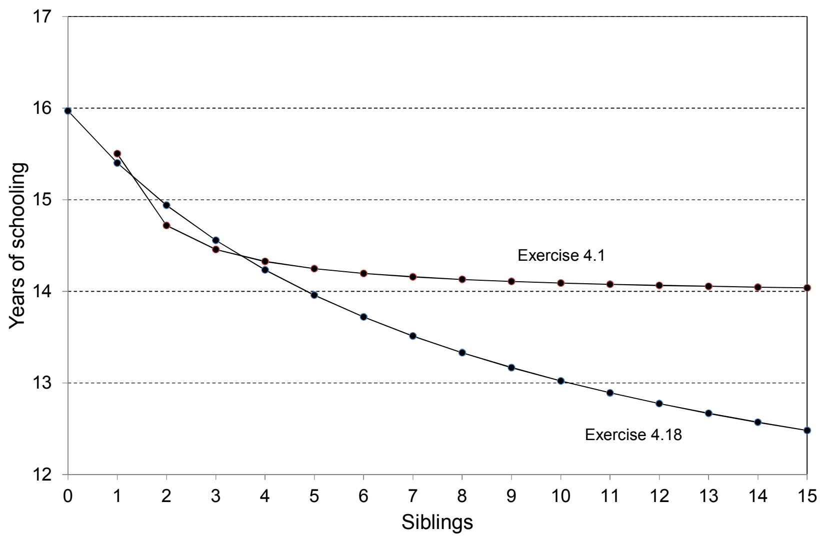
The output uses *EAWE* Data Set 21 to fit the nonlinear model:

$$S = \beta_1 + \frac{\beta_2}{\beta_3 + SIBLINGS} + u$$

where S is the years of schooling of the respondent and $SIBLINGS$ is the number of brothers and sisters. The specification is an extension of that for Exercise 4.1, with the addition of the parameter β_3. Provide an interpretation of the regression results and compare it with that for Exercise 4.1.

Answer:

As in Exercise 4.1, the estimate of β_1 provides an estimate of the lower bound of schooling, 10.46 years, when the number of siblings is large. The other parameters do not have straightforward interpretations. The figure below represents the relationship. Comparing this figure with that for Exercise 4.1, it can be seen that it gives a very different picture of the adverse effect of additional siblings. The specification in Exercise 4.1 suggests that the adverse effect is particularly large for the first few siblings, and then attenuates. The revised specification indicates that the adverse effect is more evenly spread and is more enduring. However, the relationship has been fitted with imprecision since the estimates of β_2 and β_3 are not significant.



4.6 Answers to the additional exercises

A4.1

```
. reg LGFDHOPC LGEXPPC

      Source |       SS       df       MS              Number of obs =    6334
-------------+------------------------------           F(  1,  6332) = 4757.00
       Model |  1502.58932     1  1502.58932           Prob > F      =  0.0000
    Residual |  2000.08269  6332  .315869029           R-squared     =  0.4290
-------------+------------------------------           Adj R-squared =  0.4289
       Total |  3502.67201  6333  .553082585           Root MSE      =  .56202
```

```
------------------------------------------------------------------------------
    LGFDHOPC |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
     LGEXPPC |   .6092734   .0088338    68.97   0.000     .5919562    .6265905
       _cons |   .8988291   .0703516    12.78   0.000     .7609161    1.036742
------------------------------------------------------------------------------
```

The regression implies that the income elasticity of expenditure on food is 0.61 (supposing that total household expenditure can be taken as a proxy for permanent income). In addition to testing the null hypothesis that the elasticity is equal to zero, which is rejected at a very high significance level for all the categories, one might test whether it is different from 1, as a means of classifying the categories of expenditure as luxuries (elasticity > 1) and necessities (elasticity < 1).

The table gives the results for all the categories of expenditure.

Regression of *LGCATPC* on *EXPPC*

	n	$\widehat{\beta}_2$	s.e.($\widehat{\beta}_2$)	t ($\beta_2 = 0$)	t ($\beta_2 = 1$)	R^2	*RSS*
ADM	2,815	1.098	0.030	37.20	3.33	0.330	1,383.9
CLOT	4,500	0.794	0.021	37.34	−9.69	0.237	1,394.0
DOM	1,661	0.812	0.049	16.54	−3.84	0.142	273.5
EDUC	561	1.382	0.090	15.43	4.27	0.299	238.1
ELEC	5,828	0.586	0.011	50.95	−36.05	0.308	2,596.3
FDAW	5,102	0.947	0.015	64.68	−3.59	0.451	4,183.6
FDHO	6,334	0.609	0.009	68.97	−44.23	0.429	4,757.0
FOOT	1,827	0.608	0.027	22.11	−14.26	0.211	488.7
FURN	487	0.912	0.085	10.66	−1.03	0.190	113.7
GASO	5,710	0.677	0.012	56.92	−27.18	0.362	3,240.1
HEAL	4,802	0.868	0.021	40.75	−6.22	0.257	1,660.6
HOUS	6,223	1.033	0.014	73.34	2.34	0.464	5,378.5
LIFE	1,253	0.607	0.047	13.00	−8.40	0.119	169.1
LOCT	692	0.510	0.055	9.29	−8.92	0.111	86.2
MAPP	399	0.817	0.033	9.87	−2.21	0.197	97.5
PERS	3,817	0.891	0.019	48.14	−5.88	0.378	2,317.3
READ	2,287	0.909	0.032	28.46	−2.84	0.262	809.9
SAPP	1,037	0.665	0.045	14.88	−7.49	0.176	221.3
TELE	5,788	0.710	0.012	58.30	−23.82	0.370	3,398.8
TEXT	992	0.629	0.046	13.72	−8.09	0.160	188.2
TOB	1,155	0.721	0.035	20.39	−7.87	0.265	415.8
TOYS	2,504	0.733	0.028	26.22	−9.57	0.216	687.5
TRIP	516	0.723	0.077	9.43	−3.60	0.147	88.9

A4.2

```
. reg LGFDHOPC LGEXPPC LGSIZE

----------------------------------------------------------------------------
      Source |       SS       df       MS            Number of obs =    6334
-------------+------------------------------         F(  2,  6331) = 2410.79
       Model |  1514.30728     2   757.15364         Prob> F       =  0.0000
    Residual |  1988.36473  6331  .314068035         R-squared     =  0.4323
-------------+------------------------------         Adj R-squared =  0.4321
       Total |  3502.67201  6333  .553082585         Root MSE      =  .56042
----------------------------------------------------------------------------
    LGFDHOPC |      Coef.   Std. Err.       t    P>|t|     [95% Conf. Interval]
-------------+--------------------------------------------------------------
     LGEXPPC |   .5842097   .0097174    60.12   0.000     .5651604    .6032591
      LGSIZE |  -.0814427   .0133333    -6.11   0.000    -.1075806   -.0553049
       _cons |   1.158326   .0820119    14.12   0.000     .9975545    1.319097
----------------------------------------------------------------------------
```

The income elasticity, 0.58, is now a little lower than before. The size elasticity is significantly negative, suggesting economies of scale and indicating that the model in the previous exercise was misspecified.

The specification is equivalent to that in Exercise 4.5 in the text. Writing the latter again as:

$$LGCAT = \beta_1 + \beta_2 LGEXP + \beta_3 LGSIZE + u$$

we have:

$$LGCAT - LGSIZE = \beta_1 + \beta_2(LGEXP - LGSIZE) + (\beta_3 + \beta_2 - 1)LGSIZE + u$$

and so:

$$LGCATPC = \beta_1 + \beta_2 LGEXPPC + (\beta_3 + \beta_2 - 1)LGSIZE + u.$$

Note that the estimates of the income elasticity are identical to those in Exercise 4.5 in the text. This follows from the fact that the theoretical coefficient, β_2, has not been affected by the manipulation. The specification differs from that in Exercise A4.1 in that we have not dropped the *LGSIZE* term and so we are not imposing the restriction $\beta_3 + \beta_2 - 1 = 0$.

Dependent variable $LGCATPC$								
		$LGEXPPC$		$LGSIZE$				
	n	$\widehat{\beta}_2$	s.e.$(\widehat{\beta}_2)$	$\widehat{\beta}_3$	s.e.$(\widehat{\beta}_3)$	R^2	F	RSS
ADM	2,815	1.080	0.033	−0.055	0.043	0.330	692.9	3,945.2
$CLOT$	4,500	0.842	0.024	0.146	0.032	0.240	710.1	5,766.1
DOM	1,661	0.941	0.054	0.415	0.075	0.157	154.6	4,062.5
$EDUC$	561	1.229	0.101	−0.437	0.139	0.311	125.9	1,380.1
$ELEC$	5,828	0.372	0.012	−0.362	0.017	0.359	1,627.8	2,636.3
$FDAW$	5,102	0.879	0.016	−0.213	0.022	0.461	2,176.6	3,369.1
$FDHO$	6,334	0.584	0.010	−0.081	0.013	0.432	2,410.8	1,988.4
$FOOT$	1,827	0.396	0.031	−0.560	0.042	0.281	356.1	1,373.5
$FURN$	487	0.807	0.103	−0.246	0.137	0.195	58.7	913.9
$GASO$	5,710	0.676	0.013	−0.004	0.018	0.362	1,691.8	2,879.3
$HEAL$	4,802	0.779	0.023	−0.306	0.031	0.272	894.6	6,062.5
$HOUS$	6,223	0.989	0.016	−0.140	0.021	0.467	2,729.5	4,825.6
$LIFE$	1,253	0.464	0.050	−0.461	0.065	0.154	113.4	1,559.2
$LOCT$	692	0.389	0.060	−0.396	0.086	0.138	54.9	1,075.1
$MAPP$	399	0.721	0.094	−0.264	0.123	0.206	51.5	576.8
$PERS$	3,817	0.824	0.020	−0.217	0.028	0.388	1,206.3	3,002.2
$READ$	2,287	0.764	0.034	−0.503	0.047	0.297	482.8	2,892.1
$SAPP$	1,037	0.467	0.048	−0.592	0.066	0.236	160.1	1,148.9
$TELE$	5,788	0.640	0.013	−0.222	0.018	0.386	1,816.3	3,055.1
$TEXT$	992	0.388	0.049	−0.713	0.067	0.246	161.0	1,032.9
TOB	1,155	0.563	0.037	−0.515	0.049	0.329	282.1	873.4
$TOYS$	2,504	0.638	0.031	−0.304	0.043	0.231	375.8	2,828.3
$TRIP$	516	0.681	0.083	−0.142	0.109	0.150	45.3	792.8

A4.3 A researcher is considering two regression specifications:

$$\log Y = \beta_1 + \beta_2 \log X + u \qquad (1)$$

and:

$$\log \frac{Y}{X} = \alpha_1 + \alpha_2 \log X + u \qquad (2)$$

where u is a disturbance term.

Determine whether (2) is a reparameterised or a restricted version of (1).

(2) may be rewritten:

$$\log Y = \alpha_1 + (\alpha_2 + 1) \log X + u$$

so it is a reparameterised version of (1) with $\beta_1 = \alpha_1$ and $\beta_2 = \alpha_2 + 1$.

Writing $y = \log Y$, $x = \log X$, and $z = \log \frac{Y}{X}$, and using the same sample of n observations, the researcher fits the two specifications using OLS:

$$\widehat{y} = \widehat{\beta}_1 + \widehat{\beta}_2 x \qquad (3)$$

and:

$$\widehat{z} = \widehat{\alpha}_1 + \widehat{\alpha}_2 x. \qquad (4)$$

Using the expressions for the OLS regression coefficients, demonstrate that $\widehat{\beta}_2 = \widehat{\alpha}_2 + 1$.

$$\begin{aligned}\widehat{\alpha}_2 &= \frac{\sum(x_i - \bar{x})(z_i - \bar{z})}{\sum(x_i - \bar{x})^2} = \frac{\sum(x_i - \bar{x})([y_i - x_i] - [\bar{y} - \bar{x}])}{\sum(x_i - \bar{x})^2} \\ &= \frac{\sum(x_i - \bar{x})(y_i - \bar{y})}{\sum(x_i - \bar{x})^2} - \frac{\sum(x_i - \bar{x})^2}{\sum(x_i - \bar{x})^2} = \widehat{\beta}_2 - 1.\end{aligned}$$

Similarly, using the expressions for the OLS regression coefficients, demonstrate that $\widehat{\beta}_1 = \widehat{\alpha}_1$.

$$\widehat{\alpha}_1 = \bar{z} - \widehat{\alpha}_2\bar{x} = (\bar{y} - \bar{x}) - \widehat{\alpha}_2\bar{x} = \bar{y} - (\widehat{\alpha}_2 + 1)\bar{x} = \bar{y} - \widehat{\beta}_2\bar{x} = \widehat{\beta}_1.$$

Hence demonstrate that the relationship between the fitted values of y, *the fitted values of* z, *and the actual values of* x, *is* $\widehat{y}_i - x_i = \widehat{z}_i$.

$$\widehat{z}_i = \widehat{\alpha}_1 + \widehat{\alpha}_2 x_i = \widehat{\beta}_1 + (\widehat{\beta}_2 - 1)x_i = \widehat{\beta}_1 + \widehat{\beta}_2 x_i - x_i = \widehat{y}_i - x_i.$$

Hence show that the residuals for regression (3) are identical to those for (4).

Let $\widehat{u}_i$ be the residual in (3) and $\widehat{v}_i$ the residual in (4). Then:

$$\widehat{v}_i = z_i - \widehat{z}_i = y_i - x_i - (\widehat{y}_i - x_i) = y_i - \widehat{y}_i = \widehat{u}_i.$$

Hence show that the standard errors of $\widehat{\beta}_2$ *and* $\widehat{\alpha}_2$ *are the same.*

The standard error of $\widehat{\beta}_2$ is:

$$\text{s.e.}(\widehat{\beta}_2) = \sqrt{\frac{\sum \widehat{u}_i^2/(n-2)}{\sum(x_i - \bar{x})^2}} = \sqrt{\frac{\sum \widehat{v}_i^2/(n-2)}{\sum(x_i - \bar{x})^2}} = \text{s.e.}(\widehat{\alpha}_2).$$

Determine the relationship between the t *statistic for* $\widehat{\beta}_2$ *and the* t *statistic for* $\widehat{\alpha}_2$, *and give an intuitive explanation for the relationship.*

$$t_{\widehat{\beta}_2} = \frac{\widehat{\beta}_2}{\text{s.e.}(\widehat{\beta}_2)} = \frac{\widehat{\alpha}_2 + 1}{\text{s.e.}(\widehat{\alpha}_2)}.$$

The t statistic for $\widehat{\beta}_2$ is for the test of $H_0 : \beta_2 = 0$. Given the relationship, it is also for the test of $H_0 : \alpha_2 = -1$. The tests are equivalent since both of them reduce the model to $\log Y$ depending only on an intercept and the disturbance term.

Explain whether R^2 *would be the same for the two regressions.*

R^2 will be different because it measures the proportion of the variance of the dependent variable explained by the regression, and the dependent variables are different.

A4.4 The proposed model:

$$SKILL = \beta_1 + \beta_2 \log(EXP) + \beta_3 \log(EXP^2) + u$$

cannot be fitted since:

$$\log(EXP^2) = 2\log(EXP)$$

and the specification is therefore subject to exact multicollinearity.

A4.5 In (1) R^2 is the proportion of the variance of Y explained by the regression. In (2) it is the proportion of the variance of $\log Y$ explained by the regression. Thus, although related, they are not directly comparable. In (1) *RSS* has dimension the squared units of Y. In (2) it has dimension the squared units of $\log Y$. Typically it will be much lower in (2) because the logarithm of Y tends to be much smaller than Y.

The specifications with the same dependent variable may be compared directly in terms of *RSS* (or R^2) and hence two of the specifications may be eliminated immediately. The remaining two specifications should be compared after scaling, with Y replaced by Y^* where Y^* is defined as Y divided by the geometric mean of Y in the sample. *RSS* for the scaled regressions will then be comparable.

A4.6 The *RSS* comparisons for all the categories of expenditure indicate that the logarithmic specification is overwhelmingly superior to the linear one. The differences are actually surprisingly large and suggest that some other factor may also be at work. One possibility is that the data contain many outliers, and these do more damage to the fit in linear than in logarithmic specifications. To see this, plot *CATPC* and *EXPPC* and compare with a plot of *LGCATPC* and *LGEXPPC*. (Strictly speaking, you should control for *SIZE* and *LGSIZE* using the Frisch–Waugh–Lovell method described in Chapter 3.)

The following Stata output gives the results of fitting the model for *FDHO*, assuming that both the dependent variable and the explanatory variables are subject to the Box–Cox transformation with the same value of λ. Iteration messages have been deleted. The maximum likelihood estimate of λ is 0.10, so the logarithmic specification is a better approximation than the linear specification. The latter is very soundly rejected by the likelihood-ratio test.

```
. boxcox FDHOPC EXPPC SIZE if FDHO>0, model(lambda)
                                                  Number of obs   =      6334
                                                  LR chi2(2)      =   3592.55
Log likelihood = -41551.328                       Prob > chi2     =     0.000

------------------------------------------------------------------------------
      FDHOPC |      Coef.   Std. Err.      z    P>|z|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
     /lambda |   .1019402   .0117364     8.69   0.000     .0789372    .1249432
------------------------------------------------------------------------------
Estimates of scale-variant parameters
----------------------------
             |      Coef.
-------------+--------------
Notrans      |
       _cons |   2.292828
-------------+--------------
Trans        |
       EXPPC |   .4608736
        SIZE |  -.1486856
-------------+--------------
      /sigma |   .9983288
----------------------------
```

```
------------------------------------------------------------
    Test        Restricted     LR statistic      P-value
     H0:      log likelihood       chi2        Prob > chi2
------------------------------------------------------------
lambda = -1     -50942.835     18783.01           0.000
lambda =  0     -41590.144        77.63           0.000
lambda =  1     -44053.749      5004.84           0.000
------------------------------------------------------------
```

A4.7 Let the theoretical model for the regression be written:

$$LGEARN = \beta_1 + \beta_2 S + \beta_3 EXP + \beta_4 ASVABC + \beta_5 SA + u.$$

The estimate of β_4 is negative, at first sight suggesting that cognitive ability has an adverse effect on earnings, contrary to common sense and previous results with wage equations of this kind. However, rewriting the model as:

$$LGEARN = \beta_1 + \beta_2 S + \beta_3 EXP + (\beta_4 + \beta_5 S) ASVABC + u$$

it can be seen that, as a consequence of the inclusion of the interactive term, β_4 represents the effect of a marginal year of schooling for an individual with no schooling. Since no individual in the sample had fewer than 8 years of schooling, the perverse sign of the estimate illustrates only the danger of extrapolating outside the data range. It makes better sense to evaluate the implicit coefficient for an individual with the mean years of schooling, 14.9. This is $(-0.2096 + 0.0189 \times 14.9) = 0.072$, implying a much more plausible 7.2 per cent increase in earnings for each standard deviation increase in cognitive ability. The positive sign of the coefficient of SA suggests that schooling and cognitive ability have mutually reinforcing effects on earnings.

One way of avoiding nonsense parameter estimates is to measure the variables in question from their sample means. This has been done in the regression output below, where *S1* and *ASVABC1* are schooling and *ASVABC* measured from their sample means and *SASVABC1* is their interaction. The coefficients of *S* and *ASVABC* now provide estimates of their effects when the other variable is equal to its sample mean.

```
. reg LGEARN S1 EXP ASVABC1 SASVABC1
----------------------------------------------------------------------------
    Source |       SS       df       MS              Number of obs =     500
-----------+------------------------------           F(  4,   495) =   22.68
     Model |  23.6368304     4  5.90920759           Prob > F      =  0.0000
  Residual |  128.962389   495  .260530079           R-squared     =  0.1549
-----------+------------------------------           Adj R-squared =  0.1481
     Total |   152.59922   499   .30581006           Root MSE      =  .51042
----------------------------------------------------------------------------
    LGEARN |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-----------+----------------------------------------------------------------
        S1 |   .0815188   .0116521     7.00   0.000     .0586252    .1044125
       EXP |   .0400506   .0096479     4.15   0.000     .0210948    .0590065
   ASVABC1 |   .0715084   .0298278     2.40   0.017     .0129036    .1301132
  SASVABC1 |   .0188685   .0093393     2.02   0.044     .0005189    .0372181
     _cons |   2.544783   .0675566    37.67   0.000      2.41205    2.677516
----------------------------------------------------------------------------
```

A4.8 In the first part of the output, *WEIGHT11* is regressed on *HEIGHT*, using *EAWE* Data Set 21. The predict command saves the fitted values from the most recent regression, assigning them the variable name that follows the command, in this case *YHAT*. *YHATSQ* is defined as the square of *YHAT*, and this is added to the regression specification. Somewhat surprisingly, its coefficient is not significant. A logarithmic regression of *WEIGHT11* on *HEIGHT* yields an estimated elasticity of 2.05, significantly different from 1 at a high significance level. Multicollinearity is responsible for the failure to detect nonlinearity hear. *YHAT* is very highly correlated with *HEIGHT*.

```
. reg WEIGHT11 HEIGHT
----------------------------------------------------------------------------
    Source |       SS       df       MS              Number of obs =     500
-----------+------------------------------           F(  1,   498) =  139.97
     Model |  236642.736     1  236642.736           Prob > F      =  0.0000
  Residual |  841926.912   498  1690.61629           R-squared     =  0.2194
-----------+------------------------------           Adj R-squared =  0.2178
     Total |  1078569.65   499  2161.46222           Root MSE      =  41.117
----------------------------------------------------------------------------
  WEIGHT11 |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-----------+----------------------------------------------------------------
    HEIGHT |   5.369246   .4538259    11.83   0.000     4.477597    6.260895
     _cons |  -184.7802    30.8406    -5.99   0.000    -245.3739   -124.1865
----------------------------------------------------------------------------


. predict YHAT
. gen YHATSQ = YHAT*YHAT

. reg WEIGHT11 HEIGHT YHATSQ
----------------------------------------------------------------------------
    Source |       SS       df       MS              Number of obs =     500
-----------+------------------------------           F(  2,   497) =   70.33
     Model |  237931.888     2  118965.944           Prob > F      =  0.0000
  Residual |   840637.76   497  1691.42407           R-squared     =  0.2206
-----------+------------------------------           Adj R-squared =  0.2175
     Total |  1078569.65   499  2161.46222           Root MSE      =  41.127
----------------------------------------------------------------------------
  WEIGHT11 |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-----------+----------------------------------------------------------------
    HEIGHT |  -.4995924   6.737741    -0.07   0.941    -13.73756    12.73837
    YHATSQ |   .0030233    .003463     0.87   0.383    -.0037807    .0098273
     _cons |   114.5523   344.2538     0.33   0.739    -561.8199    790.9244
----------------------------------------------------------------------------
```

Chapter 5
Dummy variables

5.1 Overview

This chapter explains the definition and use of a dummy variable, a device for allowing qualitative characteristics to be introduced into the regression specification. Although the intercept dummy may appear artificial and strange at first sight, and the slope dummy even more so, you will become comfortable with the use of dummy variables very quickly. The key is to keep in mind the graphical representation of the regression model.

5.2 Learning outcomes

After working through the corresponding chapter in the text, studying the corresponding slideshows, and doing the starred exercises in the text and the additional exercises in this subject guide, you should be able to explain:

- how the intercept and slope dummy variables are defined
- what impact they have on the regression specification
- how the choice of reference (omitted) category affects the interpretation of t tests on the coefficients of dummy variables
- how a change of reference category would affect the regression results
- how to perform a Chow test
- when and why a Chow test is equivalent to a particular F test of the joint explanatory power of a set of dummy variables.

5.3 Additional exercises

A5.1 In Exercise A1.4 the logarithm of earnings was regressed on height using *EAWE* Data Set 21 and, somewhat surprisingly, it was found that height had a highly significant positive effect. We have seen that the logarithm of earnings is more satisfactory than earnings as the dependent variable in a wage equation. Fitting the semilogarithmic specification, we obtain:

```
. reg LGEARN HEIGHT
------------------------------------------------------------------------------
    Source |       SS       df       MS              Number of obs =     500
-----------+------------------------------           F(  1,   498) =    6.27
     Model |  1.84965685     1  1.84965685           Prob > F      =  0.0126
  Residual |   146.79826   498  .294775622           R-squared     =  0.0124
-----------+------------------------------           Adj R-squared =  0.0105
     Total |  148.647917   499  .297891616           Root MSE      =  .54293
------------------------------------------------------------------------------
    LGEARN |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-----------+------------------------------------------------------------------
    HEIGHT |   .0148894    .005944     2.50   0.013      .003211    .0265678
     _cons |   1.746174   .4032472     4.33   0.000     .9538982    2.538449
------------------------------------------------------------------------------
```

The t statistic for *HEIGHT* is again significant, if only at the 5 per cent level. In Exercise A1.4 it was hypothesised that the effect might be attributable to males tending to have greater earnings than females and also tending to be taller. The output below shows the result of adding the dummy variable to the specification, to control for sex. Comment on the results.

```
. reg LGEARN HEIGHT MALE
------------------------------------------------------------------------------
    Source |       SS       df       MS              Number of obs =     500
-----------+------------------------------           F(  2,   497) =    4.20
     Model |  2.47043329     2  1.23521664           Prob > F      =  0.0155
  Residual |  146.177483   497  .294119685           R-squared     =  0.0166
-----------+------------------------------           Adj R-squared =  0.0127
     Total |  148.647917   499  .297891616           Root MSE      =  .54233
------------------------------------------------------------------------------
    LGEARN |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-----------+------------------------------------------------------------------
    HEIGHT |   .0060845   .0084844     0.72   0.474    -.0105852    .0227541
      MALE |   .1007018   .0693157     1.45   0.147    -.0354862    .2368898
     _cons |   2.292078   .5508559     4.16   0.000     1.209784    3.374371
------------------------------------------------------------------------------
```

A5.2 *Does ethnicity have an effect on household expenditure?*

The variable *REFRACE* in the *CES* data set is coded 1 if the reference individual in the household, usually the head of the household, is white and it is coded greater than 1 for other ethnicities. Define a dummy variable *NONWHITE* that is 0 if *REFRACE* is 1 and 1 if *REFRACE* is greater than 1. Regress *LGCATPC* on *LGEXPPC*, *LGSIZE*, and *NONWHITE*. Provide an interpretation of the coefficients, and perform appropriate statistical tests.

A5.3 *Does education have an effect on household expenditure?*

The variable *REFEDUC* in the *CES* data set provides information on the education of the reference individual in the household. Define dummy variables *EDUCDO* (high-school drop out or less), *EDUCSC* (some college), and *EDUCBA* (complete college or more) using the following rules:

- *EDUCDO* = 1 if *REFEDUC* < 12, 0 otherwise
- *EDUCSC* = 1 if *REFEDUC* = 13 or 14, 0 otherwise
- *EDUCBA* = 1 if *REFEDUC* > 14, 0 otherwise.

Regress *LGCATPC* on *LGEXPPC*, *LGSIZE*, *EDUCDO*, *EDUCSC*, and *EDUCBA*. Provide an interpretation of the coefficients, and perform appropriate statistical tests. Note that the reference (omitted) category for the dummy variables is high school graduate with no college (*REFEDUC* = 12).

A5.4 Using the *CES* data set, evaluate whether the education dummies as a group have significant explanatory power for expenditure on your category of expenditure by comparing the residual sums of squares in the regressions in Exercises A4.2 and A5.3.

A5.5 Repeat Exercise A5.3 making *EDUCDO* the reference (omitted) category. Introduce a new dummy variable *EDUCHSD* for high school diploma, since this is no longer the omitted category:

- *EDUCHSD* = 1 if *REFEDUC* = 12, 0 otherwise.

Evaluate the impact on the interpretation of the coefficients and the statistical tests.

A5.6 A researcher has data on hourly earnings in dollars, *EARNINGS*, years of schooling (highest grade completed), S, and sector of employment, GOV, for 1,355 male respondents in the National Longitudinal Survey of Youth 1979– for 2002. GOV is defined as a dummy variable equal to 0 if the respondent was working in the private sector and 1 if the respondent was working in the government sector. 91 per cent of the private sector workers and 95 per cent of the government sector workers had at least 12 years of schooling. The mean value of S was 13.5 for the private sector and 14.6 for the government sector. The researcher regresses *LGEARN*, the natural logarithm of *EARNINGS*:

- (1) on GOV alone
- (2) on GOV and S
- (3) on GOV, S, and $SGOV$

where the variable $SGOV$ is defined to be the product of S and GOV, with the results shown in the following table.

Standard errors are shown in parentheses and t statistics in square brackets. *RSS* = residual sum of squares.

	(1)	(2)	(3)
GOV	0.007 (0.043) [0.16]	-0.121 (0.038) [-3.22]	0.726 (0.193) [3.76]
S	—	0.116 (0.006) [21.07]	0.130 (0.006) [20.82]
$SGOV$	—	—	-0.059 (0.013) [-4.48]
constant	2.941 (0.018) [163.62]	1.372 (0.076) [18.04]	1.195 (0.085) [14.02]
R^2	0.000	0.247	0.258
RSS	487.7	367.2	361.8

- Explain verbally why the estimates of the coefficient of GOV are different in regressions (1) and (2).
- Explain the difference in the estimates of the coefficient of GOV in regressions (2) and (3).
- The correlation between GOV and $SGOV$ was 0.977. Explain the variations in the standard error of the coefficient of GOV in the three regressions.

A5.7 A researcher has data on the average annual rate of growth of employment, e, and the average annual rate of growth of GDP, x, both measured as percentages, for a sample of 27 developing countries and 23 developed ones for the period 1985–1995. He defines a dummy variable D that is equal to 1 for the developing countries and 0 for the others. Hypothesising that the impact of GDP growth on employment growth is lower in the developed countries than in the developing ones, he defines a slope dummy variable xD as the product of x and D and fits the regression (standard errors in parentheses):

$$\begin{array}{llll} \text{whole sample} & \widehat{e} = & -1.45 + 0.19x + 0.78xD & R^2 = 0.61 \\ & & (0.36)\ \ (0.10)\ \ (0.10) & RSS = 50.23 \end{array}$$

He also runs simple regressions of e on x for the whole sample, for the developed countries only, and for the developing countries only, with the following results:

$$\begin{array}{llll} \text{whole sample} & \widehat{e} = & -0.56 + 0.24x & R^2 = 0.04 \\ & & (0.53)\ \ (0.16) & RSS = 121.61 \\ \text{developed countries} & \widehat{e} = & -2.74 + 0.50x & R^2 = 0.35 \\ & & (0.58)\ (0.15) & RSS = 18.63 \\ \text{developing countries} & \widehat{e} = & -0.85 + 0.78x & R^2 = 0.51 \\ & & (0.42)\ (0.15) & RSS = 25.23 \end{array}$$

- Explain mathematically and graphically the role of the dummy variable xD in this model.

- The researcher could have included D as well as xD as an explanatory variable in the model. Explain mathematically and graphically how it would have affected the model.
- Suppose that the researcher had included D as well as xD.
  - What would the coefficients of the regression have been?
  - What would the residual sum of squares have been?
  - What would the t statistic for the coefficient of D have been?
- Perform two tests of the researcher's hypothesis. Explain why you would *not* test it with a t test on the coefficient of xD in regression (1).

A5.8 *Does going to college have an effect on household expenditure?*

Using the *CES* data set, define a dummy variable *COLLEGE* that is 0 if *REFEDUC* is less than 13 (no college education) and 1 if *REFEDUC* is greater than 12 (partial or complete college education). Regress *LGCATPC* on *LGEXPPC* and *LGSIZE*: (1) for those respondents with *COLLEGE* = 1, (2) for those respondents with *COLLEGE* = 0, and (3) for the whole sample. Perform a Chow test.

A5.9 *How does education impact on household expenditure?*

In Exercise A5.8 you defined an intercept dummy *COLLEGE* that allowed you to investigate whether going to college caused a shift in your expenditure function. Now define slope dummy variables that allow you to investigate whether going to college affects the coefficients of *LGEXPPC* and *LGSIZE*. Define *LEXPCOL* as the product of *LGEXPPC* and *COLLEGE*, and define *LSIZECOL* as the product of *LGSIZE* and *COLLEGE*. Regress *LGCATPC* on *LGEXPPC*, *LGSIZE*, *COLLEGE*, *LEXPCOL*, and *LSIZECOL*. Provide an interpretation of the coefficients, and perform appropriate tests. Include a test of the joint explanatory power of the dummy variables by comparing *RSS* in this regression with that in Exercise A4.3. Verify that the outcome of this F test is identical to that for the Chow test in Exercise A5.8.

A5.10 You are given the following data on 2,800 respondents in the National Longitudinal Survey of Youth 1979– with jobs in 2011:

- hourly earnings in the respondent's main job at the time of the 2011 interview
- educational attainment (highest grade completed)
- mother's and father's educational attainment
- *ASVABC* score
- sex
- ethnicity: black, Hispanic, or white, that is (not black nor Hispanic)
- whether the main job in 2011 was in the government sector or the private sector.

As a policy analyst, you are asked to investigate whether there is evidence of earnings discrimination, positive or negative, by sex or ethnicity in (1) the government sector, and (2) the private sector. Explain how you would do this, giving a mathematical representation of your regression specification(s).

You are also asked to investigate whether the incidence of earnings discrimination, if any, is significantly different in the two sectors. Explain how you would do this, giving a mathematical representation of your regression specification(s). In particular, discuss whether a Chow test would be useful for this purpose.

A5.11 A researcher has data from the National Longitudinal Survey of Youth 1997– for the year 2000 on hourly earnings, Y, years of schooling, S, and years of work experience, EXP, for a sample of 1,774 males and 1,468 females. She defines a dummy variable $MALE$ for being male, a slope dummy variable $SMALE$ as the product of S and $MALE$, and another slope dummy variable $EXPMALE$ as the product of EXP and $MALE$. She performs the following regressions (1) $\log Y$ on S and EXP for the entire sample, (2) $\log Y$ on S and EXP for males only, (3) $\log Y$ on S and EXP for females only, (4) $\log Y$ on S, EXP, and $MALE$ for the entire sample, and (5) $\log Y$ on S, EXP, $MALE$, $SMALE$, and $EXPMALE$ for the entire sample. The results are shown in the table, with standard errors in parentheses. RSS is the residual sum of squares and n is the number of observations.

	(1)	(2)	(3)	(4)	(5)
S	0.094	0.099	0.094	0.0967	0.094
	(0.003)	(0.004)	(0.005)	(0.003)	(0.005)
EXP	0.046	0.042	0.039	0.040	0.039
	(0.002)	(0.003)	(0.002)	(0.002)	(0.003)
$MALE$	—	—	—	0.234	0.117
				(0.016)	(0.108)
$SMALE$	—	—	—	—	0.005
					(0.007)
$EXPMALE$	—	—	—	—	0.003
					(0.004)
constant	5.165	5.283	5.166	5.111	5.166
	(0.054)	(0.083)	(0.068)	(0.052)	(0.074)
R^2	0.319	0.277	0.363	0.359	0.359
RSS	714.6	411.0	261.6	672.8	672.5
n	3,242	1,774	1,468	3,242	3,242

The correlations between $MALE$ and $SMALE$, and $MALE$ and $EXPMALE$, were both 0.96. The correlation between $SMALE$ and $EXPMALE$ was 0.93.

- Give an interpretation of the coefficients of S and $SMALE$ in regression (5).
- Give an interpretation of the coefficients of $MALE$ in regressions (4) and (5).
- The researcher hypothesises that the earnings function is different for males and females. Perform a test of this hypothesis using regression (4), and also using regressions (1) and (5).
- Explain the differences in the tests using regression (4) and using regressions (1) and (5).

- At a seminar someone suggests that a Chow test could shed light on the researcher's hypothesis. Is this correct?
- Explain which of (1), (4), and (5) would be your preferred specification.

A5.12 A researcher has data for the year 2000 from the National Longitudinal Survey of Youth 1997– on the following characteristics of the respondents: hourly earnings, *EARNINGS*, measured in dollars; years of schooling, *S*; years of work experience, *EXP*; sex; and ethnicity (blacks, hispanics, and 'whites' (those not classified as black or hispanic). She drops the hispanics from the sample, leaving 2,135 'whites' and 273 blacks, and defines dummy variables *MALE* and *BLACK*. *MALE* is defined to be 1 for males and 0 for females. *BLACK* is defined to be 1 for blacks and 0 for 'whites'. She defines *LGEARN* to be the natural logarithm of *EARNINGS*. She fits the following ordinary least squares regressions, each with *LGEARN* as the dependent variable:

- (1) Explanatory variables *S*, *EXP*, and *MALE*, whole sample
- (2) Explanatory variables *S*, *EXP*, *MALE*, and *BLACK*, whole sample
- (3) Explanatory variables *S*, *EXP*, and *MALE*, 'whites' only
- (4) Explanatory variables *S*, *EXP*, and *MALE*, blacks only.

She then defines interaction terms $SB = S \times BLACK$, $EB = EXP \times BLACK$, and $MB = MALE \times BLACK$, and runs a fifth regression, still with *LGEARN* as the dependent variable:

- (5) Explanatory variables *S*, *EXP*, *MALE*, *BLACK*, *SB*, *EB*, *MB*, whole sample.

The results are shown in the table. Unfortunately, some of those for Regression 4 are missing from the table. *RSS* = residual sum of squares. Standard errors are given in parentheses.

	(1)	(2)	(3)	(4)	(5)
	whole sample	whole sample	'whites' only	blacks only	whole sample
S	0.124	0.121	0.122	**V**	0.122
	(0.004)	(0.004)	(0.004)		(0.004)
EXP	0.033	0.032	0.033	**W**	0.033
	(0.002)	(0.002)	(0.003)		(0.003)
$MALE$	0.278	0.277	0.306	**X**	0.306
	(0.020)	(0.020)	(0.021)		(0.021)
$BLACK$	—	−0.144	—	—	0.205
		(0.032)			(0.225)
SB	—	—	—	—	−0.009
					(0.016)
EB	—	—	—	—	−0.006
					(0.007)
MB	—	—	—	—	−0.280
					(0.065)
constant	0.390	0.459	0.411	**Y**	0.411
	(0.075)	(0.076)	(0.084)		(0.082)
R^2	0.335	0.341	0.332	0.321	0.347
RSS	610.0	605.1	555.7	**Z**	600.0
n	2,408	2,408	2,135	273	2,408

- Calculate the missing coefficients **V**, **W**, **X**, and **Y** in Regression 4 (just the coefficients, not the standard errors) and **Z**, the missing *RSS*, giving an explanation of your computations.
- Give an interpretation of the coefficient of *BLACK* in Regression 2.
- Perform an F test of the joint explanatory power of *BLACK*, *SB*, *EB*, and *MB* in Regression 5.
- Explain whether it is possible to relate the F test in part (c) to a Chow test based on Regressions 1, 3, and 4.
- Give an interpretation of the coefficients of *BLACK* and *MB* in Regression 5.
- Explain whether a simple t test on the coefficient of *BLACK* in Regression 2 is sufficient to show that the wage equations are different for blacks and 'whites'.

A5.13 As part of a workshop project, four students are investigating the effects of ethnicity and sex on earnings using data for the year 2002 in the National Longitudinal Survey of Youth 1979–. They all start with the same basic specification:

$$\log Y = \beta_1 + \beta_2 S + \beta_3 EXP + u$$

where Y is hourly earnings, measured in dollars, S is years of schooling completed, and *EXP* is years of work experience. The sample contains 123 black males, 150 black females, 1,146 white males, and 1,127 white females. (All respondents were either black or white. The Hispanic subsample was dropped.) The output from fitting this basic specification is shown in column 1 of the table (standard errors in parentheses; RSS is residual sum of squares, n is the number of observations in the regression).

	Basic	Student C		Student D			
	(1)	(2)	(3)	(4a)	(4b)	(5a)	(5b)
	All	All	All	Males	Females	Whites	Blacks
S	0.126	0.121	0.121	0.133	0.112	0.126	0.112
	(0.004)	(0.004)	(0.004)	(0.006)	(0.006)	(0.005)	(0.012)
EXP	0.040	0.032	0.032	0.032	0.035	0.041	0.028
	(0.002)	(0.002)	(0.002)	(0.004)	(0.003)	(0.003)	(0.005)
$MALE$	—	0.277	0.308	—	—	—	—
		(0.020)	(0.021)				
$BLACK$	—	−0.144	−0.011	—	—	—-	—
		(0.032)	(0.043)				
$MALEBLACK$	—	—	−0.290	—	—	—	—
			(0.063)				
constant	0.376	0.459	0.447	0.566	0.517	0.375	0.631
	(0.078)	(0.076)	(0.076)	(0.124)	(0.097)	(0.087)	(0.172)
R^2	0.285	0.341	0.346	0.287	0.275	0.271	0.320
RSS	659	608	603	452	289	609	44
n	2,546	2,546	2,546	1,269	1,277	2,273	273

Student A divides the sample into the four ethnicity/sex categories. He chooses white females as the reference category and fits a regression that includes three dummy variables BM, WM, and BF. BM is 1 for black males, 0 otherwise; WM is 1 for white males, 0 otherwise, and BF is 1 for black females, 0 otherwise.

Student B simply fits the basic specification separately for the four ethnicity/sex subsamples.

Student C defines dummy variables $MALE$, equal to 1 for males and 0 for females, and $BLACK$, equal to 1 for blacks and 0 for whites. She also defines an interactive dummy variable $MALEBLACK$ as the product of $MALE$ and $BLACK$. She fits a regression adding $MALE$ and $BLACK$ to the basic specification, and a further regression adding $MALEBLACK$ as well. The output from these regressions is shown in columns 2 and 3 in the table.

Student D divides the sample into males and females and performs the regression for both sexes separately, using the basic specification. The output is shown in columns 4a and 4b. She also divides the sample into whites and blacks, and again runs separate regressions using the basic specification. The output is shown in columns 5a and 5b.

Reconstruction of missing output.

Students A and B left their output on a bus on the way to the workshop. This is why it does not appear in the table.

- State what the missing output of Student A would have been, as far as this is can be done exactly, given the results of Students C and D. (Coefficients, standard errors, R^2, RSS.)

- Explain why it is not possible to reconstruct any of the output of Student B.

Tests of hypotheses.

The approaches of the students allowed them to perform different tests, given the output shown in the table and the corresponding output for Students A and B. Explain the tests relating to the effects of sex and ethnicity that could be performed by each student, giving a clear indication of the null hypothesis in each case. (Remember, all of them started with the basic specification (1), before continuing with their individual regressions.) In the case of F tests, state the test statistic in terms of its components.

- Student A (assuming he had found his output)
- Student B (assuming he had found his output)
- Student C
- Student D.

If you had been participating in the project and had had access to the data set, what regressions and tests would you have performed?

5.4 Answers to the starred exercises in the textbook

5.2 The Stata output for Data Set 21 shows the result of regressing weight in 2004, measured in pounds, on height, measured in inches, first with a linear specification, then with a logarithmic one, in both cases including a dummy variable *MALE*, defined as in Exercise 5.1. Give an interpretation of the coefficients and perform appropriate statistical tests. See Box 5.1 for a guide to the interpretation of dummy variable coefficients in logarithmic regressions.

```
. reg WEIGHT04 HEIGHT MALE
----------------------------------------------------------------------------
    Source |       SS       df       MS              Number of obs =     500
-----------+------------------------------           F(  2,   497) =   90.45
     Model |   215264.34     2   107632.17           Prob > F      =  0.0000
  Residual |   591434.61   497  1190.00927           R-squared     =  0.2668
-----------+------------------------------           Adj R-squared =  0.2639
     Total |   806698.95   499  1616.63116           Root MSE      =  34.497
----------------------------------------------------------------------------
  WEIGHT04 |      Coef.   Std. Err.       t    P>|t|     [95% Conf. Interval]
-----------+----------------------------------------------------------------
    HEIGHT |   4.424345   .5213809     8.49   0.000     3.399962    5.448727
      MALE |   7.702828   4.225065     1.82   0.069     -.598363    16.00402
     _cons |  -136.9713    33.9953    -4.03   0.000    -203.7635   -70.17904
----------------------------------------------------------------------------
```

```
. reg LGWT04 LGHEIGHT MALE
----------------------------------------------------------------------------
    Source |       SS       df       MS              Number of obs =     500
-----------+------------------------------           F(  2,   497) =  109.53
     Model |  8.12184709     2  4.06092355           Prob > F      =  0.0000
  Residual |  18.4269077   497  .037076273           R-squared     =  0.3059
-----------+------------------------------           Adj R-squared =  0.3031
     Total |  26.5487548   499  .053203918           Root MSE      =  .19255
----------------------------------------------------------------------------
    LGWT04 |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-----------+----------------------------------------------------------------
  LGHEIGHT |     1.7814   .1978798     9.00   0.000     1.392616    2.170185
      MALE |   .0566894   .0236289     2.40   0.017     .0102645    .1031142
     _cons |   -2.44656   .8261259    -2.96   0.003     -4.06969   -.8234307
----------------------------------------------------------------------------
```

Answer:

The first regression indicates that weight increase by 4.4 pounds for each inch of stature and that males tend to weigh 7.7 pounds more than females, controlling for height, but the coefficient of $MALE$ is not significant. The second regression indicates that the elasticity of weight with respect to height is 1.78, and that males weigh 5.7 per cent more than females, the latter effect now being significantly different from zero at the 5 per cent level.

The null hypothesis that the elasticity is zero is not worth testing, except perhaps in a negative sense, for if the result were not highly significant there would have to be something seriously wrong with the model specification. Two other hypotheses might be of greater interest: the elasticity being equal to 1, weight growing proportionally with height, and the elasticity being equal to 3, all dimensions increasing proportionally with height. The t statistics are 4.27 and -8.37, respectively, so both hypotheses are rejected.

5.5 Suppose that the relationship:

$$Y_i = \beta_1 + \beta_2 X_i + u_i$$

is being fitted and that the value of X is missing for some observations. One way of handling the missing values problem is to drop those observations. Another is to set $X = 0$ for the missing observations and include a dummy variable D defined to be equal to 1 if X is missing, 0 otherwise. Demonstrate that the two methods must yield the same estimates of β_1 and β_2. Write down an expression for RSS using the second approach, decompose it into the RSS for observations with X present and RSS for observations with X missing, and determine how the resulting expression is related to RSS when the missing value observations are dropped.

Answer:

Let the fitted model, with D included, be:

$$\widehat{Y}_i = \widehat{\beta}_1 + \widehat{\beta}_2 X_i + \widehat{\beta}_3 D_i.$$

If X is missing for observations $m+1$ to n, then:

$$\begin{aligned} RSS &= \sum_{i=1}^{n}(Y_i - \widehat{Y}_i)^2 = \sum_{i=1}^{n}(Y_i - (\widehat{\beta}_1 + \widehat{\beta}_2 X_i + \widehat{\beta}_3 D_i))^2 \\ &= \sum_{i=1}^{m}(Y_i - (\widehat{\beta}_1 + \widehat{\beta}_2 X_i + \widehat{\beta}_3 D_i))^2 + \sum_{i=m+1}^{n}(Y_i - (\widehat{\beta}_1 + \widehat{\beta}_2 X_i + \widehat{\beta}_3 D_i))^2 \\ &= \sum_{i=1}^{m}(Y_i - (\widehat{\beta}_1 + \widehat{\beta}_2 X_i))^2 + \sum_{i=m+1}^{n}(Y_i - (\widehat{\beta}_1 + \widehat{\beta}_3))^2. \end{aligned}$$

The normal equation for $\widehat{\beta}_3$ will yield:

$$\widehat{\beta}_3 = \widehat{\beta}_1 - \overline{Y}_{\text{missing}}$$

where $\overline{Y}_{\text{missing}}$ is the mean value of Y for those observations for which X is missing. This relationship means that $\widehat{\beta}_1$ and $\widehat{\beta}_2$ may be chosen so as to minimise the first term in RSS. This, of course, is RSS for the regression omitting the observations for which X is missing, and hence $\widehat{\beta}_1$ and $\widehat{\beta}_2$ will be the same as for that regression.

5.7

```
. reg LGEARN EDUCPROF EDUCPHD EDUCMAST EDUCBA EDUCAA EDUCGED EDUCDO EXP MALE
---------------------------------------------------------------------------
    Source |       SS       df       MS              Number of obs =     500
-----------+------------------------------           F(  8,   491) =   17.75
     Model |  34.2318979     8  4.27898724           Prob > F      =  0.0000
  Residual |  118.367322   491  .241073975           R-squared     =  0.2243
-----------+------------------------------           Adj R-squared =  0.2117
     Total |   152.59922   499   .30581006           Root MSE      =  .49099
---------------------------------------------------------------------------
    LGEARN |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-----------+---------------------------------------------------------------
  EDUCPROF |   1.233278   .1920661     6.42   0.000     .8559049    1.610651
   EDUCPHD |  (dropped)
  EDUCMAST |   .7442879   .0875306     8.50   0.000     .5723071    .9162686
    EDUCBA |   .3144576   .0578615     5.43   0.000     .2007709    .4281443
    EDUCAA |   .2076079    .084855     2.45   0.015     .0408843    .3743316
   EDUCGED |  -.2000523   .0886594    -2.26   0.024     -.374251   -.0258537
    EDUCDO |  -.2216305    .132202    -1.68   0.094    -.4813819     .038121
       EXP |   .0261946   .0085959     3.05   0.002     .0093054    .0430839
      MALE |   .1756002   .0445659     3.94   0.000     .0880369    .2631636
     _cons |   2.385391   .0804166    29.66   0.000     2.227388    2.543394
---------------------------------------------------------------------------
```

The Stata output shows the result of a semilogarithmic regression of earnings on highest educational qualification obtained, work experience, and the sex of the respondent, the educational qualifications being a professional degree, a PhD (no respondents in this sample), a Master's degree, a Bachelor's degree, an Associate of Arts degree, the GED certification, and no qualification (high school drop-out). The high school diploma was the reference category. Provide an interpretation of the coefficients and perform t tests.

Answer:

The regression results indicate that those with professional degrees earn 123 per cent more than high school graduates, or 243 per cent more if calculated as $100(e^{1.233} - 1)$, the coefficient being significant at the 0.1 per cent level. There was no respondent with a PhD in this subsample. For the other qualifications the corresponding figures are:

- Master's: 74.4, 110.4, 0.1 per cent.
- Bachelor's: 31.4, 36.9, 0.1 per cent.
- Associate's: 20.8, 23.1, 5 per cent.
- GED: −20.0, −18.1, 5 per cent.
- Drop-out: −22.2, −19.9, 5 per cent, using a one-sided test, as seems reasonable.

Males earn 17.6 per cent (19.2 per cent) more than females, and every year of work experience increases earnings by 2.6 per cent. The coefficient of those with a professional degree should be treated cautiously since there were only seven such individuals in the subsample (*EAWE* 21). For the other categories the numbers of observations were: Master's 42; Bachelor's 168; Associate's 44; High school diploma 187; GED 37; and drop-out 15.

5.8 Given a hierarchical classification such as that of educational qualifications in Exercise 5.7, some researchers unthinkingly choose the bottom category as the omitted category. In the case of Exercise 5.7, this would be *EDUCDO*, the high school drop-outs. Explain why this procedure may be undesirable (and, in the case of Exercise 5.7, definitely would not be recommended).

Answer:

The use of drop-outs as the reference category would make the tests of the coefficients of the other categories of little interest. If one wishes to evaluate the earnings premium for a bachelor's or associate's degree, it is much more sensible to use high school diploma as the benchmark. There is also the consideration that the drop-out category is tiny and unrepresentative.

5.16 Column (1) of the table shows the result of regressing *WEIGHT04* on *HEIGHT*, *MALE*, and ethnicity dummy variables, using *EAWE* Data Set 21. The omitted ethnicity category was *ETHWHITE*. Column (2) shows in abstract the result of the same regression, using *ETHBLACK* as the omitted ethnicity category instead of *ETHWHITE*. As far as this is possible, determine the numbers represented by the letters.

	(1)	(2)
HEIGHT	4.45	**A**
	(0.53)	**(B)**
MALE	7.68	**C**
	(4.26)	**(D)**
ETHBLACK	4.08	—
	(4.52)	
ETHHISP	0.07	**E**
	(4.90)	**(F)**
ETHWHITE	—	**G**
		(H)
constant	−139.41	**I**
	(34.64)	**(J)**
R^2	0.27	**K**
RSS	590,443	**L**
n	500	500

Answer:

The parts of the output unrelated to the dummy variables will not be affected, so A, B, C, D, K, and L are as in column (1). G = −4.08 and H = 4.52.
E = 0.07 − 4.08 = −4.01. I = −139.41 + 4.08 = −135.33. F and J cannot be determined.

5.19 Is the effect of education on earnings different for members of a union? In the output below, *COLLBARG* is a dummy variable defined to be 1 for workers whose wages are determined by collective bargaining and 0 for the others. *SBARG* is a slope dummy variable defined as the product of S and *COLLBARG*. Provide an interpretation of the regression coefficients, comparing them with those in Exercise 5.10, and perform appropriate statistical tests.

```
. gen SBARG=S*COLLBARG
. reg LGEARN S EXP MALE COLLBARG SBARG
----------------------------------------------------------------------------
    Source |       SS       df       MS              Number of obs =     500
-----------+------------------------------           F(  5,   494) =   23.88
     Model |  29.6989993     5  5.93979987           Prob > F      =  0.0000
  Residual |   122.90022   494  .248785871           R-squared     =  0.1946
-----------+------------------------------           Adj R-squared =  0.1865
     Total |   152.59922   499   .30581006           Root MSE      =  .49878
----------------------------------------------------------------------------
    LGEARN |      Coef.   Std. Err.      t    P>|t|      [95% Conf. Interval]
-----------+----------------------------------------------------------------
         S |    .093675    .010815     8.66   0.000       .072426    .1149241
       EXP |   .0423016   .0094148     4.49   0.000      .0238037    .0607995
      MALE |   .1713487   .0453584     3.78   0.000      .0822295    .2604679
  COLLBARG |   .2982818   .3573731     0.83   0.404     -.4038769    1.000441
     SBARG |  -.0026071   .0226557    -0.12   0.908     -.0471205    .0419064
     _cons |   1.034781   .2049246     5.05   0.000      .6321502    1.437413
----------------------------------------------------------------------------
```

Answer:

In this specification, the coefficient of S is an estimate of the effect of schooling on the earnings of those whose earnings are not subject to collective bargaining (henceforward, for short, unionised workers, though obviously the category includes some who do not actually belong to unions), and the coefficient of *SBARG* is the extra effect in the case of those whose earnings are. One might have anticipated a negative coefficient, since seniority and skills are often thought to be more important than schooling for the earnings of union workers, but in fact there is no significant difference.

5.23 Column (1) of the table shows the result of regressing *HOURS*, hours worked per week, on S, *MALE*, and *MALES* using *EAWE* Data Set 21. *MALES* is defined as the product of *MALE* and S. Provide an interpretation of the coefficients.

Column (2) gives the output in abstract when *FEMALE* is used instead of *MALE* and *FEMALES* instead of *MALES*. *FEMALES* is the product of *FEMALE* and S. As far as this is possible, determine the numbers represented by the letters.

	(1)	(2)
S	0.79	**A**
	(0.24)	**(B)**
$MALE$	14.00	—
	(4.99)	
$FEMALE$	—	**C**
		(D)
$MALES$	−0.69	—
	(0.33)	
$FEMALES$	—	**E**
		(F)
constant	25.56	**G**
	(3.71)	**(H)**
R^2	0.05	**I**
RSS	49,384	**J**
n	500	500

Answer:

The coefficient of $MALE$ indicates that a male with no schooling works 14 hours longer than a similar female. The coefficient of S indicates that a female works an extra 0.79 hours per year of schooling. For males, the corresponding figure would be 0.10 hours, taking account of the interactive effect.

A = 0.79 − 0.69 = 0.10. C = −14.00. D = 4.99. E = 0.69.
G = 25.56 + 14.00 = 39.56. I and J are not affected. B, F and H cannot be determined.

5.29 The first paragraph of Section 5.4 used the words 'satisfactory' and 'better'. Such intuitive terms have no precise meaning in econometrics. What ideas were they trying to express?

Answer:

The Chow test is effectively an F test of the joint explanatory power of a full set of dummy variables. If the joint explanatory power is significant, this implies that the model is misspecified if they are omitted. In this sense, it is 'better' to include them.

5.5 Answers to the additional exercises

A5.1 As was to be expected, the coefficient of $HEIGHT$ falls with the addition of $MALE$ to the specification and is no longer significant. However, the coefficient of $MALE$ is not significant, either. This is because $MALE$ and $HEIGHT$ are sufficiently correlated (correlation coefficient 0.71) to give rise to a problem of multicollinearity.

A5.2

```
. reg LGFDHOPC LGEXPPC LGSIZE NONWHITE
----------------------------------------------------------------------------
    Source |       SS       df       MS              Number of obs =    6334
-----------+------------------------------           F(  3,  6330) = 1607.67
     Model |  1514.69506     3  504.898354           Prob > F      =  0.0000
  Residual |  1987.97695  6330   .31405639           R-squared     =  0.4324
-----------+------------------------------           Adj R-squared =  0.4322
     Total |  3502.67201  6333  .553082585           Root MSE      =  .56041
----------------------------------------------------------------------------
  LGFDHOPC |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-----------+----------------------------------------------------------------
   LGEXPPC |   .5831052   .0097679    59.70   0.000     .5639568    .6022535
    LGSIZE |  -.0814498   .0133331    -6.11   0.000    -.1075871   -.0553124
  NONWHITE |  -.0195916   .0176311    -1.11   0.267    -.0541544    .0149713
     _cons |   1.171052   .0828062    14.14   0.000     1.008723     1.33338
----------------------------------------------------------------------------
```

The regression indicates that, controlling for total household expenditure per capita and size of household, non-whites spend 2.0 per cent less per year than whites on food consumed at home. However, the effect is not significant. The coefficients of *LGEXPPC* and *LGSIZE* are not affected by the introduction of the dummy variable.

Summarising the effects for all the categories of expenditure, one finds:

- Positive, significant at the 1 per cent level: *HOUS*, *LOCT*, *PERS*.
- Positive, significant at the 5 per cent level: *FOOT*, *TELE*.
- Negative, significant at the 1 per cent level: *HEAL*, *TOB*.
- Not significant: the rest.

Under the hypothesis that non-whites tend to live in urban areas, some of these effects may have more to do with residence than ethnicity – for example, the positive effect on *LOCT*. The results for all the categories are shown in the table.

	Dependent variable *LGCATPC*								
		LGEXPPC		*LGSIZE*		*NONWHITE*			
	n	$\widehat{\beta}_2$	s.e.($\widehat{\beta}_2$)	$\widehat{\beta}_3$	s.e.($\widehat{\beta}_3$)	$\widehat{\beta}_4$	s.e.($\widehat{\beta}_4$)	R^2	F
ADM	2,815	1.078	0.033	−0.053	0.043	−0.084	0.061	0.331	462.7
CLOT	4,500	0.843	0.024	0.146	0.032	0.006	0.042	0.240	473.3
DOM	1,661	0.927	0.055	0.420	0.075	−0.152	0.096	0.159	104.0
EDUC	561	1.231	0.101	−0.436	0.139	0.107	0.166	0.312	84.0
ELEC	5,828	0.475	0.012	−0.363	0.017	0.042	0.022	0.359	1,086.9
FDAW	5,102	0.879	0.016	−0.213	0.022	−0.010	0.029	0.461	1,450.9
FDHO	6,334	0.583	0.010	−0.081	0.013	−0.020	0.018	0.432	1,607.7
FOOT	1,827	0.404	0.031	−0.555	0.042	0.119	0.050	0.283	239.9
FURN	487	0.826	0.104	−0.251	0.137	0.248	0.159	0.199	40.1
GASO	5,710	0.676	0.013	−0.004	0.018	0.008	0.024	0.362	1,079.7
HEAL	4,802	0.773	0.023	−0.306	0.031	−0.142	0.042	0.273	601.4
HOUS	6,223	1.001	0.016	−0.140	0.021	0.206	0.028	0.472	1,853.6
LIFE	1,253	0.470	0.050	−0.460	0.065	0.082	0.081	0.154	75.9
LOCT	692	0.418	0.061	−0.390	0.086	−0.390	0.100	0.150	40.3
MAPP	399	0.725	0.094	−0.266	0.124	0.073	0.157	0.207	34.3
PERS	3,817	0.834	0.020	−0.224	0.028	0.188	0.038	0.391	817.5
READ	2,287	0.760	0.034	−0.504	0.047	−0.127	0.068	0.298	323.4
SAPP	1,037	0.465	0.049	−0.591	0.066	−0.036	0.085	0.237	106.7
TELE	5,788	0.642	0.013	−0.222	0.018	0.053	0.024	0.386	1,213.3
TEXT	992	0.384	0.049	−0.712	0.067	−0.072	0.083	0.246	107.5
TOB	1,155	0.552	0.037	−0.531	0.049	−0.257	0.067	0.337	195.2
TOYS	2,504	0.639	0.031	−0.306	0.043	0.032	0.062	0.231	250.6
TRIP	516	0.691	0.084	−0.146	0.109	0.158	0.136	0.152	30.7

A5.3

```
. reg LGFDHOPC LGEXPPC LGSIZE EDUCBA EDUCSC EDUCDO;
----------------------------------------------------------------------------
    Source |       SS       df       MS              Number of obs =    6334
-----------+------------------------------           F(  5,  6328) = 1012.42
     Model |  1556.69485     5  311.33897            Prob > F      =  0.0000
  Residual |  1945.97716  6328  .307518514           R-squared     =  0.4444
-----------+------------------------------           Adj R-squared =  0.4440
     Total |  3502.67201  6333  .553082585           Root MSE      =  .55454
----------------------------------------------------------------------------
  LGFDHOPC |      Coef.   Std. Err.       t    P>|t|     [95% Conf. Interval]
-----------+----------------------------------------------------------------
   LGEXPPC |   .6268014   .0102972    60.87   0.000      .6066154    .6469874
    LGSIZE |  -.0660179   .0132808    -4.97   0.000     -.0920527   -.0399831
    EDUCBA |  -.1639669   .0193625    -8.47   0.000      -.201924   -.1260097
    EDUCSC |  -.0702103   .0189683    -3.70   0.000     -.1073947   -.0330259
    EDUCDO |   .1022739   .0245346     4.17   0.000      .0541778      .15037
     _cons |   .8718572   .0854964    10.20   0.000      .7042553    1.039459
----------------------------------------------------------------------------
```

The dummies have been defined with high school graduate as the reference category. Their coefficients indicate a significant negative association between level of education and expenditure on food consumed at home, controlling for expenditure per person and the size of the household. The finding does not shed light on the reason for the negative association. Possibly those with greater education tend to eat less. There is also a negative association between level of education and expenditure on tobacco.

	Dependent variable *LGCATPC*							
Category	*ADM*	*CLOT*	*DOM*	*EDUC*	*ELEC*	*FDAW*	*FDHO*	*FOOT*
LGEXPPC	1.049	0.832	0.040	1.132	0.541	0.882	0.627	0.307
	(0.034)	(0.026)	(0.058)	(0.107)	(0.013)	(0.017)	(0.010)	(0.033)
LGSIZE	−0.060	0.141	0.386	−0.448	−0.334	−0.214	−0.066	−0.560
	(0.043)	(0.033)	(0.076)	(0.139)	(0.017)	(0.022)	(0.013)	(0.043)
EDUCBA	0.239	0.072	0.187	0.601	−0.319	0.011	−0.164	0.005
	(0.065)	(0.047)	(0.113)	(0.214)	(0.024)	(0.031)	(0.019)	(0.058)
EDUCSC	0.193	0.055	−0.035	0.320	−0.114	−0.014	−0.070	0.012
	(0.068)	(0.048)	(0.120)	(0.218)	(0.024)	(0.032)	(0.019)	(0.057)
EDUCDO	0.000	0.035	0.075	0.133	0.055	0.065	0.102	0.009
	(0.116)	(0.062)	(0.163)	(0.320)	(0.031)	(0.044)	(0.025)	(0.077)
R^2	0.334	0.240	0.160	0.323	0.384	0.461	0.444	0.281
F	281.8	284.5	63.3	52.8	724.7	871.5	1,012.4	142.2
n	2,815	4,500	1,661	461	5,828	5,102	6,334	1,827

	Dependent variable *LGCATPC*							
Category	*FURN*	*GASO*	*HEAL*	*HOUS*	*LIFE*	*LOCT*	*MAPP*	*PERS*
LGEXPPC	0.875	0.719	0.822	0.960	0.468	0.464	0.728	0.826
	(0.107)	(0.014)	(0.024)	(0.017)	(0.053)	(0.067)	(0.100)	(0.021)
LGSIZE	−0.228	0.015	−0.279	−0.155	−0.453	−0.394	−0.268	−0.213
	(0.137)	(0.018)	(0.031)	(0.021)	(0.066)	(0.086)	(0.124)	(0.028)
EDUCBA	−0.345	−0.215	−0.222	0.190	0.045	−0.325	−0.058	−0.043
	(0.174)	(0.026)	(0.044)	(0.031)	(0.087)	(0.143)	(0.171)	(0.039)
EDUCSC	−0.363	−0.010	−0.152	0.127	−0.031	−0.404	−0.375	−0.002
	(0.177)	(0.025)	(0.045)	(0.030)	(0.089)	(0.146)	(0.167)	(0.041)
EDUCDO	0.071	−0.004	0.002	0.084	0.190	0.558	−0.150	−0.087
	(0.297)	(0.034)	(0.061)	(0.039)	(0.134)	(0.167)	(0.214)	(0.057)
R^2	0.206	0.373	0.276	0.471	0.156	0.154	0.219	0.388
F	24.9	679.8	366.1	1,105.8	46.0	25.0	22.1	483.4
n	487	5,710	4,802	6,223	1,253	692	399	3,817

Dependent variable *LGCATPC*							
Category	*READ*	*SAPP*	*TELE*	*TEXT*	*TOB*	*TOYS*	*TRIP*
LGEXPPC	0.748	0.486	0.676	0.376	0.667	0.644	0.652
	(0.036)	(0.052)	(0.014)	(0.052)	(0.038)	(0.033)	(0.087)
LGSIZE	−0.512	−0.586	−0.204	−0.718	−0.483	−0.300	−0.155
	(0.047)	(0.066)	(0.018)	(0.068)	(0.048)	(0.043)	(0.110)
EDUCBA	0.112	−0.150	−0.205	0.015	−0.593	−0.030	0.092
	(0.066)	(0.093)	(0.026)	(0.093)	(0.075)	(0.059)	(0.175)
EDUCSC	0.169	−0.180	−0.017	0.038	−0.258	0.031	−0.031
	(0.069)	(0.094)	(0.026)	(0.096)	(0.061)	(0.059)	(0.189)
EDUCDO	−0.036	−0.093	−0.056	−0.095	0.117	−0.021	−0.147
	(0.113)	(0.138)	(0.033)	(0.135)	(0.077)	(0.085)	(0.299)
R^2	0.300	0.239	0.394	0.246	0.375	0.232	0.153
F	195.1	64.9	752.8	64.5	137.7	150.5	18.4
n	2,287	1,037	5,788	992	1,155	2,504	516

A5.4 For *FDHO*, *RSS* was 1,988.4 without the education dummy variables and 1,946.0 with them. 3 degrees of freedom were consumed when adding them, and $6334 - 6 = 6328$ degrees of freedom remained after they had been added. The F statistic is, therefore:

$$F(3,\,6328) = \frac{(1988.4 - 1946.0)/3}{1946.0/6328} = 45.98.$$

The critical value of $F(3,\,1000)$ at the 5 per cent level is 2.61. The critical value of $F(3,\,6328)$ must be lower. Hence we reject the null hypothesis that the dummy variables have no explanatory power (that is, that all their coefficients are jointly equal to zero).

F test of dummy variables as a group				
	n	*RSS* without dummies	*RSS* with dummies	F
ADM	2,815	3,945.2	3,922.3	5.47
CLOT	4,500	5,766.1	5,763.0	0.81
DOM	1,661	4,062.5	4,047.0	2.12
EDUC	561	1,380.1	1,356.9	3.16
ELEC	5,828	2,636.3	2,533.2	79.01
FDAW	5,102	3,369.1	3,366.7	1.23
FDHO	6,334	1,988.4	1,946.0	45.98
FOOT	1,827	1,373.5	1,373.5	0.01
FURN	487	913.9	902.0	2.12
GASO	5,710	2,879.3	2,828.4	34.23
HEAL	4,802	6,062.5	6,023.7	10.30
HOUS	6,223	4,825.6	4,795.7	12.91
LIFE	1,253	1,559.2	1,555.2	1.08
LOCT	692	1,075.1	1,054.7	4.41
MAPP	399	576.8	567.4	2.18
PERS	3,817	3,002.2	2,999.2	1.25
READ	2,287	2,892.1	2,882.2	2.61
SAPP	1,037	1,148.9	1,144.5	1.31
TELE	5,788	3,055.1	3,012.4	27.31

TEXT	992	1,032.9	1,031.8	0.36
TOB	1,155	873.4	813.5	28.18
TOYS	2,504	2,828.3	2,826.7	0.48
TRIP	516	792.8	790.6	0.48

A5.5

```
. reg LGFDHOPC LGEXPPC LGSIZE EDUCBA EDUCSC EDUCHSD;
----------------------------------------------------------------------------
    Source |       SS       df       MS            Number of obs =    6334
-----------+------------------------------         F(  5,  6328) = 1012.42
     Model |  1556.69485     5   311.33897         Prob > F      =  0.0000
  Residual |  1945.97716  6328  .307518514         R-squared     =  0.4444
-----------+------------------------------         Adj R-squared =  0.4440
     Total |  3502.67201  6333  .553082585         Root MSE      =  .55454
----------------------------------------------------------------------------
  LGFDHOPC |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-----------+----------------------------------------------------------------
   LGEXPPC |   .6268014   .0102972    60.87   0.000     .6066154    .6469874
    LGSIZE |  -.0660179   .0132808    -4.97   0.000    -.0920527   -.0399831
    EDUCBA |  -.2662408   .0246636   -10.79   0.000    -.3145898   -.2178917
    EDUCSC |  -.1724842   .0239688    -7.20   0.000    -.2194713   -.1254972
   EDUCHSD |  -.1022739   .0245346    -4.17   0.000      -.15037   -.0541778
     _cons |   .9741311   .0845451    11.52   0.000     .8083941    1.139868
----------------------------------------------------------------------------
```

The results for all the categories of expenditure have not been tabulated but are easily summarised:

- The analysis of variance in the upper half of the output is unaffected.
- The results for variables other than the dummy variables are unaffected.
- The results for *EDUCHSD* are identical to those for *EDUCDO* in the first regression, except for a change of sign in the coefficient, the t statistic, and the limits of the confidence interval.
- The constant is equal to the old constant plus the coefficient of *EDUCDO* in the first regression.
- The coefficients of the other dummy variables are equal to their values in the first regression minus the coefficient of EDUCDO in the first regression.
- One substantive change is in the standard errors of *EDUCIC* and *EDUCCO*, caused by the fact that the comparisons are now between these categories and *EDUCDO*, not *EDUCHSD*.
- The other is that the t statistics are for the new comparisons, not the old ones.

A5.6 *Explain verbally why the estimates of the coefficient of GOV are different in regressions (1) and (2).*

The second specification indicates that earnings are positively related to schooling and negatively related to working in the government sector. S has a significant coefficient in (2) and therefore ought to be in the model. If S is omitted from the specification the estimate of the coefficient of GOV will be biased upwards because schooling is positively correlated with working in the government sector. (We are told in the question that government workers on average have an extra year of schooling.) The bias is sufficiently strong to make the negative coefficient disappear.

Explain the difference in the estimates of the coefficient of GOV in regressions (2) and (3).

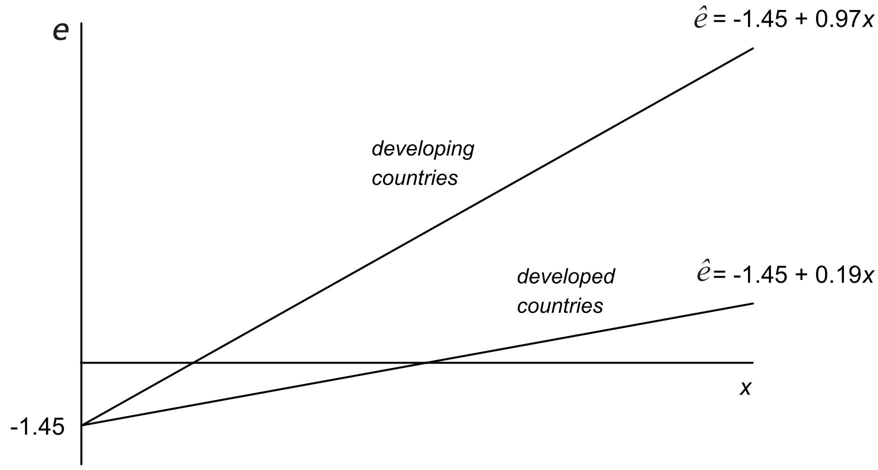
The coefficient of GOV in the third regression is effectively a linear function of S: $0.726 - 0.059S$. The coefficient of the GOV intercept dummy is therefore an estimate of the extra earnings of a government worker *with no schooling*. The premium disappears for $S = 12$ and becomes negative for higher values of S. The second regression does not take account of the variation of the coefficient of GOV with S and hence yields an average effect of GOV. The average effect was negative since only a small minority of government workers had fewer than 12 years of schooling.

The correlation between GOV and SGOV was 0.977. Explain the variations in the standard error of the coefficient of GOV in the three regressions.

The standard error in the first regression is meaningless given severe omitted variable bias. For comparing the standard errors in (2) and (3), it should be noted that the same problem in principle applies in (2), given that the coefficient of $SGOV$ in (3) is highly significant. However, part of the reason for the huge increase must be the high correlation between GOV and $SGOV$.

A5.7 1. The dummy variable allows the slope coefficient to be different for developing and developed countries. From equation (1) one may derive the following relationships:

$$\begin{aligned}
\text{developed countries} \quad \widehat{e} &= -1.45 + 0.19x \\
\text{developing countries} \quad \widehat{e} &= -1.45 + 0.19x + 0.78x \\
&= -1.45 + 0.97x.
\end{aligned}$$

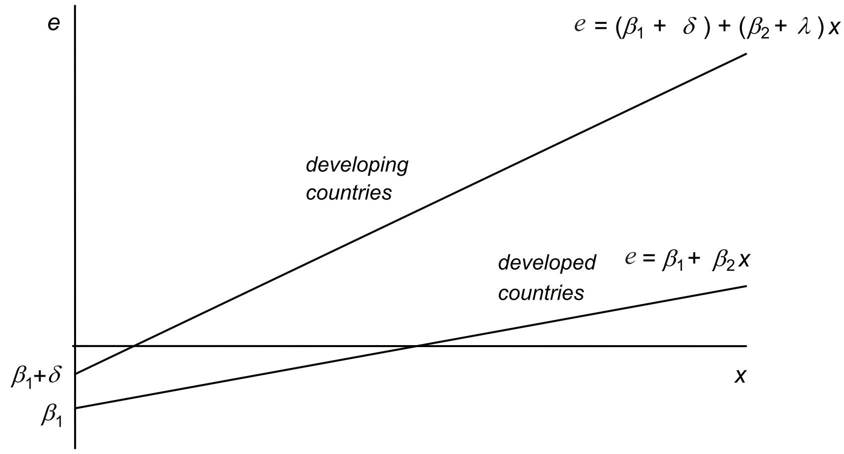


2. The inclusion of D would allow the intercept to be different for the two types of country. If the model was written as:

$$e = \beta_1 + \beta_2 x + \delta D + \lambda Dx + u$$

the implicit relationships for the two types of country would be:

$$\begin{aligned} \text{developed countries} \quad e &= \beta_1 + \beta_2 x + u \\ \text{developing countries} \quad e &= \beta_1 + \beta_2 x + \delta + \lambda x + u \\ &= (\beta_1 + \delta) + (\beta_2 + \lambda)x + u. \end{aligned}$$



3. When the specification includes both an intercept dummy and a slope dummy, the coefficients for the two categories will be the same as in the separate regressions (2) and (3). Hence the intercept and coefficient of x will be the same as in the regression for the reference category, regression (3), and the coefficients of the dummies will be such that they modify the intercept and slope coefficient so that they are equal to their counterparts in regression (4):

$$\widehat{e} = -2.74 + 0.50x + 1.89D + 0.28xD.$$

Since the coefficients are the same, the overall fit for this regression will be the same as that for regressions (2) and (3). Hence $RSS = 18.63 + 25.23 = 43.86$.

The t statistic for the coefficient of x will be the square root of the F statistic for the test of the marginal explanatory power of D when it is included in the equation. The F statistic is:

$$F(1,46) = \frac{(50.23 - 43.86)/1}{43.86/46} = 6.6808.$$

The t statistic is therefore 2.58.

4. One method is to use a Chow test comparing RSS for the pooled regression, regression (2), with the sum of RSS regressions (3) and (4):

$$F(2,46) = \frac{(121.61 - 43.86)/2}{43.86/46} = 40.8.$$

The critical value of $F(2,40)$ at the 0.1 per cent significance level is 8.25. The critical value of $F(2,46)$ must be lower. Hence the null hypothesis that the coefficients are the same for developed and developing countries is rejected.

We should also consider t tests on the coefficients of D and xD. We saw in (3) that the t statistic for the coefficient of D was 2.58, so we would reject the null hypothesis of no intercept shift at the 5 per cent level, and nearly at the 1 per cent level. We do not have enough information to derive the t statistic for xD. We would not perform a t test on the coefficient of xD in regression (1) because that regression is clearly misspecified.

A5.8

Chow test

	n	*RSS* All	*RSS* $COLLEGE = 0$	*RSS* $COLLEGE = 1$	F
ADM	2,815	3,945.2	789.5	3,129.9	6.15
CLOT	4,500	5,766.1	1,837.9	3,913.8	3.77
DOM	1,661	4,062.5	1,048.5	2,984.0	4.10
EDUC	561	1,380.1	278.0	1,087.0	2.05
ELEC	5,828	2,636.3	962.6	1,594.6	60.02
FDAW	5,102	3,369.1	1,114.8	2,251.7	1.32
FDHO	6,334	1,988.4	751.9	1,205.3	33.63
FOOT	1,827	1,373.5	513.1	858.5	0.82
FURN	487	913.9	238.7	662.1	2.32
GASO	5,710	2,879.3	1,043.2	1,811.7	16.27
HEAL	4,802	6,062.5	2,211.7	3,796.6	14.42
HOUS	6,223	4,825.6	2,234.6	2,566.5	10.55
LIFE	1,253	1,559.2	424.0	1,119.6	4.20
LOCT	692	1,075.1	283.3	769.3	4.88
MAPP	399	576.8	205.6	367.5	0.84
PERS	3,817	3,002.2	918.5	2,081.1	1.10
READ	2,287	2,892.1	752.6	2,129.1	2.75
SAPP	1,037	1,148.9	342.9	802.1	1.18
TELE	5,788	3,055.1	1,132.8	1,903.2	12.10

TEXT	992	1,032.9	278.0	754.1	0.25
TOB	1,155	873.4	351.3	476.8	20.91
TOYS	2,504	2,828.3	862.5	1,964.2	0.46
TRIP	516	792.8	114.2	675.6	0.66

For *FDHO*, *RSS* for the logarithmic regression without college in Exercise A4.2 was 1,988.4. When the sample is split, *RSS* for $COLLEGE = 0$ is 751.9 and for $COLLEGE = 1$ is 1,205.3. Three degrees of freedom are consumed because the coefficients of *LGEXPPC* and *LGSIZE* and the constant have to be estimated twice. The number of degrees of freedom remaining after splitting the sample is $6334 - 6 = 6328$. Hence the F statistic is:

$$F(3,\, 6328) = \frac{(1988.4 - (751.9 + 1205.3))/3}{(751.9 + 1205.3)/6328} = 33.63.$$

The critical value of $F(3,\, 1000)$ at the 1 per cent level is 2.62 and so we reject the null hypothesis of no difference in the expenditure functions at that significance level. The results for all the categories are shown in the table.

A5.9

```
. gen LEXPCOL  = LGEXPPC*COLLEGE
. gen LSIZECOL = LGSIZE*COLLEGE
. reg LGFDHOPC LGEXPPC LGSIZE COLLEGE LEXPCOL LSIZECOL
----------------------------------------------------------------------------
    Source |       SS       df       MS              Number of obs =    6334
-----------+------------------------------           F(  5,  6328) =  999.36
     Model |  1545.47231     5  309.094462           Prob > F      =  0.0000
  Residual |   1957.1997  6328  .309291987           R-squared     =  0.4412
-----------+------------------------------           Adj R-squared =  0.4408
     Total |  3502.67201  6333  .553082585           Root MSE      =  .55614
----------------------------------------------------------------------------
  LGFDHOPC |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-----------+----------------------------------------------------------------
   LGEXPPC |    .648295   .0171599    37.78   0.000     .6146559    .6819342
    LGSIZE |  -.0559735   .0216706    -2.58   0.010    -.0984552   -.0134917
   COLLEGE |   .3046012   .1760486     1.73   0.084    -.0405137    .6497161
   LEXPCOL |  -.0558931   .0211779    -2.64   0.008     -.097409   -.0143772
  LSIZECOL |  -.0198021   .0274525    -0.72   0.471    -.0736182     .034014
     _cons |   .7338499   .1403321     5.23   0.000     .4587514    1.008948
----------------------------------------------------------------------------
```

The example output is for *FDHO*. In Exercise A4.2, *RSS* was 1,988.4 for the same regression without the dummy variables. To perform the F test of the explanatory power of the intercept dummy variable and the two slope dummy variables as a group, we evaluate whether *RSS* for this regression is significantly lower. *RSS* has fallen from 1,988.4 to 1,957.2. 3 degrees of freedom are consumed by adding the dummy variables, and $6334 - 6 = 6328$ degrees of freedom remain after adding the dummy variables. The F statistic is therefore:

$$F(3,\, 6328) = \frac{(1988.4 - 1957.2)/3}{1957.2/6328} = 33.63.$$

This is highly significant. This F test is, of course, equivalent to the Chow test in the previous exercise. One possible explanation was offered there. The present regression suggests another. The slope dummy variable *LGEXPCOL* has a significant negative coefficient, implying that the elasticity falls as income rises. This is plausible for a basic necessity such as food.

A5.10 (a) You should fit models such as:

$$LGEARN = \beta_1 + \beta_2 S + \beta_3 ASVABC + \beta_4 MALE + \beta_5 ETHBLACK + \beta_6 ETHHISP + u$$

separately for the private and government sectors. To investigate discrimination, for each sector t tests should be performed on the coefficients of *MALE*, *ETHBLACK*, and *ETHHISP* and an F test on the joint explanatory power of *ETHBLACK* and *ETHHISP*.

(b) You should combine the earnings functions for the two sectors, while still allowing their parameters to differ, by fitting a model such as:

$$\begin{aligned} LGEARN &= \beta_1 + \beta_2 S + \beta_3 ASVABC + \beta_4 MALE + \beta_5 ETHBLACK + \beta_6 ETHHISP \\ &\quad + \delta_1 GOV + \delta_2 GOVS + \delta_3 GOVASV + \delta_4 GOVMALE + \delta_5 GOVBLACK \\ &\quad + \delta_6 GOVHISP + u \end{aligned}$$

where *GOV* is equal to 1 if the respondent works in the government sector and 0 otherwise, and *GOVS*, *GOVASV*, *GOVMALE*, *GOVBLACK*, and *GOVHISP* are slope dummy variables defined as the product of *GOV* and the respective variables. To investigate whether the level of discrimination is different in the two sectors, one should perform t tests on the coefficients of *GOVMALE*, *GOVBLACK*, and *GOVHISP* and an F test on the joint explanatory power of *GOVBLACK* and *GOVHISP*.

A Chow test would not be appropriate because if it detected a significant difference in the earnings functions, this could be due to differences in the coefficients of S and *ASVABC* rather than the discrimination variables.

A5.11 *Give an interpretation of the coefficients of S and SMALE in regression (5).*

An extra year of schooling increases female earnings by 9.4 per cent. (Strictly, $100(e^{0.094} - 1) = 9.9$ per cent.) For males, an extra year of schooling leads to an increase in earnings 0.5 per cent greater than for females, i.e. 9.9 per cent.

Give an interpretation of the coefficients of MALE in regressions (4) and (5).

(4): males earn 23.4 per cent more than females (controlling for other factors). (5): males with no schooling or work experience earn 11.7 per cent more than similar females.

The researcher hypothesises that the earnings function is different for males and females. Perform a test of this hypothesis using regression (4), and also using regressions (1) and (5).

Looking at regression (4), the coefficient of *MALE* is highly significant, indicating that the earnings functions are indeed different. Looking at regression (5), and comparing it with (1), the null hypothesis is that the coefficients of the male dummy variables in (5) are all equal to zero.

$$F(3, 3236) = \frac{(714.6 - 672.5)/3}{672.5/3236} = 67.5.$$

The critical value of $F(3, 1000)$ at the 1 per cent level is 3.80. The corresponding critical value for $F(3, 3236)$ must be lower, so we reject the null hypothesis and conclude that the earnings functions are different.

Explain the differences in the tests using regression (4) and using regressions (1) and (5).

In regression (4) the coefficient of $MALE$ is highly significant. In regression (5) it is not. Likewise the coefficients of the slope dummies are not significant. This is (partly) due to the effect of multicollinearity. The male dummy variables are very highly correlated and as a consequence the standard error of the coefficient of $MALE$ is much larger than in regression (4). Nevertheless the F test reveals that their joint explanatory power is highly significant.

At a seminar someone suggests that a Chow test could shed light on the researcher's hypothesis. Is this correct?

Yes. Using regressions (1)–(3):

$$F(3, 3236) = \frac{(714.6 - (411.0 + 261.6))/3}{(411.0 + 261.6)/3236} = 67.4.$$

The null hypothesis that the coefficients are the same for males and females is rejected at the 1 per cent level. The test is, of course, equivalent to the dummy variable test comparing (1) and (5).

Explain which of (1), (4), and (5) would be your preferred specification.

(4) seems best, given that the coefficients of S and EXP are fairly similar for males and females and that introducing the slope dummies causes multicollinearity. The F statistic of their joint explanatory power is only 0.72, not significant at any significance level.

A5.12 *Calculate the missing coefficients* **V**, **W**, **X**, *and* **Y** *in Regression 4 (just the coefficients, not the standard errors) and* **Z**, *the missing RSS, giving an explanation of your computations.*

Since Regression 5 includes a complete set of black intercept and slope dummy variables, the basic coefficients will be the same as for a regression using the 'whites' only subsample and the coefficients modified by the dummies will give the counterparts for the blacks only subsample. Hence $\mathbf{V} = 0.122 - 0.009 = 0.113$; $\mathbf{W} = 0.033 - 0.006 = 0.027$; $\mathbf{X} = 0.306 - 0.280 = 0.026$; and $\mathbf{Y} = 0.411 + 0.205 = 0.616$. The residual sum of squares for Regression 5 will be equal to the sum of RSS for the 'whites' and blacks subsamples. Hence $\mathbf{Z} = 600.0 - 555.7 = 44.3$.

Give an interpretation of the coefficient of BLACK in Regression 2.

It suggests that blacks earn 14.4 per cent less than whites, controlling for other characteristics.

Perform an F test of the joint explanatory power of BLACK, SB, EB, and MB in Regression 5.

Write the model as:

$$LGEARN = \beta_1 + \beta_2 S + \beta_3 EXP + \beta_4 MALE + \beta_5 BLACK + \beta_6 SB + \beta_7 EB + \beta_8 MB + u.$$

The null hypothesis for the test is if $H_0 : \beta_5 = \beta_6 = \beta_7 = \beta_8 = 0$, and the alternative hypothesis is H_1: at least one coefficient different from 0. The F statistic is:

$$F(4, 2400) = \frac{(610.0 - 600.0)/4}{600.0/2400} = \frac{2400}{240} = 10.0.$$

This is significant at the 0.1 per cent level (critical value 4.65) and so the null hypothesis is rejected.

Explain whether it is possible to relate the F test in part (c) to a Chow test based on Regressions 1, 3, and 4.

The Chow test would be equivalent to the F test in this case.

Give an interpretation of the coefficients of BLACK and MB in Regression 5.

Re-write the model as:

$$LGEARN = \beta_1 + \beta_2 S + \beta_3 EXP + \beta_4 MALE + (\beta_5 + \beta_6 S + \beta_7 EXP + \beta_8 MALE) BLACK + u.$$

From this it follows that β_5 is the extra proportional earnings of a black, compared with a white, when $S = EXP = MALE = 0$. Thus the coefficient of $BLACK$ indicates that a black female with no schooling or experience earns 20.5 per cent more than a similar white female. The interpretation of the coefficient of any interactive term requires care. Holding $S = EXP = MALE = 0$, the coefficients of $MALE$ and $BLACK$ indicate that black males will earn $30.6 + 20.5 = 51.1$ per cent more than white females. The coefficient of MB modifies this estimate, reducing it by 28.0 per cent to 23.1 per cent.

Explain whether a simple t test on the coefficient of BLACK in Regression 2 is sufficient to show that the wage equations are different for blacks and whites.

Regression 2 is misspecified because it embodies the restriction that the effect of being black is the same for males and females, and that is contradicted by Regression 5. Hence any test is in principle invalid. However, the fact that the coefficient has a very high t statistic is suggestive that something associated with being black is affecting the wage equation.

A5.13 *Reconstruction of missing output*

Students A and B left their output on a bus on the way to the workshop. This is why it does not appear in the table.

State what the missing output of Student A would have been, as far as this can be done exactly, given the results of Students C and D. (Coefficients, standard errors, R^2, RSS.)

The output would be as for column (3) (coefficients, standard errors, R^2), with the following changes:

- the row label $MALE$ should be replaced with WM
- the row label $BLACK$ should be replaced with BF
- the row label $MALEBLACK$ should be replaced with BM and the coefficient for that row should be the sum of the coefficients in column (3): $0.308 - 0.011 - 0.290 = 0.007$, and the standard error would not be known.

Explain why it is not possible to reconstruct any of the output of Student B.

One could not predict the coefficients of either S or EXP in the four regressions performed by Student B. They will, except by coincidence, be different from any of the estimates of the other students because the coefficients for S and EXP in the other specifications are constrained in some way. As a consequence, one cannot predict exactly any part of the rest of the output, either.

Tests of hypotheses

- *Student A (assuming he had found his output)*

  Student A could perform tests of the differences in earnings between white males and white females, black males and white females, and black females and white females, through simple t tests on the coefficients of WM, BM, and BF.

  He could also test the null hypothesis that there are no sex/ethnicity differences with an F test, comparing RSS for his regression with that of the basic regression:

$$F(3,\ 2540) = \frac{(922 - 603)/3}{603/2540}.$$

  This would be compared with the critical value of F with 3 and 2,540 degrees of freedom at the significance level chosen and the null hypothesis of no sex/ethnicity effects would be rejected if the F statistic exceeded the critical value.

- *Student B (assuming he had found his output)*

  In the case of Student B, with four separate subsample regressions, candidates are expected say that no tests would be possible because no relevant standard errors would be available. We have covered Chow tests only for two categories. However, a four-category test could be performed, with:

$$F(9,\ 2534) = \frac{(922 - X)/9}{X/2534}$$

  where $RSS = 922$ for the basic regression and X is the sum of RSS in the four separate regressions.

- *Student C*

  Student C could perform the same t tests and the same F test as Student A, with one difference: the t test of the difference between the earnings of black males and white females would not be available. Instead, the t statistic of $MALEBLACK$ would allow a test of whether there is any interactive effect of being black and being male on earnings.

- *Student D*

  Student D could perform a Chow test to see if the wage equations of males and females differed:

$$F(3,\ 2540) = \frac{(659 - (322 + 289))/3}{(322 + 289)/2540}.$$

  $RSS = 322$ for males and 289 for females. This would be compared with the critical value of F with 3 and 2,540 degrees of freedom at the significance level chosen and the null hypothesis of no sex/ethnicity effects would be rejected if the F statistic exceeded the critical value. She could also perform a corresponding Chow test for blacks and whites:

$$F(3,\ 2540) = \frac{(659 - (609 + 44))/3}{(609 + 44)/2540}.$$

If you had been participating in the project and had had access to the data set, what regressions and tests would you have performed?

The most obvious development would be to relax the sex/ethnicity restrictions on the coefficients of S and EXP by including appropriate interaction terms. This could be done by interacting these variables with the dummy variables defined by Student A or those defined by Student C.

Chapter 6
Specification of regression variables

6.1 Overview

This chapter treats a variety of topics relating to the specification of the variables in a regression model. First there are the consequences for the regression coefficients, their standard errors, and R^2 of failing to include a relevant variable, and of including an irrelevant one. This leads to a discussion of the use of proxy variables to alleviate a problem of omitted variable bias. Next come F and t tests of the validity of a restriction, the use of which was advocated in Chapter 3 as a means of improving efficiency and perhaps mitigating a problem of multicollinearity. The chapter concludes by outlining the potential benefit to be derived from examining observations with large residuals after fitting a regression model.

6.2 Learning outcomes

After working through the corresponding chapter in the text, studying the corresponding slideshows, and doing the starred exercises in the text and the additional exercises in this subject guide, you should be able to:

- derive the expression for the bias in an OLS estimator of a slope coefficient when the true model has two explanatory variables but the regression model has only one
- determine the likely direction of omitted variable bias, given data on the correlation between the explanatory variables
- explain the consequence of omitted variable bias for the standard errors of the coefficients and for t tests and F tests
- explain the consequences of including an irrelevant variable for the regression coefficients, their standard errors, and t and F tests
- explain how the regression results are affected by the substitution of a proxy variable for a missing explanatory variable
- perform an F test of a restriction, stating the null hypothesis for the test
- perform a t test of a restriction, stating the null hypothesis for the test.

6.3 Additional exercises

A6.1 *Does the omission of total household expenditure or household size give rise to omitted variable bias in your CES regressions?*

Regress *LGCATPC* (1) on both *LGEXPPC* and *LGSIZE*, (2) on *LGEXPPC* only, and (3) on *LGSIZE* only. Assuming that (1) is the correct specification, analyse the likely direction of the bias in the estimate of the coefficient of *LGEXPPC* in (2) and that of *LGSIZE* in (3). Check whether the regression results are consistent with your analysis.

A6.2 A school has introduced an extra course of reading lessons for children starting school and a researcher is evaluating the impact of the course on the scores on a literacy test taken at the age of seven. In the first year of its implementation, those children whose surnames begin A–M are assigned to the extra course, while the rest have the normal curriculum. The researcher hypothesises that:

$$Y = \beta_1 + \beta_2 D + \beta_3 A + u$$

where Y is the score on the literacy test, D is a dummy variable that is equal to 1 for those assigned to the extra course and 0 for the others, A is a measure of the cognitive ability of the child when starting school, and u is an iid (independently and identically distributed) disturbance term assumed to have a normal distribution. Unfortunately, the researcher has no data on A. Using OLS (ordinary least squares), she fits the regression:

$$\widehat{Y} = \widehat{\beta}_1 + \widehat{\beta}_2 D.$$

- Demonstrate that $\widehat{\beta}_2$ is an unbiased estimator of β_2.
- A commentator says that the standard error of $\widehat{\beta}_2$ will be invalid because an important variable, A, has been omitted from the specification. The researcher replies that the standard error will remain valid if A can be assumed to have a normal distribution. Explain whether the commentator or the researcher is correct.
- Another commentator says that whether the distribution of A is normal or not makes no difference to the validity of the standard error. Evaluate this assertion.

A6.3 A researcher obtains data on household annual expenditure on books, B, and annual household income, Y, for 100 households. He hypothesises that B is related to Y and the average cognitive ability of adults in the household, IQ, by the relationship:

$$\log B = \beta_1 + \beta_2 \log Y + \beta_3 \log IQ + u \qquad \text{(A)}$$

where u is a disturbance term that satisfies the regression model assumptions. He also considers the possibility that $\log B$ may be determined by $\log Y$ alone:

$$\log B = \beta_1 + \beta_2 \log Y + u. \qquad \text{(B)}$$

He does not have data on IQ and decides to use average years of schooling of the adults in the household, S, as a proxy in specification (A). It may be assumed that Y and S are both nonstochastic. In the sample the correlation between $\log Y$ and $\log S$ is 0.86. He performs the following regressions: (1) $\log B$ on both $\log Y$ and $\log S$, and (2) $\log B$ on $\log Y$ only, with the results shown in the table (standard errors in parentheses):

	(1)	(2)
$\log Y$	1.10	2.10
	(0.69)	(0.35)
$\log S$	0.59	—
	(0.35)	
constant	−6.89	−3.37
	(2.28)	(0.89)
R^2	0.29	0.27

- Assuming that (A) is the correct specification, explain, with a mathematical proof, whether you would expect the coefficient of $\log Y$ to be greater in regression (2).
- Assuming that (A) is the correct specification, describe the various benefits from using $\log S$ as a proxy for $\log IQ$, as in regression (1), if $\log S$ is a good proxy.
- Explain whether the low value of R^2 in regression (1) implies that $\log S$ is not a good proxy.
- Assuming that (A) is the correct specification, provide an explanation of why the coefficients of $\log Y$ and $\log S$ in regression (1) are not significantly different from zero, using two-sided t tests.
- Discuss whether the researcher would be justified in using one-sided t tests in regression (1).
- Assuming that (B) is the correct specification, explain whether you would expect the coefficient of $\log Y$ to be lower in regression (1).
- Assuming that (B) is the correct specification, explain whether the standard errors in regression (1) are valid estimates.

A6.4 A researcher has the following data for the year 2012: T, annual total sales of cinema tickets per household, and P, the average price of a cinema ticket in the city. She believes that the true relationship is:

$$\log T = \beta_1 + \beta_2 \log P + \beta_3 \log Y + u$$

where Y is average household income, but she lacks data on Y and fits the regression (standard errors in parentheses):

$$\widehat{\log T} = \underset{(0.52)}{13.74} + \underset{(0.23)}{0.17} \log P \qquad R^2 = 0.01$$

Explain analytically whether the slope coefficient is likely to be biased. You are told that if the researcher had been able to obtain data on Y, her regression would have been:

$$\widehat{\log T} = \underset{(2.93)}{-1.63} - \underset{(0.21)}{0.48} \log P + \underset{(0.35)}{1.83} \log Y \qquad R^2 = 0.44$$

You are also told that Y and P are positively correlated.

The researcher is not able to obtain data on Y but, from local records, she is able to obtain data on H, the average value of a house in each city, and she decides to use it as a proxy for Y. She fits the following regression (standard errors in parentheses):

$$\widehat{\log T} = \underset{(3.22)}{-0.63} - \underset{(0.22)}{0.37} \log P + \underset{(0.38)}{1.69} \log H \qquad R^2 = 0.36$$

Describe the theoretical benefits from using H as a proxy for Y, discussing whether they appear to have been obtained in this example.

A6.5 A researcher has data on years of schooling, S, weekly earnings in dollars, W, hours worked per week, H, and hourly earnings, E (computed as W/H) for a sample of 1,755 white males in the United States in the year 2000. She calculates *LW*, *LE*, and *LH* as the natural logarithms of W, E, and H, respectively, and fits the following regressions, with the results shown in the table below (standard errors in parentheses; *RSS* = residual sum of squares):

- Column 1: a regression of *LE* on S.
- Column 2: a regression of *LW* on S and *LH*.
- Column 3: a regression of *LE* on S and *LH*.

The correlation between S and *LH* is 0.06.

	(1)	(2)	(3)	(4)	(5)
Respondents	All	All	All	FT	PT
Dependent variable	*LE*	*LW*	*LE*	*LW*	*LW*
S	0.099	0.098	0.098	0.101	0.030
	(0.006)	(0.006)	(0.006)	(0.006)	(0.049)
LH	—	1.190	0.190	0.980	0.885
		(0.065)	(0.065)	(0.088)	(0.325)
constant	6.111	5.403	5.403	6.177	7.002
	(0.082)	(0.254)	(0.254)	(0.345)	(1.093)
RSS	741.5	737.9	737.9	626.1	100.1
Observations	1,755	1,755	1,755	1,669	86

- Explain why specification (1) is a restricted version of specification (2), stating and interpreting the restriction.
- Supposing the restriction to be valid, explain whether you expect the coefficient of S and its standard error to differ, or be similar, in specifications (1) and (2).

- Supposing the restriction to be invalid, how would you expect the coefficient of S and its standard error to differ, or be similar, in specifications (1) and (2)?
- Perform an F test of the restriction.
- Perform a t test of the restriction.
- Explain whether the F test and the t test could lead to different conclusions.
- At a seminar, a commentator says that part-time workers tend to be paid worse than full-time workers and that their earnings functions are different. Defining full-time workers as those working at least 35 hours per week, the researcher divides the sample and fits the earnings functions for full-time workers (column 4) and part-time workers (column 5). Test whether the commentator's assertion is correct.
- What are the implications of the commentator's assertion for the test of the restriction?

A6.6 A researcher investigating whether government expenditure tends to crowd out investment has data on government recurrent expenditure, G, investment, I, and gross domestic product, Y, all measured in US$ billion, for 30 countries in 2012. She fits two regressions (standard errors in parentheses; t statistics in square brackets; RSS = residual sum of squares).

(1) A regression of $\log I$ on $\log G$ and $\log Y$:

$$\begin{array}{rcll} \widehat{\log I} & = & -2.44 - 0.63 \log G + 1.60 \log Y & R^2 = 0.98 \qquad (1) \\ & & (0.26) \quad (0.12) \qquad\quad (0.12) & RSS = 0.90 \\ & & [9.42] \;\; [-5.23] \qquad\; [12.42] & \end{array}$$

(2) a regression of $\log(I/Y)$ on $\log(G/Y)$:

$$\begin{array}{rcll} \widehat{\log\left(\frac{I}{Y}\right)} & = & 2.65 - 0.63 \log\left(\frac{G}{Y}\right) & R^2 = 0.48 \qquad (2) \\ & & (0.23) \; (0.12) & RSS = 0.99 \\ & & [11.58] \; [-5.07] & \end{array}$$

The correlation between $\log G$ and $\log Y$ in the sample is 0.98. The table gives some further basic data on $\log G$, $\log Y$, and $\log(G/Y)$.

	Sample mean	Mean square deviation
$\log G$	3.75	2.00
$\log Y$	5.57	1.95
$\log(G/Y)$	−1.81	0.08

- Explain why the second specification is a restricted version of the first. State the restriction.
- Perform a test of the restriction.

- The researcher expected the standard error of the coefficient of $\log(G/Y)$ in (2) to be smaller than the standard error of the coefficient of $\log G$ in (1). Explain why she expected this.
- However, the standard error is the same, at least to two decimal places. Give an explanation.
- Show how the restriction could be tested using a t test in a reparameterised version of the specification for (1).

A6.7 *Is expenditure per capita on your CES category related to total household expenditure per capita?*

The model specified in Exercise A4.1 is a restricted version of that in Exercise 4.5 in the text. Perform an F test of the restriction. Also perform a t test of the restriction.

[Exercise 4.5: regress *LGCAT* on *LGEXP* and *LGSIZE*; Exercise A4.1: regress *LGCATPC* on *LGEXPPC*.]

A6.8 A researcher is considering two regression specifications:

$$\log Y = \beta_1 + \beta_2 \log X + u \qquad (1)$$

and:

$$\log \frac{Y}{X} = \alpha_1 + \alpha_2 \log X + u \qquad (2)$$

where u is a disturbance term. Determine whether (2) is a reparameterised or a restricted version of (1).

A6.9 Three researchers investigating the determinants of hourly earnings have the following data for a sample of 104 male workers in the United States in 2006: E, hourly earnings in dollars; S, years of schooling; *NUM*, score on a test of numeracy; and *VERB*, score on a test of literacy. The *NUM* and *VERB* tests are marked out of 100. The correlation between them is 0.81. Defining *LGE* to be the natural logarithm of E, Researcher 1 fits the following regression (standard errors in parentheses; *RSS* = residual sum of squares):

$$\begin{aligned}\widehat{LGE} &= \underset{(1.81)}{2.02} + \underset{(0.007)}{0.063}S + \underset{(0.0011)}{0.0044}NUM + \underset{(0.0010)}{0.0026}VERB \qquad RSS = 2{,}000\end{aligned}$$

Researcher 2 defines a new variable *SCORE* as the average of *NUM* and *VERB*. She fits the regression:

$$\widehat{LGE} = \underset{(1.78)}{1.72} + \underset{(0.005)}{0.050}S + \underset{(0.0010)}{0.0068}SCORE \qquad RSS = 2{,}045$$

Researcher 3 fits the regression:

$$\widehat{LGE} = \underset{(1.81)}{2.02} + \underset{(0.007)}{0.063}S + \underset{(0.0022)}{0.0088}SCORE - \underset{(0.0012)}{0.0018}VERB \qquad RSS = 2{,}000$$

- Show that the specification of Researcher 2 is a restricted version of the specification of Researcher 1, stating the restriction.
- Perform an F test of the restriction.
- Show that the specification of Researcher 3 is a reparameterised version of the specification of Researcher 1 and hence perform a t test of the restriction in the specification of Researcher 2.
- Explain whether the F test and the t test could have led to different results.
- Perform a test of the hypothesis that the numeracy score has a greater effect on earnings than the literacy score.
- Compare the regression results of the three researchers.

A6.10 It is assumed that manufacturing output is subject to the production function:

$$Q = AK^{\alpha}L^{\beta} \qquad (1)$$

where Q is output and K and L are capital and labour inputs. The cost of production is:

$$C = \rho K + wL \qquad (2)$$

where ρ is the cost of capital and w is the wage rate. It can be shown that, if the cost is minimised, the wage bill wL will be given by the relationship:

$$\log wL = \frac{1}{\alpha+\beta}\log Q + \frac{\alpha}{\alpha+\beta}\log\rho + \frac{\beta}{\alpha+\beta}\log w + \text{constant}. \qquad (3)$$

(Note: You are not expected to prove this.)

A researcher has annual data for 2002 for Q, K, L, ρ, and w (all monetary measures being converted into US dollars) for the manufacturing sectors of 30 industrialised countries and regresses $\log wL$ on $\log Q$, $\log\rho$, and $\log w$.

- Demonstrate that relationship (3) embodies a testable restriction and show how the model may be reformulated to take advantage of it.
- Explain how the restriction could be tested using an F test.
- Explain how the restriction could be tested using a t test.
- Explain the theoretical benefits of making use of a valid restriction. How could the researcher assess whether there are any benefits in practice, in this case?
- At a seminar, someone suggests that it is reasonable to hypothesise that manufacturing output is subject to constant returns to scale, so that $\alpha+\beta=1$. Explain how the researcher could test this hypothesis (1) using an F test, (2) using a t test.

A6.11 A researcher hypothesises that the net annual growth of private sector purchases of government bonds, B, is positively related to the nominal rate of interest on the bonds, I, and negatively related to the rate of price inflation, P:

$$B = \beta_1 + \beta_2 I + \beta_3 P + u$$

where u is a disturbance term. The researcher anticipates that $\beta_2 > 0$ and $\beta_3 < 0$. She also considers the possibility that B depends on the real rate of interest on the bonds, R, where $R = I - P$. Using a sample of observations for 40 countries, she regresses B:

- (1) on I and P
- (2) on R
- (3) on I
- (4) on P and R

with the results shown in the corresponding columns of the table below (standard errors in parentheses; RSS is the residual sum of squares). The correlation coefficient for I and P was 0.97.

	(1)	(2)	(3)	(4)
I	2.17	—	0.69	—
	(1.04)		(0.25)	
P	−3.19	—	—	−1.02
	(2.17)			(1.19)
R	—	1.37	—	2.17
		(0.44)		(1.04)
constant	−5.14	−3.15	−1.53	−5.14
	(2.62)	(1.21)	(0.92)	(2.62)
R^2	0.22	0.20	0.17	0.22
RSS	967.9	987.1	1,024.3	967.9

- Explain why the researcher was dissatisfied with the results of regression (1).
- Demonstrate that specification (2) may be considered to be a restricted version of specification (1).
- Perform an F test of the restriction, stating carefully your null hypothesis and conclusion.
- Perform a t test of the restriction.
- Demonstrate that specification (3) may also be considered to be a restricted version of specification (1).
- Perform both an F test and a t test of the restriction in specification (3), stating your conclusion in each case.
- At a seminar, someone suggests that specification (4) is also a restricted version of specification (1). Is this correct? If so, state the restriction.
- State, with an explanation, which would be your preferred specification.

A6.12 A researcher has a sample of 43 observations on a dependent variable, Y, and two potential explanatory variables, X and Z. He defines two further variables V and W as the sum of X and Z and the difference between them:

$$V_i = X_i + Z_i$$
$$W_i = X_i - Z_i.$$

He fits the following four regressions:

(1) A regression of Y on X and Z.

(2) A regression of Y on V and W.

(3) A regression of Y on V.

(4) A regression of Y on Z and V.

The table shows the regression results (standard errors in parentheses; RSS = residual sum of squares; there was an intercept, not shown, in each regression). Unfortunately, a goat ate part of the regression output and some of the numbers are missing. These are indicated by letters.

	(1)	(2)	(3)	(4)
X	0.60	—	—	—
	(0.04)			
Z	0.80	—	—	**H**
	(0.04)		**(I)**	
V	—	**A**	0.72	**J**
		(B)	(0.02)	**(K)**
W	—	**C**	—	—
		(D)		
R^2	0.60	**E**	**G**	**L**
RSS	200	**F**	220	**M**

Each regression included an intercept (not shown).

Reconstruct each missing number if this is possible, giving a brief explanation. If it is not possible to reconstruct a number, give a brief explanation.

A6.13 In Exercise A6.7, a researcher proposes to test the restriction using variations in R^2 instead of variations in RSS. For food consumed at home, the unrestricted regression of $LGFDHO$ on $LGEXP$ and $LGSIZE$ had $R^2 = 0.4831$. For the regression of $LGFDHOPC$ on $LGEXPPC$, $R^2 = 0.4290$. Hence the researcher's statistic is:

$$F = \frac{(0.4831 - 0.4290)/1}{(1 - 0.4290)/6331} = 599.8.$$

Explain why this is different from the F statistic reported for food consumed at home in the answer to Exercise A6.7.

6.4 Answers to the starred exercises in the textbook

6.4 The table gives the results of multiple and simple regressions of $LGFDHO$, the logarithm of annual household expenditure on food eaten at home, on $LGEXP$, the logarithm of total annual household expenditure, and $LGSIZE$, the logarithm of the number of persons in the household, using a sample of 6,334 households in the 2013 Consumer Expenditure Survey. The correlation coefficient for $LGEXP$ and $LGSIZE$ was 0.32. Explain the variations in the regression coefficients.

	(1)	(2)	(3)
$LGEXP$	0.58	0.67	—
	(0.01)	(0.01)	
$LGSIZE$	0.33	—	0.58
	(0.01)		(0.02)
constant	1.16	0.70	6.04
	(0.08)	(0.08)	(0.01)
R^2	0.48	0.43	0.19

Answer:

If the model is written as:

$$LGFDHO = \beta_1 + \beta_2 LGEXP + \beta_3 LGSIZE + u$$

the expected value of $\widehat{\beta}_2$ in the second regression is given by:

$$E(\widehat{\beta}_2) = \beta_2 + \beta_3 \frac{\sum \left(LGEXP_i - \overline{LGEXP}\right)\left(LGSIZE_i - \overline{LGSIZE}\right)}{\sum \left(LGEXP_i - \overline{LGEXP}\right)^2}.$$

We know that the covariance is positive because the correlation is positive, and it is reasonable to suppose that β_3 is also positive, especially given the highly significant positive estimate in the first regression, and so $\widehat{\beta}_2$ is biased upwards. This accounts for the large increase in its size in the second regression. In the third regression:

$$E(\widehat{\beta}_3) = \beta_3 + \beta_2 \frac{\sum \left(LGEXP_i - \overline{LGEXP}\right)\left(LGSIZE_i - \overline{LGSIZE}\right)}{\sum \left(LGSIZE_i - \overline{LGSIZE}\right)^2}.$$

β_2 is certainly positive, especially given the highly significant positive estimate in the first regression, and so $\widehat{\beta}_3$ is also biased upwards. As a consequence, the estimate in the third regression is greater than that in the first.

6.7 A researcher thinks that the level of activity in the shadow economy, Y, depends either positively on the level of the tax burden, X, or negatively on the level of government expenditure to discourage shadow economy activity, Z. Y might also depend on both X and Z. International cross-sectional data on Y, X, and Z, all measured in US$ million, are obtained for a sample of 30 industrialised countries and a second sample of 30 developing countries. The researcher regresses (1) $\log Y$ on both $\log X$ and $\log Z$, (2) $\log Y$ on $\log X$ alone, and (3) $\log Y$ on $\log Z$ alone, for each sample, with the following results (standard errors in parentheses):

	Industrialised countries			Developing countries		
	(1)	(2)	(3)	(1)	(2)	(3)
$\log X$	0.699	0.201	—	0.806	0.727	—
	(0.154)	(0.112)		(0.137)	(0.090)	
$\log Z$	−0.646	—	−0.053	−0.091	—	0.427
	(0.162)		(0.124)	(0.117)		(0.116)
constant	−1.137	−1.065	1.230	−1.122	−1.024	2.824
	(0.863)	(1.069)	(0.896)	(0.873)	(0.858)	(0.835)
R^2	0.44	0.10	0.01	0.71	0.70	0.33

X was positively correlated with Z in both samples. Having carried out the appropriate statistical tests, write a short report advising the researcher how to interpret these results.

Answer:

One way to organise an answer to this exercise is, for each sample, to consider the evidence for and against each of the three specifications in turn. The t statistics for the slope coefficients are given in the following table. * indicates significance at the 5 per cent level, ** at the 1 per cent level, and *** at the 0.1 per cent level, using one-sided tests. (Justification for one-sided tests: one may rule out a negative coefficient for X and a positive one for Y.)

	Industrialised countries			Developing countries		
	(1)	(2)	(3)	(1)	(2)	(3)
$\log X$	4.54***	1.79*	—	5.88****	8.08***	—
$\log Z$	−3.99***	—	−0.43	−0.78	—	3.68***

Industrialised countries:

The first specification is clearly the only satisfactory one for this sample, given the t statistics. Writing the model as:

$$\log Y = \beta_1 + \beta_2 \log X + \beta_3 \log Z + u$$

in the second specification:

$$E(\widehat{\beta}_2) = \beta_2 + \beta_3 \frac{\sum \left(\log X_i - \overline{\log X}\right)\left(\log Z_i - \overline{Z}\right)}{\sum \left(\log X_i - \overline{\log X}\right)^2}.$$

Anticipating that β_3 is negative, and knowing that X and Z are positively correlated, the bias term should be negative. The estimate of β_2 is indeed lower in the second specification. In the third specification:

$$E(\widehat{\beta}_3) = \beta_3 + \beta_2 \frac{\sum \left(\log X_i - \overline{\log X}\right)\left(\log Z_i - \overline{Z}\right)}{\sum \left(\log Z_i - \overline{\log Z}\right)^2}$$

and the bias should be positive, assuming β_2 is positive. $\widehat{\beta}_3$ is indeed less negative than in the first specification.

Note that the sum of the R^2 statistics for the second and third specifications is less than R^2 in the first. This is because the bias terms undermine the apparent explanatory power of X and Z in the second and third specifications. In the third specification, the bias term virtually neutralises the true effect and R^2 is very low indeed.

Developing countries:

In principle the first specification is acceptable. The failure of the coefficient of Z to be significant might be due to a combination of a weak effect of Z and a relatively small sample.

The second specification is also acceptable since the coefficient of Z and its t statistic in the first specification are very low. Because the t statistic of Z is low, R^2 is virtually unaffected when it is omitted.

The third specification is untenable because it cannot account for the highly significant coefficient of X in the first. The omitted variable bias is now so large that it overwhelms the negative effect of Z with the result that the estimated coefficient is positive.

6.11 A researcher has data on output per worker, Y, and capital per worker, K, both measured in thousands of dollars, for 50 firms in the textiles industry in 2012. She hypothesises that output per worker depends on capital per worker and perhaps also the technological sophistication of the firm, $TECH$:

$$Y = \beta_1 + \beta_2 K + \beta_3 TECH + u$$

where u is a disturbance term. She is unable to measure $TECH$ and decides to use expenditure per worker on research and development in 2012, $R\&D$, as a proxy for it. She fits the following regressions (standard errors in parentheses):

$$\begin{aligned} \widehat{Y} &= \underset{(0.45)}{1.02} + \underset{(0.04)}{0.32}K \qquad R^2 = 0.749 \\ \widehat{Y} &= \underset{(0.61)}{0.34} + \underset{(0.22)}{0.29}K + \underset{(0.15)}{0.05}R\&D \qquad R^2 = 0.750 \end{aligned}$$

The correlation coefficient for K and $R\&D$ is 0.92. Discuss these regression results:

1. assuming that Y does depend on both K and $TECH$
2. assuming that Y depends only on K.

Answer:

If Y depends on both K and $TECH$, the first specification is subject to omitted variable bias, with the expected value of $\widehat{\beta}_2$ being given by:

$$E(\widehat{\beta}_2) = \beta_2 + \beta_3 \frac{\sum \left(K_i - \overline{K}\right)\left(TECH_i - \overline{TECH}\right)}{\sum \left(K_i - \overline{K}\right)^2}.$$

Since K and $R\&D$ have a high positive correlation, it is reasonable to assume that K and $TECH$ are positively correlated. It is also reasonable to assume that β_3 is positive. Hence one would expect $\widehat{\beta}_2$ to be biased upwards. It is indeed greater than in the second equation, but not by much. The second specification is clearly subject to multicollinearity, with the consequence that, although the estimated coefficients remain unbiased, they are erratic, this being reflected in large standard errors. The large variance of the estimate of the coefficient of K means that much of the difference between it and the estimate in the first specification is likely to be purely random, and this could account for the fact that the omitted variable bias appears to be so small.

If Y depends only on K, the inclusion of *R&D* in the second specification gives rise to inefficiency. Since the standard errors in both equations remain valid, they can be compared and it is evident that the loss of efficiency is severe. As expected in this case, the coefficient of *R&D* is not significantly different from zero and the increase in R^2 in the second specification is minimal.

6.14 The first regression shows the result of regressing *LGFDHO*, the logarithm of annual household expenditure on food eaten at home, on *LGEXP*, the logarithm of total annual household expenditure, and *LGSIZE*, the logarithm of the number of persons in the household, using a sample of 6,334 households in the 2013 Consumer Expenditure Survey. In the second regression, *LGFDHOPC*, the logarithm of food expenditure per capita (*FDHO/SIZE*), is regressed on *LGEXPPC*, the logarithm of total expenditure per capita (*EXP/SIZE*). In the third regression *LGFDHOPC* is regressed on *LGEXPPC* and *LGSIZE*.

```
. reg LGFDHO LGEXP LGSIZE
-----------------------------------------------------------------------------
    Source |       SS       df       MS              Number of obs =    6334
-----------+------------------------------           F(  2,  6331) = 2958.94
     Model |  1858.61471     2  929.307357           Prob> F       =  0.0000
  Residual |  1988.36474  6331  .314068037           R-squared     =  0.4831
-----------+------------------------------           Adj R-squared =  0.4830
     Total |  3846.97946  6333   .60744978           Root MSE      =  .56042
-----------------------------------------------------------------------------
    LGFDHO |      Coef.   Std. Err.      t    P>|t|      [95% Conf. Interval]
-----------+-----------------------------------------------------------------
     LGEXP |   .5842097   .0097174    60.12   0.000      .5651604    .6032591
    LGSIZE |   .3343475   .0127587    26.21   0.000      .3093362    .3593589
     _cons |   1.158326   .0820119    14.12   0.000      .9975545    1.319097
-----------------------------------------------------------------------------
```

```
. gen LGFDHOPC = ln(FDHO/SIZE)
. gen LGEXPPC = ln(EXP/SIZE)

. reg LGFDHOPC LGEXPPC
-----------------------------------------------------------------------------
    Source |       SS       df       MS              Number of obs =    6334
-----------+------------------------------           F(  1,  6332) = 4757.00
     Model |  1502.58928     1  1502.58928           Prob> F       =  0.0000
  Residual |   2000.0827  6332   .31586903           R-squared     =  0.4290
-----------+------------------------------           Adj R-squared =  0.4289
     Total |  3502.67197  6333  .553082579           Root MSE      =  .56202
-----------------------------------------------------------------------------
  LGFDHOPC |      Coef.   Std. Err.      t    P>|t|      [95% Conf. Interval]
-----------+-----------------------------------------------------------------
   LGEXPPC |   .6092734   .0088338    68.97   0.000      .5919562    .6265905
     _cons |   .8988292   .0703516    12.78   0.000      .7609162    1.036742
-----------------------------------------------------------------------------
```

```
. reg LGFDHOPC LGEXPPC LGSIZE
-----------------------------------------------------------------------------
    Source |       SS       df       MS            Number of obs =     6334
-----------+------------------------------         F(  2,  6331) = 2410.79
     Model |  1514.30723     2  757.153617         Prob> F       =   0.0000
  Residual |  1988.36474  6331  .314068037         R-squared     =   0.4323
-----------+------------------------------         Adj R-squared =   0.4321
     Total |  3502.67197  6333  .553082579         Root MSE      =   .56042
-----------------------------------------------------------------------------
  LGFDHOPC |      Coef.   Std. Err.       t     P>|t|     [95% Conf. Interval]
-----------+-----------------------------------------------------------------
   LGEXPPC |   .5842097   .0097174    60.12   0.000     .5651604    .6032591
    LGSIZE |  -.0814427   .0133333    -6.11   0.000    -.1075805   -.0553049
     _cons |   1.158326   .0820119    14.12   0.000     .9975545    1.319097
-----------------------------------------------------------------------------
```

1. Explain why the second model is a restricted version of the first, stating the restriction.
2. Perform an F test of the restriction.
3. Perform a t test of the restriction.
4. Summarise your conclusions from the analysis of the regression results.

Answer:

Write the first specification as:

$$LGFDHO = \beta_1 + \beta_2 LGEXP + \beta_3 LGSIZE + u.$$

Then the restriction implicit in the second specification is $\beta_3 = 1 - \beta_2$, for:

$$\begin{aligned}
LGFDHO &= \beta_1 + \beta_2 LGEXP + (1 - \beta_2) LGSIZE + u \\
LGFDHO - LGSIZE &= \beta_1 + \beta_2 (LGEXP - LGSIZE) + u \\
\log \frac{FDHO}{SIZE} &= \beta_1 + \beta_2 \log \frac{EXP}{SIZE} + u \\
LGFDHOPC &= \beta_1 + \beta_2 LGEXPPC + u
\end{aligned}$$

the last equation being the second specification. The F statistic for the null hypothesis $H_0 : \beta_3 = 1 - \beta_2$ is:

$$F(1,\, 6331) = \frac{(2000.1 - 1988.4)/1}{1988.4/6331} = 37.3.$$

The critical value of $F(1,\, 1000)$ at the 0.1 per cent level is 10.9, and hence the restriction is rejected at that significance level.

Alternatively, we could use the t test approach. Under the null hypothesis that the restriction is valid, $\theta = 1 - \beta_2 - \beta_3 = 0$. Substituting for β_3, the unrestricted version may be rewritten:

$$LGFDHO = \beta_1 + \beta_2 LGEXP + (1 - \beta_2 - \theta) LGSIZE + u.$$

This may be rewritten:

$$\log \frac{FDHO}{SIZE} = \beta_1 + \beta_2 \log \frac{EXP}{SIZE} - \theta \log SIZE + u$$

that is:

$$LGFDHOPC = \beta_1 + \beta_2 LGEXPPC - \theta LGSIZE + u.$$

The t statistic for the coefficient of $LGSIZE$ is -6.11, so we reject the restriction at a very high significance level. Note that the t statistic is the square root of the F statistic and the critical value of t at the 0.1 per cent level will be the square root of the critical value of F.

6.5 Answers to the additional exercises

A6.1 The output below gives the results of a simple regression of $LGFDHOPC$ on $LGSIZE$. See Exercise A4.1 for the simple regression of $LGFDHOPC$ on $LGEXPPC$ and Exercise A4.2 for the multiple regression of $LGFDHOPC$ on $LGEXPPC$ and $LGSIZE$.

```
. reg LGFDHOPC LGSIZE
----------------------------------------------------------------------------
   Source |       SS       df       MS                  Number of obs =    6334
----------+------------------------------               F(  1,  6332) =  768.56
    Model |  379.128845     1  379.128845               Prob> F       =  0.0000
 Residual |  3123.54316  6332  .493294877               R-squared     =  0.1082
----------+------------------------------               Adj R-squared =  0.1081
    Total |  3502.67201  6333  .553082585               Root MSE      =  .70235
----------------------------------------------------------------------------
 LGFDHOPC |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
----------+-----------------------------------------------------------------
   LGSIZE |  -.4199282   .0151473   -27.72   0.000     -.449622   -.3902344
    _cons |   6.040547   .0143586   420.69   0.000     6.012399    6.068695
----------------------------------------------------------------------------
```

If the true model is assumed to be:

$$LGFDHOPC = \beta_1 + \beta_2 LGEXPPC + \beta_3 LGSIZE + u$$

the expected value of $\widehat{\beta}_2$ in the simple regression of $LGFDHOPC$ on $LGEXPPC$ is given by:

$$E(\widehat{\beta}_2) = \beta_2 + \beta_3 \frac{\sum \left(LGEXPPC_i - \overline{LGEXPPC} \right) \left(LGSIZE_i - \overline{LGSIZE} \right)}{\sum \left(LGEXPPC_i - \overline{LGEXPPC} \right)^2}.$$

We know that the numerator of the second factor in the bias term is negative because the correlation is negative:

```
. cor LGEXPPC LGSIZE
(obs=6334)
           |  LGEXPPC   LGSIZE
-----------+------------------
   LGEXPPC |   1.0000
    LGSIZE |  -0.4223   1.0000
```

It is reasonable to suppose that economies of scale will cause β_3 to be negative, and the highly significant negative estimate in the multiple regression provides empirical support, so $\widehat{\beta}_2$ is biased upwards. This accounts for the increase in its size in the second regression. In the third regression:

$$E(\widehat{\beta}_3) = \beta_3 + \beta_2 \frac{\sum \left(LGEXPPC_i - \overline{LGEXPPC} \right) \left(LGSIZE_i - \overline{LGSIZE} \right)}{\sum \left(LGSIZE_i - \overline{LGSIZE} \right)^2}.$$

β_2 is certainly positive, especially given the highly significant positive estimate in the first regression, and so $\widehat{\beta}_3$ is biased downwards. As a consequence, the estimate in the third regression is lower than that in the first.

Similar results are obtained for the other categories of expenditure. The correlation between *LGEXPPC* and *LGSIZE* varies because the missing observations are different for different categories.

Omitted variable bias, dependent variable *LGCATPC*					
		Multiple regression		Simple regressions	
	n	*LGEXPPC*	*LGSIZE*	*LGEXPPC*	*LGSIZE*
ADM	2,815	1.080	−0.055	1.098	−0.678
CLOT	4,500	0.842	0.146	0.794	−0.375
DOM	1,661	0.941	0.415	0.812	−0.150
EDUC	561	1.229	−0.437	1.382	−1.243
ELEC	5,828	0.472	−0.362	0.586	−0.645
FDAW	5,102	0.879	−0.213	0.947	−0.735
FDHO	6,334	0.584	−0.081	0.609	−0.420
FOOT	1,827	0.396	−0.560	0.608	−0.842
FURN	487	0.807	−0.246	0.912	−0.848
GASO	5,710	0.676	−0.004	0.677	−0.410
HEAL	4,802	0.779	−0.306	0.868	−0.723
HOUS	6,223	0.989	−0.140	1.033	−0.716
LIFE	1,253	0.464	−0.461	0.607	−0.701
LOCT	692	0.389	−0.396	0.510	−0.639
MAPP	399	0.721	−0.264	0.817	−0.717
PERS	3,817	0.824	−0.217	0.891	−0.703
READ	2,287	0.764	−0.503	0.909	−0.923
SAPP	1,037	0.467	−0.592	0.665	−0.879
TELE	5,788	0.640	−0.222	0.710	−0.603
TEXT	992	0.388	−0.713	0.629	−0.959
TOB	1,155	0.563	−0.515	0.721	−0.822
TOYS	2,504	0.638	−0.304	0.733	−0.691
TRIP	516	0.681	−0.142	0.723	−0.492

A6.2 *Demonstrate that $\widehat{\beta}_2$ is an unbiased estimator of β_2.*

$$\begin{aligned}\widehat{\beta}_2 &= \frac{\sum\left(D_i-\overline{D}\right)\left(Y_i-\overline{Y}\right)}{\sum\left(D_i-\overline{D}\right)^2} \\ &= \frac{\sum\left(D_i-\overline{D}\right)\left((\beta_1+\beta_2 D_i+\beta_3 A_i+u_i)-(\beta_1+\beta_2\overline{D}+\beta_3\overline{A}+\overline{u})\right)}{\sum\left(D_i-\overline{D}\right)^2} \\ &= \beta_2+\beta_3\frac{\sum\left(D_i-\overline{D}\right)\left(A_i-\overline{A}\right)}{\sum\left(D_i-\overline{D}\right)^2}+\frac{\sum\left(D_i-\overline{D}\right)(u_i-\overline{u})}{\sum\left(D_i-\overline{D}\right)^2}.\end{aligned}$$

Hence:

$$\widehat{\beta}_2=\beta_2+\beta_3\sum d_i(A_i-\overline{A})+\sum d_i(u_i-\overline{u})$$

where:

$$d_i=\frac{D_i-\overline{D}}{\sum(D_j-\overline{D})^2}.$$

Hence:

$$E(\widehat{\beta}_2)=\beta_2+\beta_3\sum E(d_i(A_i-\overline{A}))+\sum E(d_i(u_i-\overline{u})).$$

Now, since the assignment to the course was random, D is distributed independently of both A and u, and hence:

$$E(d_i(A_i-\overline{A}))=E(d_i)E(A_i-\overline{A})=0$$

and:

$$E(d_i(u_i-\overline{u}))=E(d_i)E(u_i-\overline{u})=0.$$

Hence $\widehat{\beta}_2$ is an unbiased estimator of β_2.

A commentator says that the standard error of $\widehat{\beta}_2$ will be invalid because an important variable, A, has been omitted from the specification. The researcher replies that the standard error will remain valid if A can be assumed to have a normal distribution. Explain whether the commentator or the researcher is correct.

The researcher is nearly correct. Given the random selection of the sample, A will be distributed independently of D and so it can be treated as part of the disturbance term and the standard error will remain valid. The requirement that A have a normal distribution is too strong, since the expression for the standard error does not depend on it. However, if the standard error is to be used for t tests, then it is important that the enlarged standard error should have a normal distribution, and this will be the case if an only if A has a normal distribution (assuming that u has one). If both A and u have normal distributions, a linear combination will also have one.

Another commentator says that whether the distribution of A is normal or not makes no difference to the validity of the standard error. Evaluate this assertion.

The commentator is correct for the reasons just explained.

A6.3 *Assuming that (A) is the correct specification, explain, with a mathematical proof, whether you would expect the coefficient of log Y to be greater in regression (2).*

To simplify the algebra, throughout this answer $\log B$, $\log Y$, $\log S$ and log IQ will be written as B, Y, S and IQ, it being understood that these are logarithms.

$$\begin{aligned}
\widehat{\beta}_2 &= \frac{\sum\left(B_i-\overline{B}\right)\left(Y_i-\overline{Y}\right)}{\sum\left(Y_i-\overline{Y}\right)^2} \\
&= \frac{\sum\left(\beta_1+\beta_2Y_i+\beta_3IQ_i+u_i-\beta_1-\beta_2\overline{Y}-\beta_3\overline{IQ}-\bar{u}\right)\left(Y_i-\overline{Y}\right)}{\sum\left(Y_i-\overline{Y}\right)^2} \\
&= \frac{\sum\left(\beta_2Y_i-\beta_2\overline{Y}\right)\left(Y_i-\overline{Y}\right)+\sum\left(\beta_3IQ_i-\beta_3\overline{IQ}\right)\left(Y_i-\overline{Y}\right)+\sum\left(u_i-\bar{u}\right)\left(Y_i-\overline{Y}\right)}{\sum\left(Y_i-\overline{Y}\right)^2} \\
&= \beta_2+\beta_3\frac{\sum\left(IQ_i-\overline{IQ}\right)\left(Y_i-\overline{Y}\right)}{\sum\left(Y_i-\overline{Y}\right)^2}+\frac{\sum\left(u_i-\bar{u}\right)\left(Y_i-\overline{Y}\right)}{\sum\left(Y_i-\overline{Y}\right)^2}.
\end{aligned}$$

Hence:

$$\begin{aligned}
E(\widehat{\beta}_2) &= \beta_2+\beta_3\frac{\sum\left(IQ_i-\overline{IQ}\right)\left(Y_i-\overline{Y}\right)}{\sum\left(Y_i-\overline{Y}\right)^2}+\frac{1}{\sum\left(Y_i-\overline{Y}\right)^2}E\left(\sum(u_i-\bar{u})(Y_i-\overline{Y})\right) \\
&= \beta_2+\beta_3\frac{\sum\left(IQ_i-\overline{IQ}\right)\left(Y_i-\overline{Y}\right)}{\sum\left(Y_i-\overline{Y}\right)^2}+\frac{1}{\sum\left(Y_i-\overline{Y}\right)^2}\sum E\left((u_i-\bar{u})(Y_i-\bar{Y})\right) \\
&= \beta_2+\beta_3\frac{\sum\left(IQ_i-\overline{IQ}\right)\left(Y_i-\overline{Y}\right)}{\sum\left(Y_i-\overline{Y}\right)^2}+\frac{1}{\sum\left(Y_i-\overline{Y}\right)^2}\sum(Y_i-\overline{Y})E(u_i-\bar{u}) \\
&= \beta_2+\beta_3\frac{\sum\left(IQ_i-\overline{IQ}\right)\left(Y_i-\overline{Y}\right)}{\sum\left(Y_i-\overline{Y}\right)^2}
\end{aligned}$$

assuming that Y and IQ are nonstochastic. Thus $\widehat{\beta}_2$ is biased, the direction of the bias depending on the signs of β_3 and $\sum\left(IQ_i-\overline{IQ}\right)\left(Y_i-\overline{Y}\right)$. We would expect the former to be positive and we expect the latter to be positive since we are told that the correlation between S and Y is positive and S is a proxy for IQ. So we would expect an upward bias in regression (2).

Assuming that (A) is the correct specification, describe the various benefits from using log S as a proxy for log IQ, as in regression (1), if log S is a good proxy.

The use of S as a proxy for IQ will alleviate the problem of omitted variable bias. In particular, comparing the results of regression (1) with those that would have been obtained if B had been regressed on Y and IQ:

- the coefficient of Y will be approximately the same
- its standard error will be approximately the same
- the t statistic for S will be approximately equal to that of IQ
- R^2 will be approximately the same.

Explain whether the low value of R^2 in regression (1) implies that log S is not a good proxy.

Not necessarily. It could be that S is a poor proxy for IQ, but it could also be that the original model had low explanatory power.

Assuming that (A) is the correct specification, provide an explanation of why the coefficients of log Y and log S in regression (1) are not significantly different from zero, using two-sided t tests.

The high correlation between Y and S has given rise to multicollinearity, the standard errors being so large that the coefficients are not significantly different from zero.

Discuss whether the researcher would be justified in using one-sided t tests in regression (1).

Yes. It is reasonable to suppose that expenditure on books should not be negatively influenced by either income or cognitive ability. (Note that one should *not* say that it is reasonable to suppose that expenditure on books is positively influenced by them. This rules out the null hypothesis.)

Assuming that (B) is the correct specification, explain whether you would expect the coefficient of log Y to be lower in regression (1).

No. It would be randomly higher or lower, if S is an irrelevant variable.

Assuming that (B) is the correct specification, explain whether the standard errors in regression (1) are valid estimates.

Yes. The inclusion of an irrelevant variable in general does not invalidate the standard errors. It causes them to be larger than those in the correct specification.

A6.4 *Explain analytically whether the slope coefficient is likely to be biased.*

If the fitted model is:

$$\widehat{\log T} = \widehat{\beta}_1 + \widehat{\beta}_2 \log P$$

then:

$$\begin{aligned}
\widehat{\beta}_2 &= \frac{\sum \left(\log P_i - \overline{\log P}\right)\left(\log T_i - \overline{\log T}\right)}{\sum \left(\log P_i - \overline{\log P}\right)^2} \\
&= \frac{\sum \left(\log P_i - \overline{\log P}\right)\left(\beta_1 + \beta_2 \log P_i + \beta_3 \log Y_i + u_i - \beta_1 - \beta_2 \overline{\log P} - \beta_3 \overline{\log Y} - \bar{u}\right)}{\sum \left(\log P_i - \overline{\log P}\right)^2} \\
&= \beta_2 + \beta_3 \frac{\sum \left(\log P_i - \overline{\log P}\right)\left(\log Y_i - \overline{\log Y}\right)}{\sum \left(\log P_i - \overline{\log P}\right)^2} + \frac{\sum \left(\log P_i - \overline{\log P}\right)(u_i - \bar{u})}{\sum \left(\log P_i - \overline{\log P}\right)^2}.
\end{aligned}$$

Hence:

$$E(\widehat{\beta}_2) = \beta_2 + \beta_3 \frac{\sum \left(\log P_i - \overline{\log P}\right)\left(\log Y_i - \overline{\log Y}\right)}{\sum \left(\log P_i - \overline{\log P}\right)^2}$$

provided that any random component of $\log P$ is distributed independently of u. Since it is reasonable to assume $\beta_3 > 0$, and since we are told that Y and P are positively correlated, the bias will be upwards. This accounts for the nonsensical positive price elasticity in the fitted equation.

Describe the theoretical benefits from using H as a proxy for Y, discussing whether they appear to have been obtained in this example.

Suppose that H is a perfect proxy for Y:

$$\log Y = \lambda + \mu \log H.$$

Then the relationship may be rewritten:

$$\log T = \beta_1 + \beta_3\lambda + \beta_2 \log P + \beta_3\mu \log H + u.$$

The coefficient of $\log P$ ought to be the same as in the true relationship. However in this example it is not the same. However it is of the right order of magnitude and much more plausible than the estimate in the first regression. The standard error of the coefficient ought to be the same as in the true relationship, and this is the case.

The coefficient of $\log H$ will be an estimate of $\beta_3\mu$, and since μ is unknown, β_3 is not identified. However, if it can be assumed that the average household income in a city is proportional to average house values, it could be asserted that μ is equal to 1, in which case the coefficient of $\log H$ will be a direct estimate of β_3 after all. The coefficient of $\log H$ is indeed quite close to that of $\log Y$. The t statistic for the coefficient of $\log H$ ought to be the same as that for $\log Y$, and this is approximately true, being a little lower. R^2 ought to be the same, but it is somewhat lower, suggesting that H appears to have been a good proxy, but not a perfect one.

A6.5 *Explain why specification (1) is a restricted version of specification (2), stating and interpreting the restriction.*

First note that, since $E = W/H$, $LE = \log(W/H) = LW - LH$.

Write specification (2) as:

$$LW = \beta_1 + \beta_2 S + \beta_3 LH + u.$$

If one imposes the restriction $\beta_3 = 1$, the model becomes specification (1):

$$LW - LH = \beta_1 + \beta_2 S + u.$$

The restriction implies that weekly earnings are proportional to hours worked, controlling for schooling.

Supposing the restriction to be valid, explain whether you expect the coefficient of S and its standard error to differ, or be similar, in specifications (1) and (2).

If the restriction is valid, the coefficient of S should be similar in the restricted specification (1) and the unrestricted specification (2). Both estimates will be unbiased, but that in specification (1) will be more efficient. The gain in efficiency in specification (1) should be reflected in a smaller standard error. However, the gain will be small, given the low correlation.

Supposing the restriction to be invalid, how would you expect the coefficient of S and its standard error to differ, or be similar, in specifications (1) and (2)?

The estimate of the coefficient of S would be biased. The standard error in specification (1) would be invalid and so a comparison with the standard error in specification (2) would be illegitimate.

Perform an F test of the restriction.

The null and alternative hypotheses are $H_0 : \beta_3 = 1$ and $H_1 : \beta_3 \neq 1$.

$$F(1, 1752) = \frac{(741.5 - 737.9)/1}{737.9/1752} = 8.5.$$

The critical value of $F(1, 1000)$ at the 1 per cent level is 6.66. The critical value of $F(1, 1752)$ must be lower. Thus we reject the restriction at the 1 per cent level. (The critical value at the 0.1 per cent level is about 10.8.)

Perform a t test of the restriction.

The restriction is so simple that it can be tested with no reparameterisation: a simple t test on the coefficient of LH in specification (2), $H_0 : \beta_3 = 1$.

Alternatively, mechanically following the standard procedure, we rewrite the restriction as $\beta_3 - 1 = 0$. The reparameterisation will be:

$$\theta = \beta_3 - 1$$

and so:

$$\beta_3 = \theta + 1.$$

Substituting this into the unrestricted specification, the latter may be rewritten:

$$LW = \beta_1 + \beta_2 S + (\theta + 1)LH + u.$$

Hence:

$$LW - LH = \beta_1 + \beta_2 S + \theta LH + u.$$

This is regression specification (3) and the restriction may be tested with a t test on the coefficient of LH, the null hypothesis being $H_0 : \theta = \beta_3 - 1 = 0$. The t statistic is 2.92, which is significant at the 1 per cent level, implying that the restriction should be rejected.

Explain whether the F test and the t test could lead to different conclusions.

The tests must lead to the same conclusion since the F statistic is the square of the t statistic and the critical value of F is the square of the critical value of t.

At a seminar, a commentator says that part-time workers tend to be paid worse than full-time workers and that their earnings functions are different. Defining full-time workers as those working at least 35 hours per week, the researcher divides the sample and fits the earnings functions for full-time workers (column 4) and part-time workers (column 5). Test whether the commentators assertion is correct.

The appropriate test is a Chow test. The test statistic under the null hypothesis of no difference in the earnings functions is:

$$F(3,1749) = \frac{(737.9 - 626.1 - 100.1)/3}{(626.1 + 100.1)/1749} = 9.39.$$

The critical value of $F(3,1000)$ at the 0.1 per cent level is 5.46. Hence we reject the null hypothesis and conclude that the commentator is correct.

What are the implications of the commentators assertion for the test of the restriction?

The elasticity of LH is now not significantly different from 1 for either full-time or part-time workers, so the restriction is no longer rejected.

A6.6 *Explain why the second specification is a restricted version of the first. State the restriction.*

Write the second equation as:

$$\log \frac{I}{Y} = \beta_1 + \beta_2 \log \left(\frac{G}{Y}\right) + u.$$

It may be re-written as:

$$\log I = \beta_1 + \beta_2 \log G + (1 - \beta_2) \log Y + u.$$

This is a special case of the specification of the first equation:

$$\log I = \beta_1 + \beta_2 \log G + \beta_3 \log Y + u$$

with the restriction $\beta_3 = 1 - \beta_2$.

Perform a test of the restriction.

The null hypothesis is $H_0 : \beta_2 + \beta_3 = 1$. The test statistic is:

$$F(1,27) = \frac{(0.99 - 0.90)/1}{0.90/27} = 2.7.$$

The critical value of $F(1,27)$ is 4.21 at the 5 per cent level. Hence we do not reject the null hypothesis that the restriction is valid.

The researcher expected the standard error of the coefficient of log (G/Y) in (2) to be smaller than the standard error of the coefficient of log G in (1). Explain why she expected this.

The imposition of the restriction, if valid, should lead to a gain in efficiency and this should be reflected in lower standard errors.

However the standard error is the same, at least to two decimal places. Give an explanation.

The standard errors of the coefficients of G in (1) and G/Y in (2) are given by:

$$\sqrt{\frac{\widehat{\sigma}_u^2}{n\text{MSD}(G)} \times \frac{1}{1 - r_{G,Y}^2}} \qquad \text{and} \qquad \sqrt{\frac{\widehat{\sigma}_u^2}{n\text{MSD}(G/Y)}}$$

respectively, where $\widehat{\sigma}_u^2$ is an estimate of the variance of the disturbance term, n is the number of observations, MSD is the mean square deviation in the sample, and $r_{G,Y}$ is the sample correlation coefficient of G and Y. n is the same for both standard errors and $\widehat{\sigma}_u^2$ will be very similar. We are told that $r_{G,Y} = 0.98$, so its square is 0.96 and the second factor in the expression for the standard error of G is $(1/0.04) = 25$. Hence, other things being equal, the standard error of G/Y should be much lower than that of G. However the table shows that the MSD of G/Y is only 1/25 as great as that of G. This just about exactly negates the gain in efficiency attributable to the elimination of the correlation between G and Y.

Show how the restriction could be tested using a t test in a reparameterised version of the specification for (1).

Define $\theta = \beta_2 + \beta_3 - 1$, so that the restriction may be written $\theta = 0$. Then $\beta_3 = \theta - \beta_2 + 1$. Use this to substitute for β_3 in the unrestricted model:

$$\begin{aligned}\log I &= \beta_1 + \beta_2 \log G + \beta_3 \log Y + u \\ &= \beta_1 + \beta_2 \log G + (\theta - \beta_2 + 1) \log Y + u.\end{aligned}$$

Then:

$$\log I - \log Y = \beta_1 + \beta_2(\log G - \log Y) + \theta \log Y + u$$

and:

$$\log\left(\frac{I}{Y}\right) = \beta_1 + \beta_2\left(\frac{G}{Y}\right) + \theta \log Y + u.$$

Hence the restriction may be tested by a t test of the coefficient of $\log Y$ in a regression using this specification.

A6.7 This is a generalisation of the example with *FDHO* in Exercise 6.14 in the text. The reason for the discrepancy in the number of observations is not known. Possibly it used an earlier version of the data set.

```
. reg LGFDHO LGEXP LGSIZE
------------------------------------------------------------------------------
    Source |       SS       df       MS              Number of obs =    6334
-----------+------------------------------           F(  2,  6331) = 2958.94
     Model |  1858.61471     2  929.307357           Prob> F       =  0.0000
  Residual |  1988.36474  6331  .314068037           R-squared     =  0.4831
-----------+------------------------------           Adj R-squared =  0.4830
     Total |  3846.97946  6333   .60744978           Root MSE      =  .56042
------------------------------------------------------------------------------
    LGFDHO |      Coef.   Std. Err.       t     P>|t|      [95% Conf. Interval]
-----------+------------------------------------------------------------------
     LGEXP |   .5842097   .0097174     60.12   0.000      .5651604    .6032591
    LGSIZE |   .3343475   .0127587     26.21   0.000      .3093362    .3593589
     _cons |   1.158326   .0820119     14.12   0.000      .9975545    1.319097
------------------------------------------------------------------------------
```

```
. reg LGFDHOPC LGEXPPC
------------------------------------------------------------------------------
    Source |       SS       df       MS              Number of obs =    6334
-----------+------------------------------           F(  1,  6332) = 4757.00
     Model |  1502.58932     1  1502.58932           Prob> F       =  0.0000
  Residual |  2000.08269  6332  .315869029           R-squared     =  0.4290
-----------+------------------------------           Adj R-squared =  0.4289
     Total |  3502.67201  6333  .553082585           Root MSE      =  .56202
------------------------------------------------------------------------------
  LGFDHOPC |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-----------+------------------------------------------------------------------
   LGEXPPC |   .6092734   .0088338    68.97   0.000     .5919562    .6265905
     _cons |   .8988291   .0703516    12.78   0.000     .7609161    1.036742
------------------------------------------------------------------------------
```

Write the first specification as:

$$LGFDHO = \beta_1 + \beta_2 LGEXP + \beta_3 LGSIZE + u.$$

Then the restriction implicit in the second specification is $\beta_3 = 1 - \beta_2$, for then:

$$\begin{aligned}
LGFDHO &= \beta_1 + \beta_2 LGEXP + (1-\beta_2) LGSIZE + u \\
LGFDHO - LGSIZE &= \beta_1 + \beta_2 (LGEXP - LGSIZE) + u \\
\log \frac{FDHO}{SIZE} &= \beta_1 + \beta_2 \log \frac{EXP}{SIZE} + u \\
LGFDHOPC &= \beta_1 + \beta_2 LGEXPPC + u
\end{aligned}$$

the last equation being the second specification. The F statistic for the null hypothesis $H_0 : \beta_3 = 1 - \beta_2$ is:

$$F(1, 6331) = \frac{(2000.1 - 1988.4)/1}{1998.4/6331} = 37.25.$$

The critical value of F(1, 1000) at the 0.1 per cent level is 10.9, and hence the restriction is rejected at that significance level. This is not a surprising result, given that the estimates of β_2 and β_3 in the unrestricted specification were 0.58 and 0.33, respectively, their sum being well short of 1, as implied by the restriction.

Summarising the results of the test for all the categories, we have:

- Restriction rejected at the 1 per cent level: *FDHO*, *FDAW*, *HOUS*, *TELE*, *FURN*, *MAPP*, *SAPP*, *CLOT*, *HEAL*, *ENT*, *FEES*, *READ*, *TOB*.
- Restriction rejected at the 5 per cent level: *TRIP*, *LOCT*.
- Restriction not rejected at the 5 per cent level: *DOM*, *TEXT*, *FOOT*, *GASO*, *TOYS*, *EDUC*.

	n	RSS restricted	RSS unrestricted	F	t
ADM	2,815	3,947.5	3,945.2	1.6	−1.26
CLOT	4,500	5,792.0	5,766.1	20.2	4.50
DOM	1,661	4,138.0	4,062.5	30.8	5.55
EDUC	561	1,404.6	1,380.1	9.9	−3.15
ELEC	5,828	2,842.9	2,636.3	456.4	−21.36
FDAW	5,102	3,430.9	3,369.1	93.6	−9.68
FDHO	6,334	2,000.1	1,988.4	37.2	−6.11
FOOT	1,827	1,506.4	1,373.5	176.4	−13.28
FURN	487	920.0	913.9	3.2	−1.80
GASO	5,710	2,879.4	2,879.3	0.0	−0.20
HEAL	4,802	6,183.4	6,062.5	95.7	−9.79
HOUS	6,223	4,859.4	4,825.6	43.6	−6.60
LIFE	1,253	1,622.7	1,559.2	50.9	−7.13
LOCT	692	1,108.1	1,075.1	21.1	−4.60
MAPP	399	583.5	576.8	4.6	−2.14
PERS	3,817	3,049.1	3,002.2	59.6	−7.72
READ	2,287	3,038.1	2,892.1	115.3	−10.74
SAPP	1,037	1,239.6	1,148.9	81.6	−9.03
TELE	5,788	3,133.1	3,055.1	147.6	−12.15
TEXT	992	1,150.5	1,032.9	112.6	−10.61
TOB	1,155	956.3	873.4	109.4	−10.46
TOYS	2,504	2,885.4	2,828.3	50.5	−7.11
TRIP	516	795.4	792.8	1.7	−1.30

For the t test, we first rewrite the restriction as $\beta_2 + \beta_3 - 1 = 0$. The test statistic is therefore $\theta = \beta_2 + \beta_3 - 1$. This allows us to write $\beta_3 = \theta - \beta_2 + 1$. Substituting for β_3, the unrestricted version becomes:

$$LGFDHO = \beta_1 + \beta_2 LGEXP + (\theta - \beta_2 + 1) LGSIZE + u.$$

Hence the unrestricted version may be rewritten:

$$LGFDHO - LGSIZE = \beta_1 + \beta_2 (LGEXP - LGSIZE) + \theta LGSIZE + u$$

that is:

$$LGFDHOPC = \beta_1 + \beta_2 LGEXPPC + \theta LGSIZE + u.$$

We use a t test to see if the coefficient of *LGSIZE* is significantly different from 0. If it is not, we can drop the *LGSIZE* term and we conclude that the restricted specification is an adequate representation of the data. If it is, we have to stay with the unrestricted specification.

From the output for the third regression, we see that t is −6.11 and hence the null hypothesis $H_0 : \beta_2 + \beta_3 - 1 = 0$ is rejected (critical value of t at the 0.1 per cent level is 3.29). Note that the t statistic is the square root of the F statistic and the critical value of t at the 0.1 per cent level is the square root of the critical value of F. The results for the other categories are likewise identical to those for the F test.

```
. reg LGFDHOPC LGEXPPC LGSIZE
------------------------------------------------------------------------------
    Source |       SS       df       MS              Number of obs =    6334
-----------+------------------------------           F(  2,  6331) = 2410.79
     Model |  1514.30728     2   757.15364           Prob> F       =  0.0000
  Residual |  1988.36473  6331  .314068035           R-squared     =  0.4323
-----------+------------------------------           Adj R-squared =  0.4321
     Total |  3502.67201  6333  .553082585           Root MSE      =  .56042
------------------------------------------------------------------------------
  LGFDHOPC |      Coef.   Std. Err.       t     P>|t|      [95% Conf. Interval]
-----------+------------------------------------------------------------------
   LGEXPPC |   .5842097   .0097174     60.12   0.000      .5651604    .6032591
    LGSIZE |  -.0814427   .0133333     -6.11   0.000     -.1075806   -.0553049
     _cons |   1.158326   .0820119     14.12   0.000      .9975545    1.319097
------------------------------------------------------------------------------
```

A6.8 (2) may be rewritten:

$$\log Y = \alpha_1 + (\alpha_2 + 1)\log X + u$$

so it is a reparameterised version of (1) with $\beta_1 = \alpha_1$ and $\beta_2 = \alpha_2 + 1$.

A6.9 *Show that the specification of Researcher 2 is a restricted version of the specification of Researcher 1, stating the restriction.*

Let the model be written:

$$LGE = \beta_1 + \beta_2 S + \beta_3 NUM + \beta_4 VERB + u.$$

The restriction is $\beta_4 = \beta_3$ since NUM and $VERB$ are given equal weights in the construction of $SCORE$. Using the restriction, the model can be rewritten

$$\begin{aligned} LGE &= \beta_1 + \beta_2 S + \beta_3 (NUM + VERB) + u \\ &= \beta_1 + \beta_2 S + 2\beta_3 SCORE + u. \end{aligned}$$

Perform an F test of the restriction.

The null and alternative hypotheses are $H_0 : \beta_4 = \beta_3$ and $H_1 : \beta_4 \neq \beta_3$. The F statistic is:

$$F(1, 100) = \frac{(2045 - 2000)/1}{2000/100} = 2.25.$$

The critical value of $F(1, 100)$ is 3.94 at the 5 per cent level. Hence we do not reject the restriction at the 5 per cent level.

Show that the specification of Researcher 3 is a reparameterised version of the specification of Researcher 1 and hence perform a t test of the restriction in the specification of Researcher 2.

The restriction may be rewritten $\beta_4 - \beta_3 = 0$. The test statistic is therefore $\theta = \beta_4 - \beta_3$. Hence $\beta_4 = \theta + \beta_3$. Substituting for β_4 in the unrestricted model, one has:

$$\begin{aligned} LGE &= \beta_1 + \beta_2 S + \beta_3 NUM + (\theta + \beta_3) VERB + u \\ &= \beta_1 + \beta_2 S + \beta_3 (NUM + VERB) + \theta VERB + u \\ &= \beta_1 + \beta_2 S + 2\beta_3 SCORE + \theta VERB + u. \end{aligned}$$

This is the specification of Researcher 3. To test the hypothesis that the restriction is valid, we perform a t test on the coefficient of $VERB$. The t statistic is -1.5, so we do not reject the restriction at the 5 per cent level.

Explain whether the F test in (b) and the t test in (c) could have led to different results.

No, the F test and the t test must give the same result because the F statistic must be the square of the t statistic and the critical value of F must be the square of the critical value of t for any given significance level. Note that this assumes a two-sided t test. If one is in a position to perform a one-sided test, the t test would be more powerful.

Perform a test of the hypothesis that the numeracy score has a greater effect on earnings than the literacy score.

One should perform a one-sided t test on the coefficient of $VERB$ in regression 3 with the null hypothesis $H_0 : \theta = 0$ and the alternative hypothesis $H_1 : \theta < 0$. The null hypothesis is not rejected and hence one concludes that there is no significant difference.

Compare the regression results of the three researchers.

The regression results of Researchers 1 and 3 are equivalent, the only difference being that the coefficient of $VERB$ provides a direct estimate of β_4 in the specification of Researcher 1 and $(\beta_4 - \beta_3)$ in the specification of Researcher 3. Assuming the restriction is valid, there is a large gain in efficiency in the estimation of β_3 in specification (2) because its standard error is effectively 0.0005, as opposed to 0.0011 in specifications (1) and (3).

A6.10 *Demonstrate that relationship (3) embodies a testable restriction and show how the model may be reformulated to take advantage of it.*

The coefficients of $\log \rho$ and $\log w$ sum to 1. Hence the model should be reformulated as:

$$\log L = \frac{1}{\alpha + \beta} \log Q + \frac{\alpha}{\alpha + \beta} \log \frac{\rho}{w} \tag{4}$$

(plus a disturbance term).

Explain how the restriction could be tested using an F test.

Let RSS_U and RSS_R be the residual sums of squares from the unrestricted and restricted regressions. To test the null hypothesis that the coefficients of $\log \rho$ and $\log w$ sum to 1, one should calculate the F statistic:

$$F(1, 27) = \frac{(RSS_R - RSS_U)/1}{RSS_U/27}$$

and compare it with the critical values of $F(1, 27)$.

Explain how the restriction could be tested using a t test.

Alternatively, writing (3) as an unrestricted model:

$$\log wL = \gamma_1 \log Q + \gamma_2 \log \rho + \gamma_3 \log w + u \tag{5}$$

the restriction is $\gamma_2 + \gamma_3 - 1 = 0$. Define $\theta = \gamma_2 + \gamma_3 - 1$. Then $\gamma_3 = \theta - \gamma_2 + 1$ and the unrestricted model may be rewritten as:

$$\log wL = \gamma_1 \log Q + \gamma_2 \log \rho + (\theta - \gamma_2 + 1) \log w + u.$$

Hence:

$$\log wL - \log w = \gamma_1 \log Q + \gamma_2 (\log \rho - \log w) + \theta \log w + u.$$

Hence:

$$\log L = \gamma_1 \log Q + \gamma_2 \log \frac{\rho}{w} + \theta \log w + u.$$

Thus one should regress $\log L$ on $\log Q$, $\log(\rho/w)$, and $\log w$ and perform a t test on the coefficient of $\log w$.

Explain the theoretical benefits of making use of a valid restriction. How could the researcher assess whether there are any benefits in practice, in this case?

The main theoretical benefit of making use of a valid restriction is that one obtains more efficient estimates of the coefficients. The use of a restriction would eliminate the problem of duplicate estimates of the same parameter. Reduced standard errors should provide evidence of the gain in efficiency.

At a seminar, someone suggests that it is reasonable to hypothesise that manufacturing output is subject to constant returns to scale, so that $\alpha + \beta = 1$. Explain how the researcher could test this hypothesis (1) using an F test, (2) using a t test.

Under the assumption of constant returns to scale, the model becomes:

$$\log \frac{L}{Q} = \alpha \log \frac{\rho}{w}. \qquad (6)$$

One could test the hypothesis by computing the F statistic:

$$F(1, 28) = \frac{(RSS_R - RSS_U)/1}{RSS_U/28}$$

where RSS_U and RSS_R are for the specifications in (4) and (6) respectively.

Alternatively, one could perform a simple t test of the hypothesis that the coefficient of $\log Q$ in (4) is equal to 1.

A6.11 *Explain why the researcher was dissatisfied with the results of regression (1).*

The high correlation between I and P has given rise to a problem of multicollinearity. The standard errors are relatively large and the t statistics low.

Demonstrate that specification (2) may be considered to be a restricted version of specification (1).

The restriction is $\beta_3 = -\beta_2$. Imposing it, we have:

$$\begin{aligned} B &= \beta_1 + \beta_2 I + \beta_3 P + u \\ &= \beta_1 + \beta_2 I - \beta_2 P + u \\ &= \beta_1 + \beta_2 R + u. \end{aligned}$$

Perform an F test of the restriction, stating carefully your null hypothesis and conclusion.

The null hypothesis is $H_0 : \beta_3 = -\beta_2$. The test statistic is:

$$F(1, 37) = \frac{(987.1 - 967.9)/1}{967.9/37} = 0.73.$$

The null hypothesis is not rejected at any significance level since $F < 1$.

Perform a t test of the restriction

The unrestricted specification may be rewritten:

$$\begin{aligned} B &= \beta_1 + \beta_2 I + \beta_3 P + u \\ &= \beta_1 + \beta_2 (P + R) + \beta_3 P + u \\ &= \beta_1 + (\beta_2 + \beta_3) P + \beta_2 R + u. \end{aligned}$$

Thus a t test on the coefficient of P in this specification is a test of the restriction. The null hypothesis is not rejected, given that the t statistic is 0.86. Of course, the F statistic is the square of the t statistic and the tests are equivalent.

Demonstrate that specification (3) may also be considered to be a restricted version of specification (1)

The restriction is $\beta_3 = 0$.

Perform both an F test and a t test of the restriction in specification (3), stating your conclusion in each case.

$$F(1, 37) = \frac{(1024.3 - 967.9)/1}{967.9/37} = 2.16.$$

The critical value of $F(1, 37)$ at 5 per cent is approximately 4.08, so the null hypothesis that P does not influence B is not rejected. Of course, with $t = -1.47$, the t test, which is equivalent, leads to the same conclusion.

At a seminar, someone suggests that specification (4) is also a restricted version of specification (1). Is this correct? If so, state the restriction.

No, it is not correct. As shown above, it is an alternative form of the unrestricted specification.

State, with an explanation, which would be your preferred specification.

None of the specifications has been rejected. The second should be preferred because it should be more efficient than the unrestricted specification. The much lower standard error of the slope coefficient provides supportive evidence. The third specification should be eliminated on the grounds that price inflation ought to be a determinant.

A6.12 Write the original model:

$$Y = \beta_1 + \beta_2 X + \beta_3 Z + u. \qquad (1)$$

Then, with:

$$X = 0.5(V + W), \qquad Z = 0.5(V - W)$$

the other specifications are:

$$\begin{aligned} Y &= \beta_1 + 0.5(\beta_2 + \beta_3)V + 0.5(\beta_2 - \beta_3)W + u \qquad (2) \\ Y &= \beta_1 + \beta_2 V + u \qquad (3) \end{aligned}$$

with the implicit restriction $\beta_3 = \beta_2$, and, using $X = V - Z$:

$$Y = \beta_1 + \beta_2 V + (\beta_3 - \beta_2)Z + u. \qquad (4)$$

(2) and (4) are reparameterisations of (1), so the measures of fit are unchanged: **E** = **L** = 0.60, **F** = **M** = 200.

Given the relationships among the parameters, **A** = 0.70, **C** = −0.10, **J** = 0.60, **H** = 0.20.

The standard errors **B** and **D** cannot be reconstructed because the standard errors of $\widehat{\beta}_2$ and $\widehat{\beta}_3$ cannot be used (on their own) to construct standard errors of linear combinations (a loose explanation is acceptable because we have hardly touched on covariances between estimators).

K = 0.04 since **J** = coefficient of X in specification (1).

The F statistic for the restriction $\beta_3 = \beta_2$ implicit in specification (3) is:

$$F(1,\,40) = \frac{(220 - 200)/1}{200/40} = 4.0.$$

In terms of R^2 it would be:

$$F(1,\,40) = \frac{(0.60 - G)/1}{0.40/40}.$$

Hence **G** = 0.56.

A two-sided t test on the coefficient of Z in specification (4) provides an equivalent test of the restriction. The t statistic must therefore be $\sqrt{4.0} = 2.0$ and so **I** = 0.10.

[Note: One may also compute **G** using the t statistic for the coefficient of V in specification (3):

$$\frac{G}{(1 - G)/41} = t^2.$$

Yet another was of computing **G** is as follows. Since R^2 in specification (1) is 0.60, *TSS* must be 500, using:

$$R^2 = 1 - \frac{RSS}{TSS}.$$

TSS is the same in specification (3). Hence one obtains **G** = 0.56.]

A6.13 F statistics should always be computed using *RSS*, not R^2. Often the R^2 version is equivalent, but often it is not, and this is a case in point. The reason is very simple: the dependent variables in the two specifications are different, and so the R^2 for the specifications are not comparable. The *RSS* are comparable because:

$$\begin{aligned} LGFDHOPC - LGFDHOPC &= (LGFDHO - LGSIZE) - (L\widehat{GFDH}O - LGSIZE) \\ &= LGFDHO - L\widehat{GFDH}O. \end{aligned}$$

Chapter 7
Heteroskedasticity

7.1 Overview

This chapter begins with a general discussion of homoskedasticity and heteroskedasticity: the meanings of the terms, the reasons why the distribution of a disturbance term may be subject to heteroskedasticity, and the consequences of the problem for OLS estimators. It continues by presenting several tests for heteroskedasticity and methods of alleviating the problem. It shows how apparent heteroskedasticty may be caused by model misspecification. It concludes with a description of the use of heteroskedasticity-consistent standard errors.

7.2 Learning outcomes

After working through the corresponding chapter in the text, studying the corresponding slideshows, and doing the starred exercises in the text and the additional exercises in this subject guide, you should be able to:

- explain the concepts of homoskedasticity and heteroskedasticity
- describe how the problem of heteroskedasticity may arise
- explain the consequences of heteroskedasticity for OLS estimators, their standard errors, and t and F tests
- perform the Goldfeld–Quandt test for heteroskedasticity
- perform the White test for heteroskedasticity
- explain how the problem of heteroskedasticity may be alleviated
- explain why a mathematical misspecification of the regression model may give rise to a problem of apparent heteroskedasticity
- explain the use of heteroskedasticity-consistent standard errors.

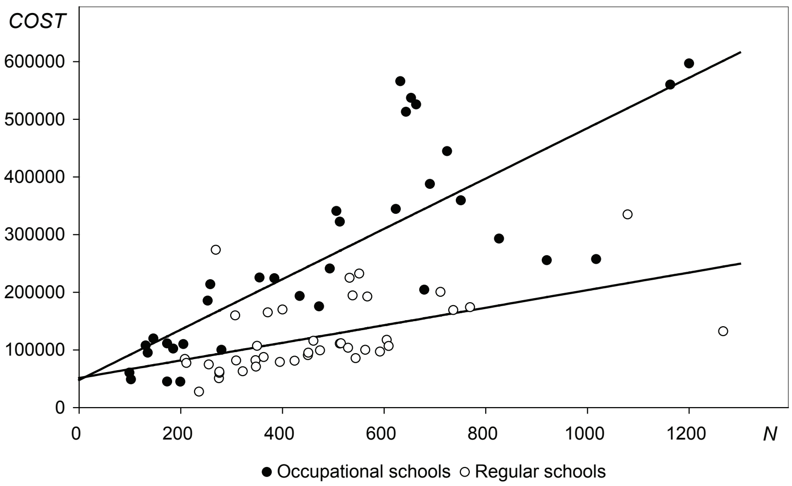
7.3 Additional exercises

A7.1 Is the disturbance term in your *CES* expenditure function heteroskedastic?

Sort the data by *EXPPC*. Excluding observations for which *EXPPC* is zero, regress *CATPC* on *EXPPC* and *SIZE* (a) for the first three-eighths of the non-zero observations, and (b) for the last three-eighths. Perform a Goldfeld–Quandt test to test for heteroskedasticity in the *EXPPC* dimension. Repeat using *LGCATPC* as the dependent variable and regressing it on *LGEXPPC* and *LGSIZE*.

A7.2 Repeat Exercise A7.1, using a White test instead of a Goldfeld–Quandt test.

A7.3 The observations for the occupational schools (see Chapter 5 in the text) in the figure suggest that a simple linear regression of cost on number of students, restricted to the subsample of these schools, would be subject to heteroskedasticity. Download the data set from the Online Resource Centre and use a Goldfeld–Quandt test to investigate whether this is the case. If the relationship is heteroskedastic, what could be done to alleviate the problem?



A7.4 A researcher hypothesises that larger economies should be more self-sufficient than smaller ones and that M/G, the ratio of imports, M, to gross domestic product, G, should be negatively related to G:

$$\frac{M}{G} = \beta_1 + \beta_2 G + u$$

with $\beta_2 < 0$. Using data for a sample of 42 countries, with M and G both measured in US$ billion, he fits the regression (standard errors in parentheses):

$$\begin{aligned}\widehat{\frac{M}{G}} &= \underset{(0.03)}{0.37} - \underset{(0.000036)}{0.000086}G \qquad R^2 = 0.12 \qquad (1)\end{aligned}$$

He plots a scatter diagram, reproduced as Figure 7.1, and notices that the ratio M/G tends to have relatively high variance when G is small. He also plots a scatter diagram for M and G, reproduced as Figure 7.2. Defining GSQ as the square of G, he regresses M on G and GSQ:

$$\widehat{M} = \underset{(10.77)}{7.27} + \underset{(0.03)}{0.30}G - \underset{(0.000009)}{0.000049}GSQ \qquad R^2 = 0.86 \qquad (2)$$

Finally, he plots a scatter diagram for $\log M$ and $\log G$, reproduced as Figure 7.3, and regresses $\log M$ on $\log G$:

$$\widehat{\log M} = -0.14 + 0.80 \log G \qquad R^2 = 0.78 \qquad (3)$$
$$\qquad (0.37) \quad (0.07)$$

Having sorted the data by G, he tests for heteroskedasticity by regressing specifications (1) – (3) first for the 16 countries with smallest G, and then for the 16 countries with the greatest G. RSS_1 and RSS_2, the residual sums of squares for these regressions, are summarised in the following table.

Specification	RSS_1	RSS_2
(1)	0.53	0.21
(2)	3178	71404
(3)	3.45	3.60

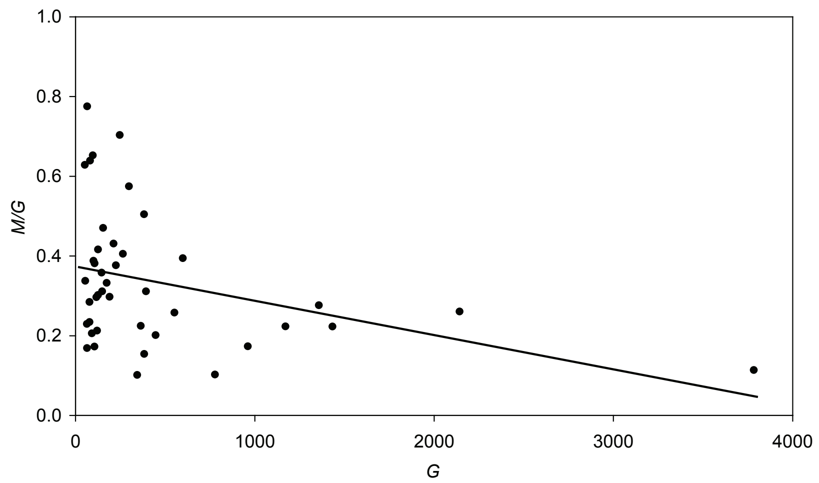


Figure 7.1: Scatter diagram of M/G against G.

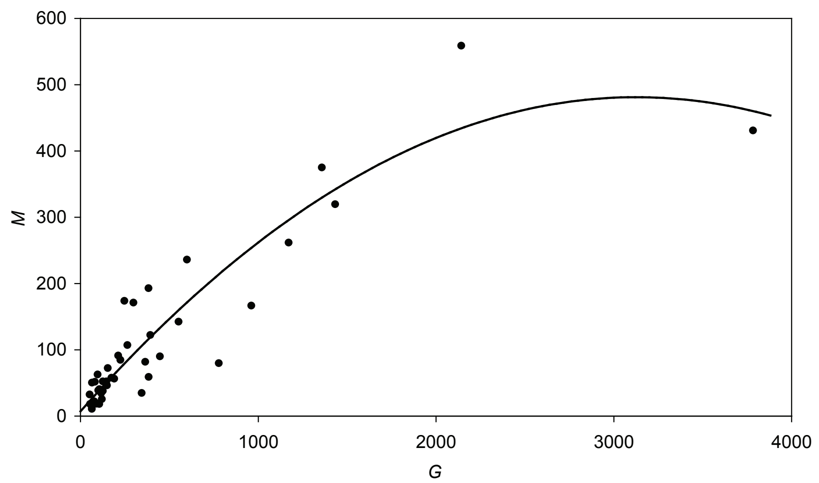


Figure 7.2: Scatter diagram of M against G.

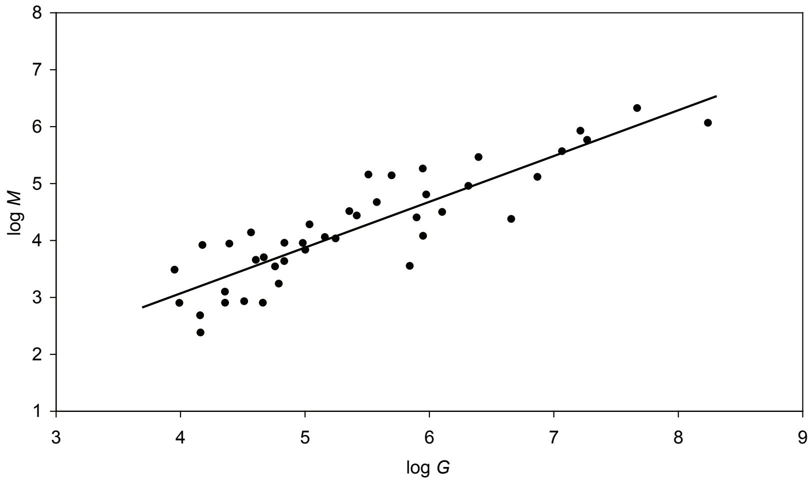


Figure 7.3: Scatter diagram of $\log M$ against $\log G$.

- Discuss whether (1) appears to be an acceptable specification, given the data in the table and Figure 7.1.
- Explain what the researcher hoped to achieve by running regression (2).
- Discuss whether (2) appears to be an acceptable specification, given the data in the table and Figure 7.2.
- Explain what the researcher hoped to achieve by running regression (3).
- Discuss whether (3) appears to be an acceptable specification, given the data in the table and Figure 7.3.
- What are your conclusions concerning the researcher's hypothesis?

A7.5 A researcher has data on the number of children attending, N, and annual recurrent expenditure, *EXP*, measured in US$, for 50 nursery schools in a US city for 2006 and hypothesises that the cost function is of the quadratic form:

$$EXP = \beta_1 + \beta_2 N + \beta_3 NSQ + u$$

where *NSQ* is the square of N, anticipating that economies of scale will cause β_3 to be negative. He fits the following equation:

$$\begin{aligned} \widehat{EXP} &= \underset{(12908)}{17999} + \underset{(133)}{1060}N - \underset{(0.30)}{1.29}NSQ \qquad R^2 = 0.74 \qquad (1) \end{aligned}$$

Suspecting that the regression was subject to heteroskedasticity, the researcher runs the regression twice more, first with the 19 schools with lowest enrolments, then with the 19 schools with the highest enrolments. The residual sums of squares in the two regressions are 8.0 million and 64.0 million, respectively.

The researcher defines a new variable, *EXPN*, expenditure per student, as *EXPN* = EXP/N, and fits the equation:

$$\begin{aligned} \widehat{EXPN} &= \underset{(90)}{1080} - \underset{(0.25)}{1.25}N + \underset{(6000)}{16114}NREC \qquad R^2 = 0.65 \qquad (2) \end{aligned}$$

where $NREC = 1/N$. He again runs regressions with the 19 smallest schools and the 19 largest schools and the residual sums of squares are 900,000 and 600,000.

- Perform a Goldfeld–Quandt test for heteroskedasticity on both of the regression specifications.
- Explain why the researcher ran the second regression.
- R^2 is lower in regression (2) than in regression (1). Does this mean that regression (1) is preferable?

A7.6 This is a continuation of Exercise A6.5.

- When the researcher presents her results at a seminar, one of the participants says that, since I and G have been divided by Y, (2) is less likely to be subject to heteroskedasticity than (1). Evaluate this suggestion.

A7.7 A researcher has data on annual household expenditure on food, F, and total annual household expenditure, E, both measured in dollars, for 400 households in the United States for 2010. The scatter plot for the data is shown as Figure 7.4. The basic model of the researcher is:

$$F = \beta_1 + \beta_2 E + u \qquad (1)$$

where u is a disturbance term. The researcher suspects heteroskedasticity and performs a Goldfeld–Quandt test and a White test. For the Goldfeld–Quandt test, she sorts the data by size of E and fits the model for the subsample with the 150 smallest values of E and for the subsample with the 150 largest values. The residual sums of squares (RSS) for these regressions are shown in column (1) of the table. She also fits the regression for the entire sample, saves the residuals, and then fits an auxiliary regression of the squared residuals on E and its square. R^2 for this regression is also shown in column (1) in the table. She performs parallel tests of heteroskedasticity for two alternative models:

$$\begin{aligned} \frac{F}{A} &= \beta_1 \frac{1}{A} + \beta_2 \frac{E}{A} + v \qquad (2) \\ \log F &= \beta_1 + \beta_2 \log E + w. \qquad (3) \end{aligned}$$

A is household size in terms of equivalent adults, giving each adult a weight of 1 and each child a weight of 0.7. The scatter plot for F/A and E/A is shown as Figure 7.5, and that for $\log F$ and $\log E$ as Figure 7.6. The data for the heteroskedasticity tests for models (2) and (3) are shown in columns (2) and (3) of the table.

Specification	(1)	(2)	(3)
Goldfeld–Quandt test			
RSS smallest 150	200 million	40 million	20.0
RSS largest 150	820 million	240 million	21.0
White test			
R^2 from auxiliary regression	0.160	0.140	0.001

- Perform the Goldfeld–Quandt test for each model and state your conclusions.

- Explain why the researcher thought that model (2) might be an improvement on model (1).
- Explain why the researcher thought that model (3) might be an improvement on model (1).
- When models (2) and (3) are tested for heteroskedasticity using the White test, auxiliary regressions must be fitted. State the specification of this auxiliary regression for model (2).
- Perform the White test for the three models.
- Explain whether the results of the tests seem reasonable, given the scatter plots of the data.

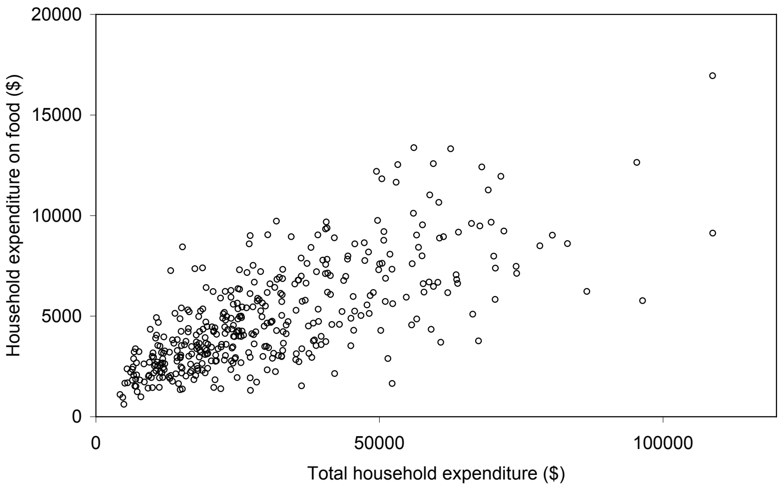


Figure 7.4: Scatter diagram of household expenditure on food against total household expenditure.

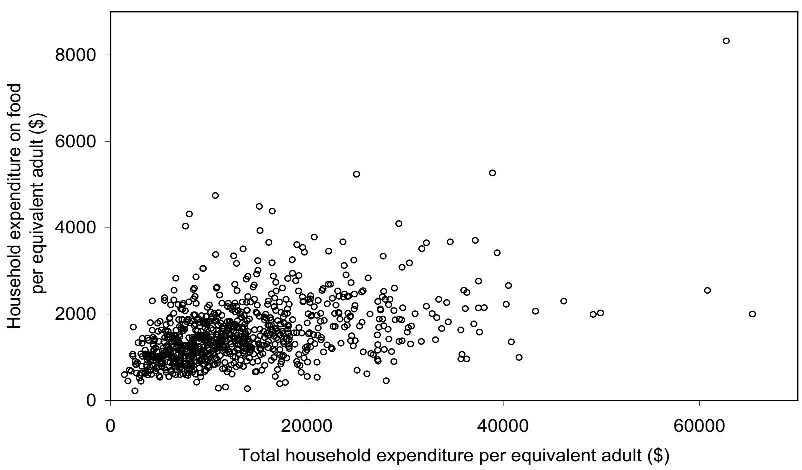


Figure 7.5: Scatter diagram of household expenditure on food per equivalent adult against total household expenditure per equivalent adult.

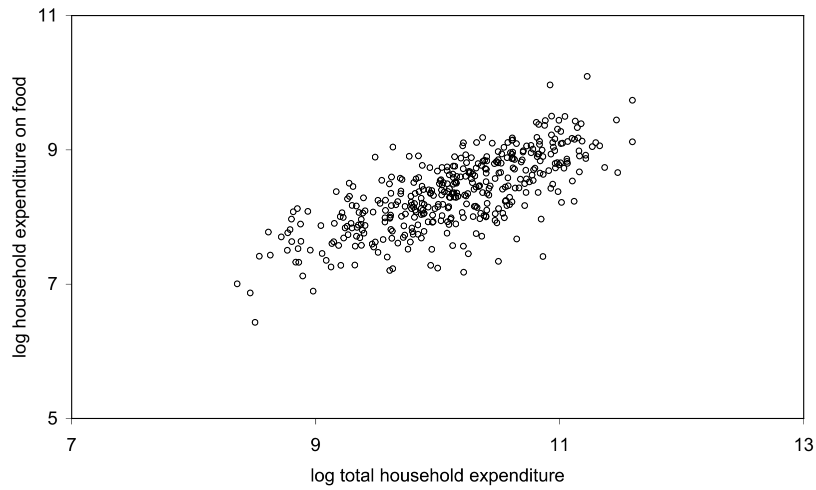


Figure 7.6: Scatter diagram of log household expenditure on food against log total household expenditure.

A7.8 Explain what is correct, mistaken, confused or in need of further explanation in the following statements relating to heteroskedasticity in a regression model:

- 'Heteroskedasticity occurs when the disturbance term in a regression model is correlated with one of the explanatory variables.'
- 'In the presence of heteroskedasticity ordinary least squares (OLS) is an inefficient estimation technique and this causes t tests and F tests to be invalid.'
- 'OLS remains unbiased but it is inconsistent.
- 'Heteroskedasticity can be detected with a Chow test.'
- 'Alternatively one can compare the residuals from a regression using half of the observations with those from a regression using the other half and see if there is a significant difference. The test statistic is the same as for the Chow test.'
- 'One way of eliminating the problem is to make use of a restriction involving the variable correlated with the disturbance term.'
- 'If you can find another variable related to the one responsible for the heteroskedasticity, you can use it as a proxy and this should eliminate the problem.'
- 'Sometimes apparent heteroskedasticity can be caused by a mathematical misspecification of the regression model. This can happen, for example, if the dependent variable ought to be logarithmic, but a linear regression is run.'

7.4 Answers to the starred exercises in the textbook

7.5 The following regressions were fitted using the Shanghai school cost data introduced in Section 6.1 (standard errors in parentheses):

$$\begin{aligned}
\widehat{COST} &= \underset{(27000)}{24000} + \underset{(50)}{339N} \qquad R^2 = 0.39 \\
\widehat{COST} &= \underset{(31000)}{51000} - \underset{(41000)}{4000OCC} + \underset{(60)}{152N} + \underset{(76)}{284NOCC} \qquad R^2 = 0.68.
\end{aligned}$$

where $COST$ is the annual cost of running a school, N is the number of students, OCC is a dummy variable defined to be 0 for regular schools and 1 for occupational schools, and $NOCC$ is a slope dummy variable defined as the product of N and OCC. There are 74 schools in the sample. With the data sorted by N, the regressions are fitted again for the 26 smallest and 26 largest schools, the residual sums of squares being as shown in the table.

	26 smallest	26 largest
First regression	7.8×10^{10}	54.4×10^{10}
Second regression	6.7×10^{10}	13.8×10^{10}

Perform a Goldfeld–Quandt test for heteroskedasticity for the two models and, with reference to Figure 6.5, explain why the problem of heteroskedasticity is less severe in the second model.

Answer:

For both regressions RSS will be denoted RSS_1 for the 26 smallest schools and RSS_2 for the 26 largest schools. In the first regression, $RSS_2/RSS_1 = (54.4 \times 10^{10})/(7.8 \times 10^{10}) = 6.97$. There are 24 degrees of freedom in each subsample (26 observations, 2 parameters estimated). The critical value of $F(24, 24)$ is approximately 3.7 at the 0.1 per cent level, and so we reject the null hypothesis of homoskedasticity at that level. In the second regression, $RSS_2/RSS_1 = (13.8 \times 10^{10})/(6.7 \times 10^{10}) = 2.06$. There are 22 degrees of freedom in each subsample (26 observations, 4 parameters estimated). The critical value of $F(22, 22)$ is 2.05 at the 5 per cent level, and so we (just) do not reject the null hypothesis of homoskedasticity at that significance level.

Why is the problem of heteroskedasticity less severe in the second regression? The figure in Exercise A7.2 reveals that the cost function is much steeper for the occupational schools than for the regular schools, reflecting their higher marginal cost. As a consequence the two sets of observations diverge as the number of students increases and the scatter is bound to appear heteroskedastic, irrespective of whether the disturbance term is truly heteroskedastic or not. The first regression takes no account of this and the Goldfeld–Quandt test therefore indicates significant heteroskedasticity. In the second regression the problem of apparent heteroskedasticity does not arise because the intercept and slope dummy variables allow separate implicit regression lines for the two types of school.

Looking closely at the diagram, the observations for the occupational schools exhibit a classic pattern of true heteroskedasticity, and this would be confirmed by a Goldfeld–Quandt test confined to the subsample of those schools (see Exercise A7.2). However the observations for the regular schools appear to be homoskedastic and this accounts for the fact that we did not (quite) reject the null hypothesis of homoskedasticity for the combined sample.

7.6 The file educ.dta on the website contains contains international cross-sectional data on aggregate expenditure on education, $EDUC$, gross domestic product, GDP, and population, POP, for a sample of 38 countries in 1997. $EDUC$ and GDP are measured in US$ million and POP is measured in thousands. Download the data set, plot a scatter diagram of $EDUC$ on GDP, and comment on whether the data set appears to be subject to heteroskedasticity. Sort the data set by GDP and perform a Goldfeld–Quandt test for heteroskedasticity, running regressions using the subsamples of 14 countries with the smallest and greatest GDP.

Answer:

The figure plots expenditure on education, $EDUC$, and gross domestic product, GDP, for the 38 countries in the sample, measured in $ billion rather than $ million. The observations exhibit heteroskedasticity. Sorting them by GDP and regressing $EDUC$ on GDP for the subsamples of 14 countries with smallest and greatest GDP, the residual sums of squares for the first and second subsamples, denoted RSS_1 and RSS_2, respectively, are 1,660,000 and 63,113,000, respectively. Hence:

$$F(12, 12) = \frac{RSS_2}{RSS_1} = \frac{63113000}{1660000} = 38.02.$$

The critical value of $F(12, 12)$ at the 0.1 per cent level is 7.00, and so we reject the null hypothesis of homoskedasticity.

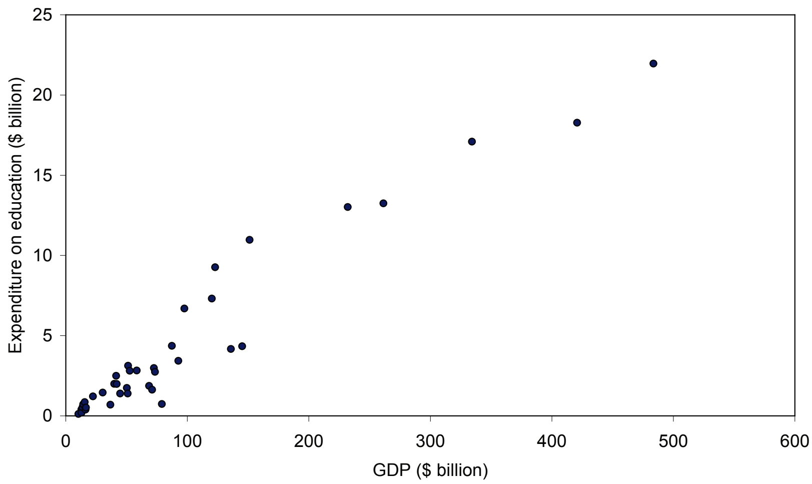


Figure 7.7: Expenditure on education and GDP ($ billion).

7.9 Repeat Exercise 7.6, using the Goldfeld–Quandt test to investigate whether scaling by population or by GDP, or whether running the regression in logarithmic form, would eliminate the heteroskedasticity. Compare the results of regressions using the entire sample and the alternative specifications.

Answer:

Dividing through by population, *POP*, the model becomes:

$$\frac{EDUC}{POP} = \beta_1 \frac{1}{POP} + \beta_2 \frac{GDP}{POP} + \frac{u}{POP}$$

with expenditure on education per capita, denoted *EDUCPOP*, hypothesised to be a function of gross domestic product per capita, *GDPPOP*, and the reciprocal of population, *POPREC*, with no intercept. Sorting the sample by *GDPPOP* and running the regression for the subsamples of 14 countries with smallest and largest *GDPPOP*, $RSS_1 = 0.006788$ and $RSS_2 = 1.415516$. Now:

$$F(12, 12) = \frac{RSS_2}{RSS_1} = \frac{1.415516}{0.006788} = 208.5.$$

Thus the model is still subject to heteroskedasticity at the 0.1 per cent level. This is evident in Figure 7.8.

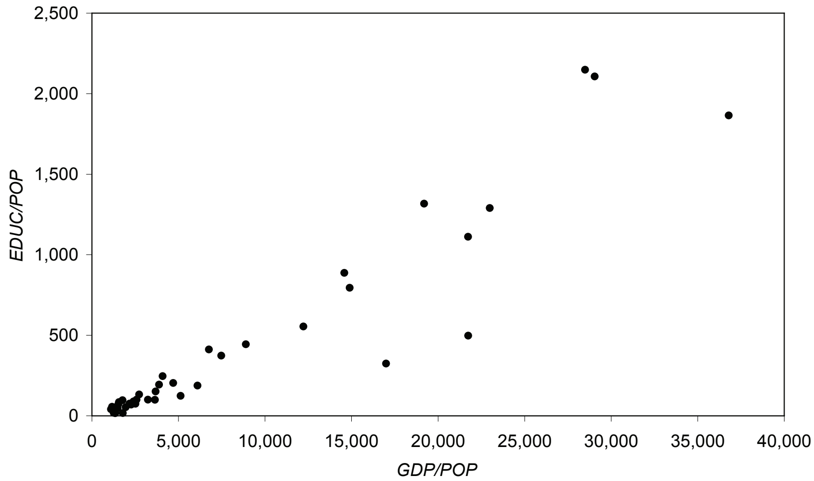


Figure 7.8: Expenditure on education per capita and GDP per capita ($ per capita).

Dividing through instead by *GDP*, the model becomes:

$$\frac{EDUC}{GDP} = \beta_1 \frac{1}{GDP} + \beta_2 + \frac{u}{GDP}$$

with expenditure on education as a share of gross domestic product, denoted *EDUCGDP*, hypothesised to be a simple function of the reciprocal of gross domestic product, *GDPREC*, with no intercept. Sorting the sample by *GDPREC* and running the regression for the subsamples of 14 countries with smallest and largest *GDPREC*, $RSS_1 = 0.00413$ and $RSS_2 = 0.00238$. Since RSS_2 is less than RSS_1, we test for heteroskedasticity under the hypothesis that the standard deviation of the disturbance term is inversely related to *GDPREC*:

$$F(12, 12) = \frac{RSS_1}{RSS_2} = \frac{0.00413}{0.00238} = 1.74.$$

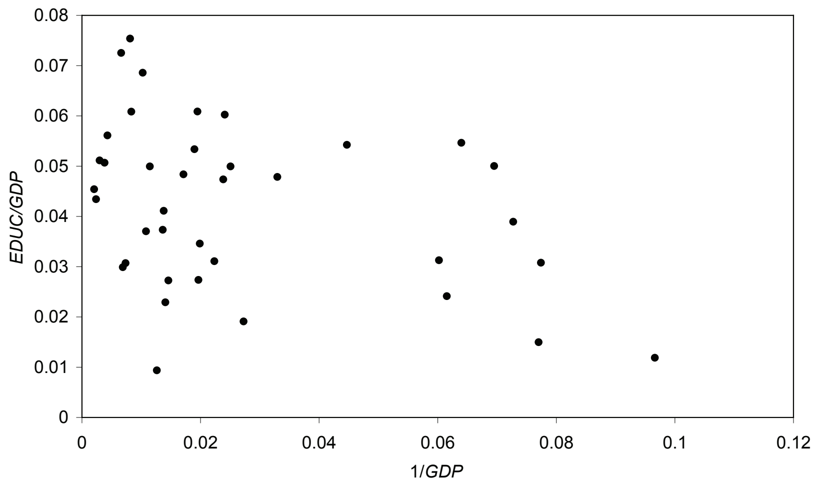


Figure 7.9: Expenditure on education as a proportion of GDP and the reciprocal of GDP (measured in \$ billion).

The critical value of $F(12, 12)$ at the 5 per cent level is 2.69, so we do not reject the null hypothesis of homoskedasticity. Could one tell this from Figure 7.9? It is a little difficult to say.

Finally, we will consider a logarithmic specification. If the true relationship is logarithmic, and homoskedastic, it would not be surprising that the linear model appeared heteroskedastic. Sorting the sample by GDP, RSS_1 and RSS_2 are 2.733 and 3.438 for the subsamples of 14 countries with smallest and greatest GDP. The F statistic is:

$$F(12, 12) = \frac{RSS_1}{RSS_2} = \frac{3.438}{2.733} = 1.26.$$

Thus again we would not reject the null hypothesis of homoskedasticity.

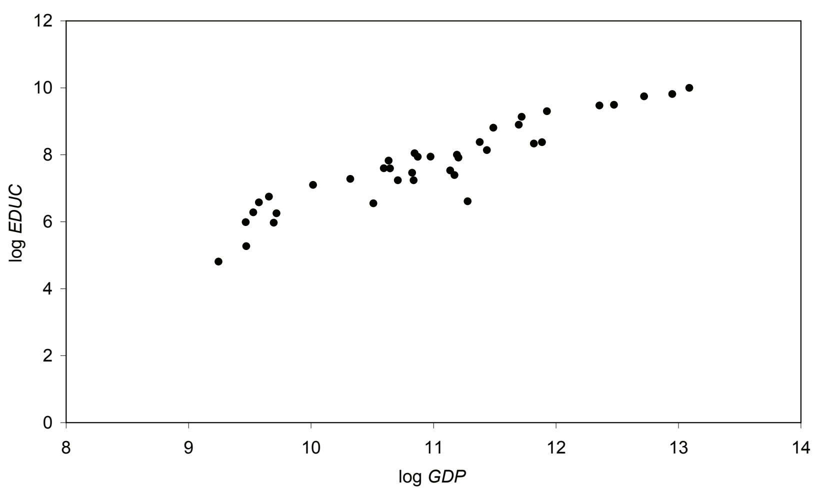


Figure 7.10: Expenditure on education and GDP, logarithmic.

The third and fourth models both appear to be free from heteroskedasticity. How do we choose between them? We will examine the regression results, shown for the two models with the full sample:

```
. reg EDUCGDP GDPREC

  Source |       SS       df       MS                  Number of obs =      38
---------+------------------------------               F(  1,  36)   =    5.62
   Model |  .001348142     1  .001348142               Prob > F      =  0.0233
Residual |  .008643037    36  .000240084               R-squared     =  0.1349
---------+------------------------------               Adj R-squared =  0.1109
   Total |  .009991179    37  .000270032               Root MSE      =  .01549

------------------------------------------------------------------------------
 EDUCGDP |      Coef.   Std. Err.       t     P>|t|       [95% Conf. Interval]
---------+--------------------------------------------------------------------
  GDPREC |  -234.0823   98.78309     -2.370   0.023      -434.4236   -33.74086
   _cons |   .0484593   .0036696     13.205   0.000       .0410169    .0559016
------------------------------------------------------------------------------
```

```
. reg LGEE LGGDP

  Source |       SS       df       MS                  Number of obs =      38
---------+------------------------------               F(  1,    36) =  246.20
   Model |  51.9905508     1  51.9905508               Prob > F      =  0.0000
Residual |   7.6023197    36  .211175547               R-squared     =  0.8724
---------+------------------------------               Adj R-squared =  0.8689
   Total |  59.5928705    37  1.61061812               Root MSE      =  .45954

------------------------------------------------------------------------------
    LGEE |      Coef.   Std. Err.       t     P>|t|       [95% Conf. Interval]
---------+--------------------------------------------------------------------
   LGGDP |   1.160594   .0739673     15.691   0.000       1.010582    1.310607
   _cons |  -5.025204   .8152239     -6.164   0.000      -6.678554   -3.371853
------------------------------------------------------------------------------
```

In equation form, the first regression is:

$$\begin{aligned}\frac{\widehat{EDUC}}{GDP} &= \underset{(0.004)}{0.048} - \underset{(98.8)}{234.1}\frac{1}{GDP} \qquad R^2 = 0.13\end{aligned}$$

Multiplying through by GDP, it may be rewritten:

$$\widehat{EDUC} = -234.1 + 0.048GDP.$$

It implies that expenditure on education accounts for 4.8 per cent of gross domestic product at the margin. The constant does not have any sensible interpretation. We will compare this with the output from an OLS regression that makes no attempt to eliminate heteroskedasticity:

```
. reg EDUC GDP

  Source |       SS       df       MS                  Number of obs =      38
---------+------------------------------               F(  1,    36) =  509.80
   Model |  1.0571e+09     1  1.0571e+09               Prob > F      =  0.0000
Residual |  74645819.2    36  2073494.98               R-squared     =  0.9340
---------+------------------------------               Adj R-squared =  0.9322
   Total |  1.1317e+09    37  30586911.0               Root MSE      =  1440.0

------------------------------------------------------------------------------
    EDUC |      Coef.   Std. Err.       t     P>|t|       [95% Conf. Interval]
---------+--------------------------------------------------------------------
     GDP |   .0480656   .0021288     22.579   0.000       .0437482     .052383
   _cons |  -160.4669    311.699     -0.515   0.610      -792.6219     471.688
------------------------------------------------------------------------------
```

The slope coefficient, 0.048, is identical to three decimal places. This is not entirely a surprise, since heteroskedasticity does not give rise to bias and so there should be no systematic difference between the estimate from an OLS regression and that from a specification that eliminates heteroskedasticity. Of course, it is a surprise that the estimates are so close. Generally there would be some random difference, and of course the OLS estimate would tend to be less accurate. In this case, the main difference is in the estimated standard error. That for the OLS regression is actually smaller than that for the regression of *EDUCGDP* on *GDPREC*, but it is misleading. It is incorrectly calculated and we know that, since OLS is inefficient, the true standard error for the OLS estimate is actually larger.

The logarithmic regression in equation form is:

$$\widehat{\log EDUC} = \underset{(0.82)}{-5.03} + \underset{(0.07)}{1.16} \log GDP \qquad R^2 = 0.87$$

implying that the elasticity of expenditure on education with regard to gross domestic product is 1.16. In substance the interpretations of the models are similar, since both imply that the proportion of GDP allocated to education increases slowly with GDP, but the elasticity specification seems a little more informative and probably serves as a better starting point for further exploration. For example, it would be natural to add the logarithm of population to see if population had an independent effect.

7.10 It was reported above that the heteroskedasticity-consistent estimate of the standard error of the coefficient of *GDP* in equation (7.18) was 0.18. Explain why the corresponding standard error in equation (7.20) ought to be lower and comment on the fact that it is not.

Answer:

(7.20), unlike (7.18) appears to be free from heteroskedasticity and therefore should provide more efficient estimates of the coefficients, reflected in lower standard errors when computed correctly. However the sample may be too small for the heteroskedasticity-consistent estimator to be a good guide.

7.11 A health economist plans to evaluate whether screening patients on arrival or spending extra money on cleaning is more effective in reducing the incidence of infections by the MRSA bacterium in hospitals. She hypothesises the following model:

$$MRSA_i = \beta_1 + \beta_2 S_i + \beta_3 C_i + u_i$$

where, in hospital i, $MRSA$ is the number of infections per thousand patients, S is expenditure per patient on screening, and C is expenditure per patient on cleaning. u_i is a disturbance term that satisfies the usual regression model assumptions. In particular, u_i is drawn from a distribution with mean zero and constant variance σ^2. The researcher would like to fit the relationship using a sample of hospitals. Unfortunately, data for individual hospitals are not available. Instead she has to use regional data to fit:

$$\overline{MRSA_j} = \beta_1 + \beta_2 \overline{S_j} + \beta_3 \overline{C_j} + \overline{u_j}$$

where $\overline{MRSA_j}$, $\overline{S_j}$, $\overline{C_j}$, and $\overline{u_j}$ are the averages of $MRSA$, S, C, and u for the hospitals in region j. There were different numbers of hospitals in the regions, there being n_j hospitals in region j.

Show that the variance of $\overline{u_j}$ is equal to σ^2/n_j and that an OLS regression using the grouped regional data to fit the relationship will be subject to heteroskedasticity.

Assuming that the researcher knows the value of n_j for each region, explain how she could re-specify the regression model to make it homoskedastic. State the revised specification and demonstrate mathematically that it is homoskedastic. Give an intuitive explanation of why the revised specification should tend to produce improved estimates of the parameters.

Answer:

$$\text{var}(\overline{u_j}) = \text{var}\left(\frac{1}{n}\sum_{k=1}^{n_j} u_{jk}\right) = \left(\frac{1}{n_j}\right)^2 \text{var}\left(\sum_{k=1}^{n_j} u_{jk}\right) = \left(\frac{1}{n_j}\right)^2 \sum_{k=1}^{n_j} \text{var}(u_{jk})$$

since the covariance terms are all 0. Hence:

$$\text{var}(\overline{u_j}) = \left(\frac{1}{n_j}\right)^2 n_j \sigma^2 = \frac{\sigma^2}{n_j}.$$

To eliminate the heteroskedasticity, multiply observation j by $\sqrt{n_j}$. The regression becomes:

$$\sqrt{n_j}\overline{MRSA_j} = \beta_1\sqrt{n_j} + \beta_2\sqrt{n_j}\overline{S_j} + \beta_3\sqrt{n_j}\overline{C_j} + \sqrt{n_j}\bar{u}_j.$$

The variance of the disturbance term is now:

$$\text{var}\left(\sqrt{n_j}\overline{u_j}\right) = \left(\sqrt{n_j}\right)^2 \text{var}(\overline{u_j}) = n_j \frac{\sigma^2}{n_j} = \sigma^2$$

and is thus the same for all observations.

From the expression for $\text{var}(\overline{u_j})$, we see that, the larger the group, the more reliable should be its observation (the closer its observation should tend to be to the population relationship). The scaling gives greater weight to the more reliable observations and the resulting estimators should be more efficient.

7.5 Answers to the additional exercises

A7.1 The first step is to drop the zero-observations from the data set and sort it by *EXPPC*. The F statistic is then computed as:

$$F(n_2 - k, n_1 - k) = \frac{RSS_2/(n_2 - k)}{RSS_1/(n_1 - k)}$$

where n_1 and n_2 are the number of available observations and k is the number of parameters in the regression specification.

```
.  drop if FDHO == 0
(0 observations deleted)
. gen EXPPC = EXP/SIZE
. sort EXPPC
. gen LGEXPPC = ln(EXPPC)
. gen LGSIZE = ln(SIZE)
. gen FDHOPC = FDHO/SIZE
. gen LGFDHOPC = ln(FDHOPC)

. reg FDHOPC EXPPC SIZE in 1/2375
-----------------------------------------------------------------------------
    Source |       SS       df       MS              Number of obs =    2375
-----------+------------------------------           F(  2,  2372) =  278.36
     Model |  7382348.18     2  3691174.09           Prob> F       =  0.0000
  Residual |  31453534.1  2372  13260.3432           R-squared     =  0.1901
-----------+------------------------------           Adj R-squared =  0.1894
     Total |  38835882.2  2374  16358.8383           Root MSE      =  115.15
-----------------------------------------------------------------------------
    FDHOPC |      Coef.   Std. Err.       t    P>|t|      [95% Conf. Interval]
-----------+-----------------------------------------------------------------
     EXPPC |   .1107869   .0051862    21.36   0.000      .1006169    .1209569
      SIZE |  -4.462209   1.438899    -3.10   0.002     -7.283838   -1.640579
     _cons |   85.38055   9.590628     8.90   0.000      66.57366    104.1874
-----------------------------------------------------------------------------
```

```
. reg FDHOPC EXPPC SIZE in 3960/6334
-----------------------------------------------------------------------------
    Source |       SS       df       MS              Number of obs =    2375
-----------+------------------------------           F(  2,  2372) =  170.94
     Model |  40643447.8     2  20321723.9           Prob> F       =  0.0000
  Residual |   281980931  2372  118878.976           R-squared     =  0.1260
-----------+------------------------------           Adj R-squared =  0.1252
     Total |   322624379  2374  135899.064           Root MSE      =  344.79
-----------------------------------------------------------------------------
    FDHOPC |      Coef.   Std. Err.       t    P>|t|      [95% Conf. Interval]
-----------+-----------------------------------------------------------------
     EXPPC |   .0286606   .0019716    14.54   0.000      .0247944    .0325268
      SIZE |  -54.33452   7.047302    -7.71   0.000     -68.15403   -40.51501
     _cons |   508.6148   22.37631    22.73   0.000      464.7356    552.4939
-----------------------------------------------------------------------------
```

```
. reg LGFDHOPC LGEXPPC LGSIZE in 1/2375
------------------------------------------------------------------------------
      Source |       SS       df       MS              Number of obs =    2375
-------------+------------------------------           F(  2,  2372) =  369.49
       Model |  207.241064     2  103.620532           Prob> F       =  0.0000
    Residual |  665.204785  2372  .280440466           R-squared     =  0.2375
-------------+------------------------------           Adj R-squared =  0.2369
       Total |  872.445849  2374  .367500357           Root MSE      =  .52957
------------------------------------------------------------------------------
    LGFDHOPC |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
     LGEXPPC |   .6510802   .0265608    24.51   0.000     .5989953     .703165
      LGSIZE |  -.0567001   .0198997    -2.85   0.004    -.0957227   -.0176775
       _cons |   .6450249   .1965331     3.28   0.001     .2596305    1.030419
------------------------------------------------------------------------------
```

```
. reg LGFDHOPC LGEXPPC LGSIZE in 3960/6334
------------------------------------------------------------------------------
      Source |       SS       df       MS              Number of obs =    2375
-------------+------------------------------           F(  2,  2372) =  138.91
       Model |  94.0495475     2  47.0247737           Prob> F       =  0.0000
    Residual |  802.969196  2372  .338519897           R-squared     =  0.1048
-------------+------------------------------           Adj R-squared =  0.1041
       Total |  897.018744  2374  .377851198           Root MSE      =  .58182
------------------------------------------------------------------------------
    LGFDHOPC |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
     LGEXPPC |   .4072631   .0297285    13.70   0.000     .3489666    .4655596
      LGSIZE |  -.1426229   .0247966    -5.75   0.000    -.1912482   -.0939976
       _cons |   2.742439   .2635057    10.41   0.000     2.225714    3.259165
------------------------------------------------------------------------------
```

The F statistic for the linear specification is:

$$F(2372,\, 2372) = \frac{281980931/2372}{31453534/2372} = 8.97.$$

This is significant at the 0.1 per cent level. The corresponding F statistic for the logarithmic specification is 1.21. The critical value of $F(200, 200)$ at the 5 per cent level is 1.26. The critical value for $F(2372,\, 2372)$ must be lower, so the null hypothesis of homoskedasticity is probably rejected at that level. However, the problem has evidently been largely eliminated.

The logarithmic specification in general appears to be much less heteroskedastic than the linear one and for some categories the null hypothesis of homoskedasticity would not be rejected. Note that for a few of these $RSS_2 < RSS_1$ for the logarithmic specification.

Goldfeld–Quandt tests								
			Linear			Logarithmic		
	n_1	n_2	$RSS_1 \times 10^{-6}$	$RSS_2 \times 10^{-6}$	F	RSS_1	RSS_2	F
ADM	1,056	1,056	1.95	62.93	32.30	1,324.96	1,593.31	1.20
CLOT	1,688	1,688	7.17	316.80	44.17	2,107.28	2,196.79	1.04
DOM	623	623	7.23	238.90	33.05	1,571.19	1,505.92	1.04*
EDUC	210	210	11.70	376.01	32.15	495.12	507.27	1.02
ELEC	2,186	2,186	7.55	33.34	4.41	1,034.70	923.18	1.12*
FDAW	1,913	1,913	9.00	278.13	30.89	1,136.09	1,361.12	1.20
FDHO	2,375	2,375	31.45	281.98	8.97	665.20	802.97	1.21
FOOT	685	685	0.55	5.74	10.37	513.08	514.24	1.00
FURN	183	183	7.17	258.26	36.00	322.50	368.42	1.14
GASO	2,141	2,141	11.06	159.54	14.43	921.26	1,245.55	1.35
HEAL	1,801	1,801	32.91	876.72	26.64	2,233.73	2,192.92	1.02*
HOUS	2,334	2,334	105.48	3,031.19	28.74	2,129.27	1,475.02	1.44*
LIFE	470	470	2.85	48.37	16.95	503.19	667.14	1.33
LOCT	260	260	0.58	5.32	9.13	366.16	409.90	1.12
MAPP	150	150	2.85	37.01	12.96	211.71	243.18	1.15
PERS	1,431	1,431	0.47	9.01	19.34	1,045.70	1,204.31	1.15
READ	858	858	0.36	4.95	13.69	1,076.35	1,085.38	1.01
SAPP	389	389	0.56	10.68	19.04	396.41	433.37	1.09
TELE	2,171	2,171	3.27	26.80	8.19	1,133.43	1,123.46	1.01*
TEXT	372	372	0.57	2.05	3.61	410.29	393.80	1.04*
TOB	433	433	1.56	27.81	17.84	312.71	338.28	1.08
TOYS	939	939	6.83	87.65	12.83	1,079.76	1,064.92	1.01*
TRIP	194	194	9.62	77.65	8.07	300.70	335.75	1.12

* indicates $RSS_2 < RSS_1$

A7.2 The table shows the construction of the White test statistics for the linear and logarithmic specifications for each category of expenditure. The regressors in the auxiliary regression were expenditure per capita and its square, size and its square, and the product of expenditure per capita and size. Hence there were five degrees of freedom for the chi-squared test. The critical values are 11.1 and 15.1 at the 5 per cent and 1 per cent levels. Thus there is strong evidence of heteroskedasticity for all of the categories in the linear specification. There is also evidence for some categories in the logarithmic specification. It is possible that the White test, being more general, is finding evidence of heteroskedasticity not detected by the Goldfeld–Quandt test.

White tests					
		Linear		Logarithmic	
	n	R^2	nR^2	R^2	nR^2
ADM	2,815	0.1710	481.4	0.0097	27.3
CLOT	4,500	0.0180	81.0	0.0074	33.3
DOM	1,661	0.0191	31.7	0.0062	10.3
EDUC	561	0.1432	80.3	0.0078	4.4
ELEC	5,828	0.0487	283.8	0.0090	52.5
FDAW	5,102	0.1072	546.9	0.0067	34.2
FDHO	6,334	0.1143	724.0	0.0129	81.7
FOOT	1,827	0.0191	34.9	0.0023	4.2
FURN	487	0.3287	160.1	0.0197	9.6
GASO	5,710	0.0575	328.3	0.0152	86.8
HEAL	4,802	0.0608	292.0	0.0021	10.1
HOUS	6,223	0.2002	1,245.8	0.0120	74.7
LIFE	1,253	0.0535	67.0	0.0132	16.5
LOCT	692	0.0388	26.8	0.0192	13.3
MAPP	399	0.0882	35.2	0.0168	6.7
PERS	3,817	0.0607	231.7	0.0086	32.8
READ	2,287	0.0158	36.1	0.0072	16.5
SAPP	1,037	0.0221	22.9	0.0032	3.3
TELE	5,788	0.0724	419.1	0.0021	12.2
TEXT	992	0.0183	18.2	0.0049	4.9
TOB	1,155	0.0235	27.1	0.0061	7.0
TOYS	2,504	0.0347	86.9	0.0026	6.5
TRIP	516	0.0571	29.5	0.0047	2.4

A7.3 Having sorted by N, the number of students, RSS_1 and RSS_2 are 2.02×10^{10} and 22.59×10^{10}, respectively, for the subsamples of the 13 smallest and largest schools. The F statistic is 11.18. The critical value of $F(11, 11)$ at the 0.1 per cent level must be a little below 8.75, the critical value for $F(10, 10)$, and so the null hypothesis of homoskedasticity is rejected at that significance level.

One possible way of alleviating the heteroskedasticity is by scaling through by the number of students. The dependent variable now becomes the unit cost per student year, and this is likely to be more uniform than total recurrent cost. Scaling through by N, and regressing *UNITCOST*, defined as *COST* divided by N, on *NREC*, the reciprocal of N, having first sorted by *NREC*, RSS_1 and RSS_2 are now 349,000 and 504,000. The F statistic is therefore 1.44, and this is not significant even at the 5 per cent level since the critical value must be a little above 2.69, the critical value for $F(12, 12)$. The regression output for this specification using the full sample is shown.

```
. reg UNITCOST NREC

  Source |       SS       df       MS              Number of obs =      34
---------+------------------------------           F(  1,    32) =    0.74
   Model |  27010.3792     1  27010.3792           Prob > F      =  0.3954
Residual |  1164624.44    32  36394.5138           R-squared     =  0.0227
---------+------------------------------           Adj R-squared = -0.0079
   Total |  1191634.82    33  36110.1461           Root MSE      =  190.77
```

```
------------------------------------------------------------------------------
UNITCOST |      Coef.   Std. Err.       t     P>|t|       [95% Conf. Interval]
---------+--------------------------------------------------------------------
    NREC |   10975.91    12740.7      0.861   0.395      -14976.04    36927.87
   _cons |    524.813   53.88367      9.740   0.000       415.0556    634.5705
------------------------------------------------------------------------------
```

In equation form, the regression is:

$$\frac{\widehat{COST}}{N} = \underset{(53.9)}{524.8} + \underset{(12741)}{10976}\frac{1}{N} \qquad R^2 = 0.03$$

Multiplying through by N, it may be rewritten:

$$\widehat{COST} = 10976 + 524.8N.$$

The estimate of the marginal cost is somewhat higher than the estimate of 436 obtained using OLS in Section 5.3 of the text.

A second possible way of alleviating the heteroskedasticity is to hypothesise that the true relationship is logarithmic, in which case the use of an inappropriate linear specification would give rise to apparent heteroskedasticity. Scaling through by N, and regressing $LGCOST$, the (natural) logarithm of $COST$, on LGN, the logarithm of N, RSS_1 and RSS_2 are 2.16 and 1.58. The F statistic is therefore 1.37, and again this is not significant even at the 5 per cent level. The regression output for this specification using the full sample is shown.

```
. reg LGCOST LGN

  Source |       SS       df       MS                  Number of obs =      34
---------+------------------------------               F(  1,    32) =  100.98
   Model |  14.7086057     1  14.7086057               Prob > F      =  0.0000
Residual |  4.66084501    32  .145651406               R-squared     =  0.7594
---------+------------------------------               Adj R-squared =  0.7519
   Total |  19.3694507    33   .58695305               Root MSE      =  .38164

------------------------------------------------------------------------------
  LGCOST |      Coef.   Std. Err.       t     P>|t|       [95% Conf. Interval]
---------+--------------------------------------------------------------------
     LGN |    .909126   .0904681     10.049   0.000       .7248485    1.093404
   _cons |   6.808312   .5435035     12.527   0.000       5.701232    7.915393
------------------------------------------------------------------------------
```

The estimate of the elasticity of cost with respect to number of students, 0.91, is less than 1 and thus suggests that the schools are subject to economies of scale. However, we are not able to reject the null hypothesis that the elasticity is equal to 1 and thus that costs are proportional to numbers, the t statistic for the null hypothesis being too low:

$$t = \frac{0.909 - 1.000}{0.091} = -1.00.$$

A7.4 *Discuss whether (1) appears to be an acceptable specification, given the data in the table and Figure 7.1.*

Using the Goldfeld–Quandt test to test specification (1) for heteroskedasticity assuming that the standard deviation of u is inversely proportional to G, we have:

$$F(14,14) = \frac{0.53}{0.21} = 2.52.$$

The critical value of $F(14,14)$ at the 5 per cent level is 2.48, so we just reject the null hypothesis of homoskedasticity at that level. Figure 7.1 does strongly suggest heteroskedasticity. Thus (1) does not appear to be an acceptable specification.

Explain what the researcher hoped to achieve by running regression (2).

If it is true that the standard deviation of u is inversely proportional to G, the heteroskedasticity could be eliminated by multiplying through by G. This is the motivation for the second specification. An intercept that in principle does not exist has been added, thereby changing the model specification slightly.

Discuss whether (2) appears to be an acceptable specification, given the data in the table and Figure 7.2.

$$F(13,13) = \frac{71404}{3178} = 22.47.$$

The critical value of $F(13,13)$ at the 0.1 per cent level is about 6.4, so the null hypothesis of homoskedasticity is rejected. Figure 7.2 confirms the heteroskedasticity.

Explain what the researcher hoped to achieve by running regression (3).

Heteroskedasticity can appear to be present in a regression in natural units if the true relationship is logarithmic. The disturbance term in a logarithmic regression is effectively increasing or decreasing the value of the dependent variable by random proportions. Its effect in absolute terms will therefore tend to be greater, the larger the value of G. The researcher is checking to see if this is the reason for the heteroskedasticity in the second specification.

Discuss whether (3) appears to be an acceptable specification, given the data in the table and Figure 7.3.

Obviously there is no problem with the Goldfeld–Quandt test, since:

$$F(14,14) = \frac{3.60}{3.45} = 1.04.$$

Figure 7.3 looks free from heteroskadasticity.

What are your conclusions concerning the researcher's hypothesis?

Evidence in support of the hypothesis is provided by (3) where, with:

$$t = \frac{0.80 - 1}{0.07} = -2.86$$

the elasticity is significantly lower than 1. Figures 7.1 and 7.2 also strongly suggest that on balance larger economies have lower import ratios than smaller ones.

A7.5 *Perform a Goldfeld–Quandt test for heteroskedasticity on both of the regression specifications.*

The F statistics for the G–Q test for the two specifications are:

$$F(16,16) = \frac{64/16}{8/16} = 8.0 \quad \text{and} \quad F(16,16) = \frac{900/16}{600/16} = 1.5.$$

The critical value of $F(16,16)$ is 2.33 at the 5 per cent level and 5.20 at the 0.1 per cent level. Hence one would reject the null hypothesis of homoskedasticity at the 0.1 per cent level for regression 1 and one would not reject it even at the 5 per cent level for regression 2.

Explain why the researcher ran the second regression.

He hypothesised that the standard deviation of the disturbance term in observation i was proportional to N_i : $\sigma_i = \lambda N_i$ for some λ. If this is the case, dividing through by N_i makes the specification homoskedastic, since:

$$\text{var}\left(\frac{u_i}{N_i}\right) = \frac{1}{N^2}\text{var}(u_i) = \frac{1}{N_i^2}(\lambda N_i)^2 = \lambda^2$$

and is therefore the same for all i.

R^2 is lower in regression (2) than in regression (1). Does this mean that regression (1) is preferable?

R^2 is not comparable because the dependent variable is different in the two regressions. Regression (2) is to be preferred since it is free from heteroskedasticity and therefore ought to tend to yield more precise estimates of the coefficients with valid standard errors.

A7.6 *When the researcher presents her results at a seminar, one of the participants says that, since I and G have been divided by Y, (2) is less likely to be subject to heteroskedasticity than (1). Evaluate this suggestion.*

If the restriction is valid, imposing it will have no implications for the disturbance term and so it could not lead to any mitigation of a potential problem of heteroskedasticity. [If there were heteroskedasticity, and if the specification were linear, scaling through by a variable proportional in observation i to the standard deviation of u_i in observation i would lead to the elimination of heteroskedasticity. The present specification is logarithmic and dividing I and G by Y does not affect the disturbance term.]

A7.7 *Perform the Goldfeld–Quandt test for each model and state your conclusions.*

The ratios are 4.1, 6.0, and 1.05. In each case we should look for the critical value of $F(148,148)$. The critical values of $F(150,150)$ at the 5 per cent, 1 per cent, and 0.1 per cent levels are 1.31, 1.46, and 1.66, respectively. Hence we reject the null hypothesis of homoskedasticity at the 0.1 per cent level (1 per cent is OK) for models (1) and (2). We do not reject it even at the 5 per cent level for model (3).

Explain why the researcher thought that model (2) might be an improvement on model (1).

If the assumption that the standard deviation of the disturbance term is proportional to household size, scaling through by A should eliminate the heteroskedasticity, since:

$$E(v^2) = E\left(\left[\frac{u}{A}\right]^2\right) = \frac{1}{A^2}E(u^2) = \lambda^2$$

if the standard deviation of $u = \lambda A$.

Explain why the researcher thought that model (3) might be an improvement on model (1).

It is possible that the (apparent) heteroskedasticity is attributable to mathematical misspecification. If the true model is logarithmic, a homoskedastic disturbance term would appear to have a heteroskedastic effect if the regression is performed in the original units.

When models (2) and (3) are tested for heteroskedasticity using the White test, auxiliary regressions must be fitted. State the specification of this auxiliary regression for model (2).

The dependent variable is the squared residuals from the model regression. The explanatory variables are the reciprocal of A and its square, E/A and its square, and the product of the reciprocal of A and E/A. (No constant.)

Perform the White test for the three models.

nR^2 is 64.0, 56.0, and 0.4 for the three models. Under the null hypothesis of homoskedasticity, this statistic has a chi-squared distribution with degrees of freedom equal to the number of terms on the right side of the regression, minus one. This is two for models (1) and (3). The critical value of chi-squared with two degrees of freedom is 5.99, 9.21, and 13.82 at the 5, 1, and 0.1 per cent levels. Hence H0 is rejected at the 0.1 per cent level for model (1), and not rejected even at the 5 per cent level for model (3). In the case of model (2), there are five terms on the right side of the regression. The critical value of chisquared with four degrees of freedom is 18.47 at the 0.1 per cent level. Hence H_0 is rejected at that level.

Explain whether the results of the tests seem reasonable, given the scatter plots of the data.

Absolutely. In Figures 7.1 and 7.2, the variances of the dispersions of the dependent variable clearly increase with the size of the explanatory variable. In Figure 7.3, the dispersion is much more even.

A7.8 *'Heteroskedasticity occurs when the disturbance term in a regression model is correlated with one of the explanatory variables.'*

This is false. Heteroskedasticity occurs when the variance of the disturbance term is not the same for all observations.

'In the presence of heteroskedasticity ordinary least squares (OLS) is an inefficient estimation technique and this causes t tests and F tests to be invalid.'

It is true that OLS is inefficient and that the t and F tests are invalid, but 'and this causes' is wrong.

'OLS remains unbiased but it is inconsistent.'

It is true that OLS is unbiased, but false that it is inconsistent.

'Heteroskedasticity can be detected with a Chow test.'

This is false.

'Alternatively one can compare the residuals from a regression using half of the observations with those from a regression using the other half and see if there is a significant difference. The test statistic is the same as for the Chow test.'

The first sentence is basically correct with the following changes and clarifications: one is assuming that the standard deviation of the disturbance term is proportional to one of the explanatory variables; the sample should first be sorted according to the size of the explanatory variable; rather than split the sample in half, it would be better to compare the first three-eighths (or one third) of the observations with the last three-eighths (or one third); 'comparing the residuals' is too vague: the F statistic is $F(n' - k, n' - k) = RSS_2/RSS_1$ assuming n' observations and k parameters in each subsample regression, and placing the larger RSS over the smaller.

The second sentence is false.

'One way of eliminating the problem is to make use of a restriction involving the variable correlated with the disturbance term.'

This is nonsense.

'If you can find another variable related to the one responsible for the heteroskedasticity, you can use it as a proxy and this should eliminate the problem.'

This is more nonsense.

'Sometimes apparent heteroskedasticity can be caused by a mathematical misspecification of the regression model. This can happen, for example, if the dependent variable ought to be logarithmic, but a linear regression is run.'

True. A homoskedastic disturbance term in a logarithmic regression, which is responsible for proportional changes in the dependent variable, may appear to be heteroskedastic in a linear regression because the absolute changes in the dependent variable will be proportional to its size.

Chapter 8
Stochastic regressors and measurement errors

8.1 Overview

Until this point it has been assumed that the only random element in a regression model is the disturbance term. This chapter extends the analysis to the case where the variables themselves have random components. The initial analysis shows that in general OLS estimators retain their desirable properties. A random component attributable to measurement error, the subject of the rest of the chapter, is however another matter. While measurement error in the dependent variable merely inflates the variances of the regression coefficients, measurement error in the explanatory variables causes OLS estimates of the coefficients to be biased and invalidates standard errors, t tests, and F tests. The analysis is illustrated with reference to the Friedman permanent income hypothesis, the most celebrated application of measurement error analysis in the economic literature. The chapter then introduces instrumental variables (IV) estimation and gives an example of its use to fit the Friedman model. The chapter concludes with a description of the Durbin–Wu–Hausman test for investigating whether measurement errors are serious enough to warrant using IV instead of OLS.

8.2 Learning outcomes

After working through the corresponding chapter in the text, studying the corresponding slideshows, and doing the starred exercises in the text and the additional exercises in this subject guide, you should be able to:

- explain the conditions under which OLS estimators remain unbiased when the variables in the regression model possess random components
- derive the large-sample expression for the bias in the slope coefficient in a simple regression model with measurement error in the explanatory variable
- demonstrate, within the context of the same model, that measurement error in the dependent variable does not cause the regression coefficients to be biased but does increase their standard errors
- describe the Friedman permanent income hypothesis and explain why OLS estimates of a conventional consumption function will be biased if it is correct
- explain what is meant by an instrumental variables estimator and state the conditions required for its use

- demonstrate that the IV estimator of the slope coefficient in a simple regression model is consistent, provided that the conditions required for its use are satisfied
- explain the factors responsible for the population variance of the IV estimator of the slope coefficient in a simple regression model
- perform the Durbin–Wu–Hausman test in the context of measurement error.

8.3 Additional exercises

A8.1 A researcher believes that a variable Y is determined by the simple regression model:

$$Y = \beta_1 + \beta_2 X + u.$$

She thinks that X is not distributed independently of u but thinks that another variable, Z, would be a suitable instrument. The instrumental estimator of the intercept, $\widehat{\beta}_1^{\text{IV}}$, is given by:

$$\widehat{\beta}_1^{\text{IV}} = \overline{Y} - \widehat{\beta}_2^{\text{IV}} \overline{X}$$

where $\widehat{\beta}_2^{\text{IV}}$ is the IV estimator of the slope coefficient. [Exercise 8.12 in the textbook asks for a proof that $\widehat{\beta}_1^{\text{IV}}$ is a consistent estimator of β_1.]

Explain, with a brief mathematical proof, why $\widehat{\beta}_1^{\text{OLS}}$, the ordinary least squares estimator of β_1, would be inconsistent, if the researcher is correct in believing that X is not distributed independently of u.

The researcher has only 20 observations in her sample. Does the fact that $\widehat{\beta}_1^{\text{IV}}$ is consistent guarantee that it has desirable small-sample properties? If not, explain how the researcher might investigate the small-sample properties.

A8.2 Suppose that the researcher in Exercise A8.1 is wrong and X is in fact distributed independently of u. Explain the consequences of using $\widehat{\beta}_1^{\text{IV}}$ instead of $\widehat{\beta}_1^{\text{OLS}}$ to estimate β_1.

Note: The population variance of $\widehat{\beta}_1^{\text{IV}}$ is given by:

$$\sigma^2_{\widehat{\beta}_1^{\text{IV}}} = \left(1 + \frac{\mu_X^2}{\sigma_X^2} \times \frac{1}{r_{XZ}^2}\right) \frac{\sigma_u^2}{n}$$

where μ_X is the population mean of X, σ_X^2 is its population variance, r_{XZ} is the correlation between X and Z, and σ_u^2 is the population variance of the disturbance term, u. For comparison, the population variance of the OLS estimator is:

$$\sigma^2_{\widehat{\beta}_1^{\text{OLS}}} = \left(1 + \frac{\mu_X^2}{\sigma_X^2}\right) \frac{\sigma_u^2}{n}$$

when the model is correctly specified and the regression model assumptions are satisfied.

A8.3 A researcher investigating the incidence of teenage knife crime has the following data for each of 35 cities for 2008:

- K = number of knife crimes per 1,000 population in 2008
- N = number of teenagers per 1,000 population living in social deprivation in 2008.

The researcher hypothesises that the relationship between K and N is given by:

$$K = \beta_1 + \beta_2 N + u \qquad (1)$$

where u is a disturbance term that satisfies the usual regression model assumptions. However, knife crime tends to be under-reported, with the degree of under-reporting worst in the most heavily afflicted boroughs, so that:

$$R = K + w \qquad (2)$$

where R = number of reported knife crimes per 1,000 population in 2008 and w is a random variable with $E(w) < 0$ and $\text{cov}(w, K) < 0$. w may be assumed to be distributed independently of u. Note that $\text{cov}(w, K) < 0$ implies $\text{cov}(w, N) < 0$. Derive analytically the sign of the bias in the estimator of β_2 if the researcher regresses R on N using ordinary least squares.

A8.4 Suppose that in the model:

$$Y = \beta_1 + \beta_2 X + u$$

where the disturbance term u satisfies the regression model assumptions, the variable X is subject to measurement error, being underestimated by a fixed amount α in all observations.

- Discuss whether it is true that the ordinary least squares estimator of β_2 will be biased downwards by an amount proportional to both α and β_2.
- Discuss whether it is true that the fitted values of Y from the regression will be reduced by an amount $\alpha\beta_2$.
- Discuss whether it is true that R^2 will be reduced by an amount proportional to α.

A8.5 A researcher believes that the rate of migration from Country B to Country A, M_t, measured in thousands of persons per year, is a linear function of the relative average wage, RW_t, defined as the average wage in Country A divided by the average wage in Country B, both measured in terms of the currency of Country A:

$$M_t = \beta_1 + \beta_2 RW_t + u_t. \qquad (1)$$

u_t is a disturbance term that satisfies the regression model assumptions. However, Country B is a developing country with limited resources for statistical surveys and the wage data for that country, derived from a small sample of social security records, are widely considered to be unrepresentative, with a tendency to overstate the true average wage because those working in the informal sector are excluded. As a consequence the measured relative wage, MRW_t, is given by

$$MRW_t = RW_t + w_t \qquad (2)$$

where w_t is a random quantity with expected value less than 0. It may be assumed to be distributed independently of u_t and RW_t.

The researcher also has data on relative GDP per capita, $RGDP_t$, defined as the ratio of GDP per capita in countries A and B, respectively, both measured in terms of the currency of Country A. He has annual observations on M_t, MRW_t, and $RGDP_t$ for a 30-year period. The correlation between MRW_t, and $RGDP_t$ in the sample period is 0.8. Analyse mathematically the consequences for the estimates of the intercept and the slope coefficient, the standard errors and the t statistics, if the migration equation (1) is fitted:

- using ordinary least squares with MRW_t as the explanatory variable.
- using OLS, with $RGDP_t$ as a proxy for RW_t.
- using instrumental variables, with $RGDP_t$ as an instrument for MRW_t.

A8.6 Suppose that in Exercise A8.5 $RGDP_t$ is subject to the same kind of measurement error as RW_t, and that as a consequence there is an exact linear relationship between $RGDP_t$ and MRW_t. Demonstrate mathematically how this would affect the IV estimator of β_2 in part (3) of Exercise A8.5 and give a verbal explanation of your result.

8.4 Answers to the starred exercises in the textbook

8.5 A variable Q is determined by the model:

$$Q = \beta_1 + \beta_2 X + v$$

where X is a variable and v is a disturbance term that satisfies the regression model assumptions. The dependent variable is subject to measurement error and is measured as Y where:

$$Y = Q + r$$

and r is the measurement error, distributed independently of v. Describe analytically the consequences of using OLS to fit this model if:

1. The expected value of r is not equal to zero (but r is distributed independently of Q).
2. r is not distributed independently of Q (but its expected value is zero).

Answer:

Substituting for Q, the model may be rewritten:

$$\begin{aligned} Y &= \beta_1 + \beta_2 X + v + r \\ &= \beta_1 + \beta_2 X + u \end{aligned}$$

where $u = v + r$. Then:

$$\widehat{\beta}_2 = \beta_2 + \frac{\left(X_i - \overline{X}\right)(u_i - \bar{u})}{\sum\left(X_i - \overline{X}\right)^2} = \beta_2 + \frac{\sum\left(X_i - \overline{X}\right)(v_i - \bar{v}) + \sum\left(X_i - \overline{X}\right)(r_i - \bar{r})}{\sum\left(X_i - \overline{X}\right)^2}$$

and:

$$
\begin{aligned}
E(\widehat{\beta}_2) &= E\left(\beta_2 + \frac{\sum\left(X_i-\overline{X}\right)(v_i-\bar{v}) + \sum\left(X_i-\overline{X}\right)(r_i-\bar{r})}{\sum\left(X_i-\overline{X}\right)^2}\right) \\
&= \beta_2 + \frac{1}{\sum\left(X_i-\overline{X}\right)^2} E\left(\sum\left(X_i-\overline{X}\right)(v_i-\bar{v}) + \sum\left(X_i-\overline{X}\right)(r_i-\bar{r})\right) \\
&= \beta_2 + \frac{1}{\sum\left(X_i-\overline{X}\right)^2} \left(\sum\left(X_i-\overline{X}\right)E(v_i-\bar{v}) + \sum\left(X_i-\overline{X}\right)E(r_i-\bar{r})\right) \\
&= \beta_2
\end{aligned}
$$

provided that X is nonstochastic. (If X is stochastic, the proof that the expected value of the error term is zero is parallel to that in Section 8.2 of the text.) Thus $\widehat{\beta}_2$ remains an unbiased estimator of β_2.

However, the estimator of the intercept is affected if $E(r)$ is not zero.

$$\widehat{\beta}_1 = \overline{Y} - \widehat{\beta}_2\overline{X} = \beta_1 + \beta_2\overline{X} + \bar{u} - \widehat{\beta}_2\overline{X} = \beta_1 + \beta_2\overline{X} + \bar{v} + \bar{r} - \widehat{\beta}_2\overline{X}.$$

Hence:

$$
\begin{aligned}
E(\widehat{\beta}_1) &= \beta_1 + \beta_2\overline{X} + E(\bar{v}) + E(\bar{r}) - E(\widehat{\beta}_2\overline{X}) \\
&= \beta_1 + \beta_2\overline{X} + E(\bar{v}) + E(\bar{r}) - \overline{X}E(\widehat{\beta}_2) \\
&= \beta_1 + E(\bar{r}).
\end{aligned}
$$

Thus the intercept is biased if $E(r)$ is not equal to zero, for then $E(\bar{r})$ is not equal to 0.

If r is not distributed independently of Q, the situation is a little bit more complicated. For it to be distributed independently of Q, it must be distributed independently of both X and v, since these are the determinants of Q. Thus if it is not distributed independently of Q, one of these two conditions must be violated. We will consider each in turn.

(a) r not distributed independently of X. We now have:

$$
\begin{aligned}
\operatorname{plim} \widehat{\beta}_2 &= \beta_2 + \frac{\operatorname{plim}\frac{1}{n}\sum\left(X_i-\overline{X}\right)(v_i-\bar{v}) + \operatorname{plim}\frac{1}{n}\sum\left(X_i-\overline{X}\right)(r_i-\bar{r})}{\operatorname{plim}\frac{1}{n}\sum\left(X_i-\overline{X}\right)^2} \\
&= \beta_2 + \frac{\sigma_{Xr}}{\sigma_X^2}.
\end{aligned}
$$

Since $\sigma_{Xr} \neq 0$, $\widehat{\beta}_2$ is an inconsistent estimator of β_2. It follows that $\widehat{\beta}_1$ will also be an inconsistent estimator of β_1:

$$\widehat{\beta}_1 = \beta_1 + \beta_2\overline{X} + \bar{v} + \bar{r} - \widehat{\beta}_2\overline{X}.$$

Hence:

$$\begin{aligned}
\text{plim } \widehat{\beta}_1 &= \beta_1 + \beta_2\overline{X} + \text{ plim } \bar{v} + \text{ plim } \bar{r} - \overline{X} \text{ plim } \widehat{\beta}_2 \\
&= \beta_1 + \overline{X}(\beta_2 - \text{ plim } \widehat{\beta}_2)
\end{aligned}$$

and this is different from β_1 if plim $\widehat{\beta}_2$ is not equal to β_2.

(b) r is not distributed independently of v. This condition is not required in the proof of the unbiasedness of either $\widehat{\beta}_1$ or $\widehat{\beta}_2$ and so both remain unbiased.

8.6 A variable Y is determined by the model:

$$Y = \beta_1 + \beta_2 Z + v$$

where Z is a variable and v is a disturbance term that satisfies the regression model conditions. The explanatory variable is subject to measurement error and is measured as X where:

$$X = Z + w$$

and w is the measurement error, distributed independently of v. Describe analytically the consequences of using OLS to fit this model if:

(1) the expected value of w is not equal to zero (but w is distributed independently of Z)

(2) w is not distributed independently of Z (but its expected value is zero).

Answer:

Substituting for Z, we have:

$$Y = \beta_1 + \beta_2(X - w) + v = \beta_1 + \beta_2 X + u$$

where $u = v - \beta_2 w$.

$$\widehat{\beta}_2 = \beta_2 + \frac{\sum\left(X_i - \overline{X}\right)(u_i - \bar{u})}{\sum\left(X_i - \overline{X}\right)^2}.$$

It is not possible to obtain a closed-form expression for the expectation of the error term since both its numerator and its denominator depend on w. Instead we take plims, having first divided the numerator and the denominator of the error term by n so that they have limits:

$$\begin{aligned}
\text{plim } \widehat{\beta}_2 &= \beta_2 + \frac{\text{plim } \frac{1}{n}\sum\left(X_i - \overline{X}\right)(u_i - \bar{u})}{\text{plim } \frac{1}{n}\sum\left(X_i - \overline{X}\right)^2} \\
&= \beta_2 + \frac{\text{cov}(X, u)}{\text{var}(X)} = \beta_2 + \frac{\text{cov}([Z + w], [v - \beta_2 w])}{\text{var}(X)} \\
&= \beta_2 + \frac{\text{cov}(Z, v) - \beta_2\text{cov}(Z, w) + \text{cov}(w, v) - \beta_2\text{cov}(w, w)}{\text{var}(X)}.
\end{aligned}$$

If $E(w)$ is not equal to zero, $\widehat{\beta}_2$ is not affected. The first three terms in the numerator are zero and:

$$\text{plim } \widehat{\beta}_2 = \beta_2 + \frac{-\beta_2 \sigma_w^2}{\sigma_X^2}$$

so $\widehat{\beta}_2$ remains inconsistent as in the standard case. If w is not distributed independently of Z, then the second term in the numerator is not 0. $\widehat{\beta}_2$ remains inconsistent, but the expression is now:

$$\text{plim } \widehat{\beta}_2 = \beta_2 + \frac{-\beta_2(\sigma_{Zw} + \sigma_w^2)}{\sigma_X^2}.$$

The OLS estimator of the intercept is affected in both cases, but like the slope coefficient, it was inconsistent anyway.

$$\widehat{\beta}_1 = \overline{Y} - \widehat{\beta}_2\overline{X} = \beta_1 + \beta_2\overline{X} + \bar{u} - \widehat{\beta}_2\overline{X} = \beta_1 + \beta_2\overline{X} + \bar{v} - \beta_2\overline{w} - \widehat{\beta}_2\overline{X}.$$

Hence:

$$\text{plim } \widehat{\beta}_1 = \beta_1 + (\beta_2 - \text{ plim } \widehat{\beta}_2)\overline{X} + \text{ plim } \bar{v} - \beta_2 \text{ plim } \overline{w}.$$

In the standard case this would reduce to:

$$\begin{aligned} \text{plim } \widehat{\beta}_1 &= \beta_1 + (\beta_2 - \text{ plim } \widehat{\beta}_2)\overline{X} \\ &= \beta_1 + \beta_2 \frac{\sigma_w^2}{\sigma_X^2}\overline{X}. \end{aligned}$$

If w has expected value μ_w, not equal to zero:

$$\text{plim } \widehat{\beta}_1 = \beta_1 + \beta_2 \left(\frac{\sigma_w^2}{\sigma_X^2}\overline{X} - \mu_w \right).$$

If w is not distributed independently of Z:

$$\text{plim } \widehat{\beta}_1 = \beta_1 + \beta_2 \frac{\sigma_{Zw} + \sigma_w^2}{\sigma_X^2}\overline{X}.$$

8.10 A researcher investigating the shadow economy using international crosssectional data for 25 countries hypothesises that consumer expenditure on shadow goods and services, Q, is related to total consumer expenditure, Z, by the relationship:

$$Q = \beta_1 + \beta_2 Z + v$$

where v is a disturbance term that satisfies the regression model assumptions. Q is part of Z and any error in the estimation of Q affects the estimate of Z by the same amount. Hence:

$$Y_i = Q_i + w_i$$

and:

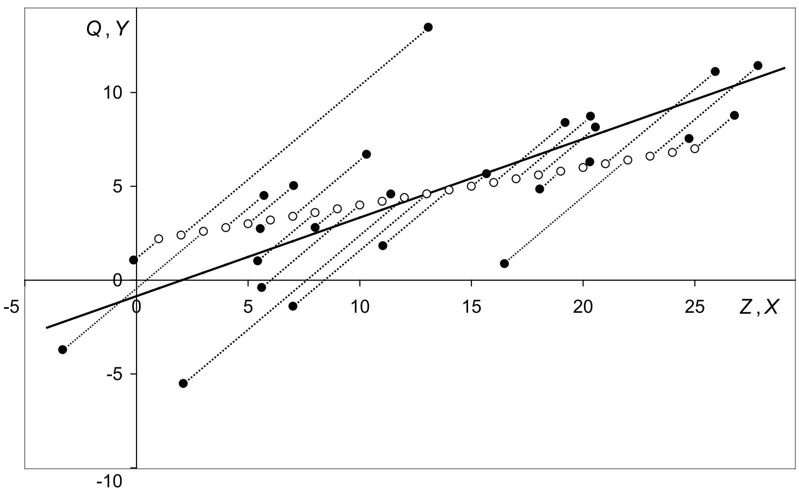
$$X_i = Z_i + w_i$$

where Y_i is the estimated value of Q_i, X_i is the estimated value of Z_i, and w_i is the measurement error affecting both variables in observation i. It is assumed that the expected value of w is 0 and that v and w are distributed independently of Z and of each other.

1. Derive an expression for the large-sample bias in the estimate of β_2 when OLS is used to regress Y on X, and determine its sign if this is possible. [Note: The standard expression for measurement error bias is not valid in this case.]
2. In a Monte Carlo experiment based on the model above, the true relationship between Q and Z is:
$$Q = 2.0 + 0.2Z.$$
A sample of 25 observations is generated using the integers 1, 2,..., 25 as data for Z. The variance of Z is 52.0. A normally distributed random variable with mean 0 and variance 25 is used to generate the values of the measurement error in the dependent and explanatory variables. The results with 10 samples are summarised in the table below. Comment on the results, stating whether or not they support your theoretical analysis.

Sample	$\widehat{\beta_1}$	s.e.$(\widehat{\beta_1})$	$\widehat{\beta_2}$	s.e.$(\widehat{\beta_2})$	R^2
1	−0.85	1.09	0.42	0.07	0.61
2	−0.37	1.45	0.36	0.10	0.36
3	−2.85	0.88	0.49	0.06	0.75
4	−2.21	1.59	0.54	0.10	0.57
5	−1.08	1.43	0.47	0.09	0.55
6	−1.32	1.39	0.51	0.08	0.64
7	−3.12	1.12	0.54	0.07	0.71
8	−0.64	0.95	0.45	0.06	0.74
9	0.57	0.89	0.38	0.05	0.69
10	−0.54	1.26	0.40	0.08	0.50

3. The figure below plots the points (Q, Z), represented as circles, and (Y, X), represented as solid markers, for the first sample, with each (Q, Z) point linked to the corresponding (Y, X) point. Comment on this graph, given your answers to parts 1 and 2.



Answer:

1. Substituting for Q and Z in the first equation:
$$(Y - w) = \beta_1 + \beta_2(X - w) + v.$$

Hence:

$$\begin{aligned} Y &= \beta_1 + \beta_2 X + v + (1 - \beta_2)w \\ &= \beta_1 + \beta_2 X + u \end{aligned}$$

where $u = v + (1 - \beta_2)w$. So:

$$\widehat{\beta}_2 = \beta_2 + \frac{\sum \left(X_i - \overline{X}\right)(u_i - \bar{u})}{\sum \left(X_i - \overline{X}\right)^2}.$$

It is not possible to obtain a closed-form expression for the expectation of the error term since both its numerator and its denominator depend on w. Instead we take plims, having first divided the numerator and the denominator of the error term by n so that they have limits:

$$\begin{aligned} \text{plim } \widehat{\beta}_2 &= \beta_2 + \frac{\text{plim } \frac{1}{n} \sum \left(X_i - \overline{X}\right)(u_i - \bar{u})}{\text{plim } \frac{1}{n} \sum \left(X_i - \overline{X}\right)^2} \\ &= \beta_2 + \frac{\text{cov}(X, u)}{\text{var}(u)} = \beta_2 + \frac{\text{cov}([Z + w], [v + (1 - \beta_2)w])}{\text{var}(X)} \\ &= \beta_2 + \frac{\text{cov}(Z, v) + (1 - \beta_2)\text{cov}(Z, w) + \text{cov}(w, v) + (1 - \beta_2)\text{cov}(w, w)}{\text{var}(X)}. \end{aligned}$$

Since v and w are distributed independently of Z and of each other, $\text{cov}(Z, v) = \text{cov}(Z, w) = \text{cov}(w, v) = 0$, and so:

$$\text{plim } \widehat{\beta}_2 = \beta_2 + (1 - \beta_2)\frac{\sigma_w^2}{\sigma_X^2}.$$

β_2 clearly should be positive and less than 1, so the bias is positive.

2. $\sigma_X^2 = \sigma_Z^2 + \sigma_w^2$, given that w is distributed independently of Z, and hence $\sigma_X^2 = 52 + 25 = 77$. Thus:

$$\text{plim } \widehat{\beta}_2 = 0.2 + \frac{(1 - 0.2) \times 25}{77} = 0.46.$$

The estimates of the slope coefficient do indeed appear to be distributed around this number.

As a consequence of the slope coefficient being overestimated, the intercept is underestimated, negative estimates being obtained in each case despite the fact that the true value is positive. The standard errors are invalid, given the severe problem of measurement error.

3. The diagram shows how the measurement error causes the observations to be displaced along 45° lines. Hence the slope of the regression line will be a compromise between the true slope, β_2 and 1. More specifically, plim $\widehat{\beta}_2$ is a

weighted average of β_2 and 1, the weights being proportional to the variances of Z and w:

$$\begin{aligned} \operatorname{plim} \widehat{\beta}_2 &= \beta_1 + (1 - \beta_2)\frac{\sigma_w^2}{\sigma_Z^2 + \sigma_w^2} \\ &= \frac{\sigma_Z^2}{\sigma_Z^2 + \sigma_w^2}\beta_2 + \frac{\sigma_w^2}{\sigma_Z^2 + \sigma_w^2}. \end{aligned}$$

8.16 It is possible that the *ASVABC* test score is a poor measure of the kind of ability relevant for earnings. Accordingly, perform an OLS regression of the logarithm of hourly earnings on *S*, *EXP*, *ASVABC*, *MALE*, *ETHBLACK*, and *ETHHISP* using your *EAWE* data set and an IV regression using *SM*, *SF*, and *SIBLINGS* as instruments for *ASVABC*. Perform a Durbin–Wu–Hausman test to evaluate whether *ASVABC* appears to be subject to measurement error.

Answer:

Contrary to expectations, the coefficient of *ASVABC* is lower in the IV regression. It is 0.048 in the OLS regression and −0.094 in the IV regression. The chi-squared statistic, 1.21, is low. One might therefore conclude that there is no serious measurement error and the change in the coefficient is random. Another possibility is that the instruments are too weak. *ASVABC* is not highly correlated with any of the instruments and the standard error of the coefficient rises from 0.028 in the OLS regression to 0.132 in the IV regression.

```
. ivreg LGEARN S EXP MALE ETHBLACK ETHHISP (ASVABC=SM SF SIBLINGS)
Instrumental variables (2SLS) regression
------------------------------------------------------------------------------
    Source |       SS       df       MS              Number of obs =     500
-----------+------------------------------           F(  6,   493) =   22.29
     Model |   27.631679     6  4.60527983            Prob> F      =  0.0000
  Residual |  121.501359   493  .246453061           R-squared     =  0.1853
-----------+------------------------------           Adj R-squared =  0.1754
     Total |  149.133038   499  .298863804           Root MSE      =  .49644
------------------------------------------------------------------------------
    LGEARN |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-----------+------------------------------------------------------------------
    ASVABC |  -.0938253   .1319694    -0.71   0.477    -.3531172    .1654666
         S |   .1203265   .0251596     4.78   0.000     .0708931    .1697599
       EXP |   .0444094   .0092246     4.81   0.000      .026285    .0625338
      MALE |   .1909863   .0456252     4.19   0.000     .1013424    .2806302
  ETHBLACK |  -.1678914   .1355897    -1.24   0.216    -.4342963    .0985136
   ETHHISP |    .075698   .0828383     0.91   0.361    -.0870617    .2384576
     _cons |   .6503199   .3570741     1.82   0.069    -.0512548    1.351895
------------------------------------------------------------------------------
Instrumented:  ASVABC
Instruments:   S EXP MALE ETHBLACK ETHHISP SM SF SIBLINGS
------------------------------------------------------------------------------
```

```
. estimates store IV1

. reg LGEARN S EXP ASVABC MALE ETHBLACK ETHHISP
------------------------------------------------------------------------------
    Source |       SS       df       MS              Number of obs =     500
-----------+------------------------------           F(  6,   493) =   23.81
     Model |  33.5095496     6  5.58492493            Prob> F      =  0.0000
  Residual |  115.623489   493  .234530403           R-squared     =  0.2247
-----------+------------------------------           Adj R-squared =  0.2153
     Total |  149.133038   499  .298863804           Root MSE      =  .48428
------------------------------------------------------------------------------
    LGEARN |      Coef.   Std. Err.      t    P>|t|      [95% Conf. Interval]
-----------+------------------------------------------------------------------
         S |   .0953713   .0106101     8.99   0.000      .0745246    .1162179
       EXP |    .043139   .0089279     4.83   0.000      .0255976    .0606805
    ASVABC |   .0477892   .0282877     1.69   0.092       -.00779    .1033685
      MALE |   .1954406   .0443323     4.41   0.000      .1083371    .2825441
  ETHBLACK |  -.0448382    .074738    -0.60   0.549     -.1916824     .102006
   ETHHISP |   .1226463   .0692577     1.77   0.077     -.0134303     .258723
     _cons |   .9766376   .1938648     5.04   0.000      .5957345    1.357541
------------------------------------------------------------------------------


. estimates store OLS1

. hausman IV1 OLS1, constant

                 ---- Coefficients ----
             |      (b)          (B)            (b-B)     sqrt(diag(V_b-V_B))
             |      IV1          OLS1         Difference          S.E.
-------------+----------------------------------------------------------------
      ASVABC |   -.0938253     .0477892       -.1416145        .1289021
           S |    .1203265     .0953713        .0249552         .022813
         EXP |    .0444094      .043139        .0012704        .0023208
        MALE |    .1909863     .1954406       -.0044543        .0107847
    ETHBLACK |   -.1678914    -.0448382       -.1230532        .1131318
     ETHHISP |     .075698     .1226463       -.0469484        .0454484
       _cons |    .6503199     .9766376       -.3263177        .2998639
------------------------------------------------------------------------------
                           b = consistent under Ho and Ha; obtained from ivreg
          B = inconsistent under Ha, efficient under Ho; obtained from regress
    Test:  Ho:  difference in coefficients not systematic
chi2(7) = (b-B)'[(V_b-V_B)^(-1)](b-B)
                           =        1.21
Prob>chi2 =      0.9908


. cor ASVABC SM SF SIBLINGS
(obs=500)
             |   ASVABC       SM       SF SIBLINGS
-------------+------------------------------------
      ASVABC |   1.0000
          SM |   0.3426   1.0000
          SF |   0.3613   0.5622   1.0000
    SIBLINGS |  -0.2360  -0.3038  -0.2516   1.0000
```

8.17 What is the difference between an instrumental variable and a proxy variable (as described in Section 6.4)? When would you use one and when would you use the other?

Answer:

An instrumental variable estimator is used when one has data on an explanatory variable in the regression model but OLS would give inconsistent estimates because the explanatory variable is not distributed independently of the disturbance term. The instrumental variable partially replaces the original explanatory variable in the estimator and the estimator is consistent.

A proxy variable is used when one has no data on an explanatory variable in a regression model. The proxy variable is used as a straight substitute for the original variable. The interpretation of the regression coefficients will depend on the relationship between the proxy and the original variable, and the properties of the other estimators in the model and the tests and diagnostic statistics will depend on the degree of correlation between the proxy and the original variable.

8.5 Answers to the additional exercises

A8.1

$$\begin{aligned}\widehat{\beta}_1^{\text{OLS}} &= \overline{Y} - \widehat{\beta}_2^{\text{OLS}}\overline{X} \\ &= \beta_1 + \beta_2\overline{X} + \bar{u} - \widehat{\beta}_2^{\text{OLS}}\overline{X}.\end{aligned}$$

Therefore:

$$\begin{aligned}\operatorname{plim} \widehat{\beta}_1^{\text{OLS}} &= \beta_1 - (\operatorname{plim} \widehat{\beta}_2^{\text{OLS}} - \beta_2) \operatorname{plim} \overline{X} \\ &\neq \beta_1.\end{aligned}$$

However:

$$\begin{aligned}\widehat{\beta}_1^{\text{IV}} &= \overline{Y} - \widehat{\beta}_2^{\text{IV}}\overline{X} \\ &= \beta_1 + \beta_2\overline{X} + \bar{u} - \widehat{\beta}_2^{\text{IV}}\overline{X} \\ &= \beta_1 - (\widehat{\beta}_2^{\text{IV}} - \beta_2)\overline{X} + \bar{u}.\end{aligned}$$

Therefore:

$$\begin{aligned}\operatorname{plim} \widehat{\beta}_1^{\text{IV}} &= \beta_1 - (\operatorname{plim} \widehat{\beta}_2^{\text{IV}} - \beta_2) \operatorname{plim} \overline{X} \\ &= \beta_1.\end{aligned}$$

Consistency does not guarantee desirable small-sample properties. The latter could be investigated with a Monte Carlo experiment.

A8.2 Both estimators will be consistent (actually, unbiased) but the IV estimator will be less efficient than the OLS estimator, as can be seen from a comparison of the expressions for the population variances.

A8.3 The regression model is:

$$R = \beta_1 + \beta_2 N + u + w.$$

Hence:

$$\widehat{\beta}_2^{\text{OLS}} = \beta_2 + \frac{\sum \left(N_i - \overline{N}\right)(u_i + w_i - \bar{u} - \overline{w})}{\sum \left(N_i - \overline{N}\right)^2}.$$

It is not possible to obtain a closed-form expression for the expectation since N and w are correlated. Hence, instead, we investigate the plim:

$$\begin{aligned}
\operatorname{plim} \widehat{\beta}_2^{\text{OLS}} &= \beta_2 + \operatorname{plim} \frac{\frac{1}{n}\sum \left(N_i - \overline{N}\right)(u_i + w_i - \bar{u} - \overline{w})}{\frac{1}{n}\sum \left(N_i - \overline{N}\right)^2} \\
&= \beta_2 + \frac{\operatorname{cov}(N, u) + \operatorname{cov}(N, w)}{\operatorname{var}(N)} < \beta_2
\end{aligned}$$

since $\operatorname{cov}(N, u) = 0$ and $\operatorname{cov}(N, w) < 0$.

A8.4 *Discuss whether it is true that the ordinary least squares estimator of β_2 will be biased downwards by an amount proportional to both α and β_2.*

It is not true. Let the measured X be X', where $X' = X - \alpha$. Then:

$$\widehat{\beta}_2^{\text{OLS}} = \frac{\sum (X_i' - X')\left(Y_i - \overline{Y}\right)}{\sum (X_i' - X')^2} = \frac{\sum \left(X_i - \alpha - [\overline{X} - \alpha]\right)\left(Y_i - \overline{Y}\right)}{\sum \left(X_i - \alpha - [\overline{X} - \alpha]\right)^2} = \frac{\sum \left(X_i - \overline{X}\right)\left(Y_i - \overline{Y}\right)}{\sum \left(X_i - \overline{X}\right)^2}.$$

Thus the measurement error has no effect on the estimate of the slope coefficient.

Discuss whether it is true that the fitted values of Y from the regression will be reduced by an amount $\alpha\beta_2$.

The estimator of the intercept will be $\overline{Y} - \widehat{\beta}_2\overline{X}' = \overline{Y} - \widehat{\beta}_2(\overline{X} - \alpha)$. Hence the fitted value in observation i will be:

$$\overline{Y} - \widehat{\beta}_2(\overline{X} - \alpha) + \widehat{\beta}_2 X_i' = \overline{Y} - \widehat{\beta}_2(\overline{X} - \alpha) + \widehat{\beta}_2(X_i - \alpha) = \overline{Y} - \widehat{\beta}_2\overline{X} + \widehat{\beta}_2 X_i$$

which is what it would be in the absence of the measurement error.

Discuss whether it is true that R^2 will be reduced by an amount proportional to α.

Since R^2 is the variance of the fitted values of Y divided by the variance of the actual values, it will be unaffected.

A8.5 *Using ordinary least squares with MRW_t as the explanatory variable.*

$$\operatorname{plim} \widehat{\beta}_2^{\text{OLS}} = \beta_2 - \beta_2 \frac{\sigma_w^2}{\sigma_{Rw}^2 + \sigma_w^2} = \beta_2 \frac{\sigma_{Rw}^2}{\sigma_{Rw}^2 + \sigma_w^2}$$

(standard theory). Hence the bias is towards zero.

$$\begin{aligned}
\widehat{\beta}_1^{\text{OLS}} &= \overline{M} - \widehat{\beta}_2^{\text{OLS}}\overline{MRW} \\
&= \beta_1 + \beta_2\overline{RW} + \bar{u} - \widehat{\beta}_2^{\text{OLS}}\left(\overline{RW} + \overline{w}\right) \\
&= \beta_1 + (\beta_2 - \widehat{\beta}_2^{\text{OLS}})\overline{RW} + \bar{u} - \widehat{\beta}_2^{\text{OLS}}\overline{w}
\end{aligned}$$

and so:

$$\text{plim } \widehat{\beta}_1^{\text{OLS}} = \beta_1 + \beta_2 \frac{\sigma_w^2}{\sigma_{Rw}^2 + \sigma_w^2} \overline{RW} - \beta_2 \frac{\sigma_{Rw}^2}{\sigma_{Rw}^2 + \sigma_w^2} \mu_w$$

where μ_w is the population mean of w. The first component of the bias will be positive and the second negative, given that μ_w is negative. It is not possible without further information to predict the direction of the bias. The standard errors and t statistics will be invalidated if there is substantial measurement error in *MRW*.

Using OLS, with $RGDP_t$ as a proxy for RW.

Suppose $RW = \alpha_1 + \alpha_2 RGDP$. Then the migration equation may be rewritten:

$$\begin{aligned} M_t &= \beta_1 + \beta_2(\alpha_1 + \alpha_2 RGDP_t) + u_t \\ &= (\beta_1 + \alpha_1\beta_2) + \alpha_2\beta_2 RGDP_t + u_t. \end{aligned}$$

In general it would not be possible to derive estimates of either β_1 or β_2. Likewise one has no information on the standard errors of either $\widehat{\beta}_1$ or $\widehat{\beta}_2$. Nevertheless the t statistic for the slope coefficient would be approximately equal to the t statistic in a regression of M on *RW*, if the proxy is a good one. R^2 will be approximately the same as it would have been in a regression of M on *RW*, if the proxy is a good one. One might hypothesise that *RGDP* might be approximately equal to *RW*, in which case $\alpha_1 = 0$ and $\alpha_2 = 1$ and one can effectively fit the original model.

Using instrumental variables, with RGDP_t *as an instrument for* MRW_t.

The IV estimator of β_2 is consistent:

$$\begin{aligned} \widehat{\beta}_2^{\text{IV}} &= \frac{\sum \left(M_i - \overline{M}\right)\left(RGDP_i - \overline{RGDP}\right)}{\sum \left(MRW_i - \overline{MRW}\right)\left(RGDP_i - \overline{RGDP}\right)} \\ &= \beta_2 + \frac{\sum \left(u_i - \beta_2 w_i - \bar{u} + \beta_2 \overline{w}\right)\left(RGDP_i - \overline{RGDP}\right)}{\sum \left(MRW_i - \overline{MRW}\right)\left(RGDP_i - \overline{RGDP}\right)}. \end{aligned}$$

Hence plim $\widehat{\beta}_2^{\text{IV}} = \beta_2$ if u and w are distributed independently of *RGDP*. Likewise the IV estimator of $\widehat{\beta}_1$ is consistent:

$$\widehat{\beta}_1^{\text{IV}} = \overline{M} - \widehat{\beta}_2^{\text{IV}}\overline{MRW} = \beta_1 + \beta_2\overline{RW} + \bar{u} - \widehat{\beta}_2^{\text{IV}}\overline{RW} - \widehat{\beta}_2^{\text{IV}}\overline{w}.$$

Hence:

$$\begin{aligned} \text{plim } \widehat{\beta}_1^{\text{IV}} &= \beta_1 + \beta_2\overline{RW} + \text{ plim } \bar{u} - \text{ plim } \widehat{\beta}_2^{\text{IV}}\overline{RW} - \text{ plim } \widehat{\beta}_2^{\text{IV}} \text{ plim } \overline{w} \\ &= \beta_1 \end{aligned}$$

since plim $\widehat{\beta}_2^{\text{IV}} = \beta_2$ and plim $u =$ plim $w = 0$. The standard errors will be higher, and hence t statistics lower, than they would have been if it had been possible to run the original regression using OLS.

A8.6 Suppose $RGDP = \theta + \phi MRW$. Then:

$$\begin{aligned}
\widehat{\beta}_2^{\mathrm{IV}} &= \frac{\sum \left(M_i - \overline{M}\right)\left(RGDP_i - \overline{RGDP}\right)}{\sum \left(MRW_i - \overline{MRW}\right)\left(RGDP_i - \overline{RGDP}\right)} \\
&= \frac{\sum \left(M_i - \overline{M}\right)\left(\phi MRW_i - \phi\overline{MRW}\right)}{\sum \left(MRW_i - \overline{MRW}\right)\left(\phi MRW_i - \phi\overline{MRW}\right)} \\
&= \widehat{\beta}_2^{\mathrm{OLS}}.
\end{aligned}$$

The instrument is no longer valid because it is correlated with the measurement error.

Chapter 9
Simultaneous equations estimation

9.1 Overview

Until this point the analysis has been confined to the fitting of a single regression equation on its own. In practice, most economic relationships interact with others in a system of simultaneous equations, and when this is the case the application of ordinary least squares (OLS) to a single relationship in isolation yields biased estimates. Having defined what is meant by an endogenous variable, an exogenous variable, a structural equation, and a reduced form equation, the first objective of this chapter is to demonstrate this. The second is to show how it may be possible to use instrumental variables (IV) estimation, with exogenous variables acting as instruments for endogenous ones, to obtain consistent estimates of the coefficients of a relationship. The conditions for exact identification, underidentification, and overidentification are discussed. In the case of overidentification, it is shown how two-stage least squares can be used to obtain estimates that are more efficient than those obtained with simple IV estimation. The chapter concludes with a discussion of the problem of unobserved heterogeneity and the use of the Durbin–Wu–Hausman test in the context of simultaneous equations estimation.

9.2 Learning outcomes

After working through the corresponding chapter in the text, studying the corresponding slideshows, and doing the starred exercises in the text and the additional exercises in this subject guide, you should be able to:

- explain what is meant by:
  - an endogenous variable
  - an exogenous variable
  - a structural equation
  - a reduced form equation
- explain why the application of OLS to a single equation in isolation is likely to yield inconsistent estimates of the coefficients if the equation is part of a simultaneous equations model
- derive an expression for the large-sample bias in the slope coefficient when OLS is used to fit a simple regression equation in a simultaneous equations model

- explain how consistent estimates of the coefficients of an equation in a simultaneous equations model might in principle be obtained using instrumental variables
- explain what is meant by exact identification, underidentification, and overidentification
- explain the principles underlying the use of two-stage least squares, and the reason why it is more efficient than simple IV estimation
- explain what is meant by the problem of unobserved heterogeneity
- perform the Durbin–Wu–Hausman test in the context of simultaneous equations estimation.

9.3 Further material

Good governance and economic development

In development economics it has long been observed that there is a positive association between economic performance, Y, and good governance, R, especially in developing countries. However, quantification of the relationship is made problematic by the fact that it is unlikely that causality is unidirectional. While good governance may contribute to economic performance, better performing countries may also develop better institutions. Hence in its simplest form one has a simultaneous equations mode:

$$\begin{aligned} Y &= \beta_1 + \beta_2 R + u \qquad (1) \\ R &= \alpha_1 + \alpha_2 Y + v \qquad (2) \end{aligned}$$

where u and v are disturbance terms. Assuming that the latter are distributed independently, an OLS regression of the first equation will lead to an upwards biased estimate of β_2, at least in large samples. The proof is left as an exercise (Exercise A9.10). Thus to fit the first equation, one needs an instrument for R. Obviously a better-specified model would have additional explanatory variables in both equations, but there is a problem. In general any variable that influences R is also likely to influence Y and is therefore unavailable as an instrument.

In a study of 64 ex-colonial countries that is surely destined to become a classic, 'The colonial origins of comparative development: an empirical investigation', *American Economic Review* 91(5): 1369–1401, December 2001, Acemoglu, Johnson, and Robinson (henceforward AJR) argue that settler mortality rates provide a suitable instrument. Put simply, the thesis is that where mortality rates were low, European colonisers founded neo-European settlements with European institutions and good governance. Such settlements eventually prospered. Examples are the United States, Canada, Australia, and New Zealand. Where mortality rates were high, on account of malaria, yellow fever and other diseases for which Europeans had little or no immunity, settlements were not viable. In such countries the main objective of the coloniser was economic exploitation, especially of mineral wealth. Institutional development was not a consideration. Post-independence regimes have often been as predatory as their predecessors, indigenous rulers taking the place of the former colonisers. Think of the Belgian Congo, first exploited by King Leopold and more recently by Mobutu.

The study is valuable as an example of IV estimation in that it places minimal technical demands on the reader. There is nothing that would not be easily comprehensible to students in an introductory econometrics course that covers IV. Nevertheless, it gives careful attention to the important technical issues. In particular, it discusses at length the validity of the exclusion restriction. To use mortality as an instrument for R in the first equation, one must be sure that it is not a determinant of Y in its own right, either directly or indirectly (other than through R).

The conclusion of the study is surprising. According to theory (see Exercise A9.10), the OLS estimate of β_2 will be biased upwards by the endogeneity of R. The objective of the study was to demonstrate that the estimate remains positive and significant even when the upward bias has been removed by using IV. However, the IV estimate turns out to be higher than the OLS estimate. In fact it is nearly twice as large. AJR suggest that this is attributable to measurement error in the measurement of R. This would cause the OLS estimate to be biased downwards, and the bias would be removed (asymptotically) by the use of IV. AJR conclude that the downward bias in the OLS estimate caused by measurement error is greater than the upward bias caused by endogeneity.

9.4 Additional exercises

A9.1 In a certain agricultural country, aggregate consumption, C, is simply equal to 2,000 plus a random quantity z that depends upon the weather:

$$C = 2000 + z.$$

z has mean zero and standard deviation 100. Aggregate investment, I, is subject to a four-year trade cycle, starting at 200, rising to 300 at the top of the cycle, and falling to 200 in the next year and to 100 at the bottom of the cycle, rising to 200 again the year after that, and so on. Aggregate income, Y, is the sum of C and I:

$$Y = C + I.$$

Data on C and I, and hence Y, are given in the table. z was generated by taking normally distributed random numbers with mean zero and unit standard deviation and multiplying them by 100.

t	C	I	Y	t	C	I	Y
1	1,813	200	2,013	11	1,981	200	2,181
2	1,893	300	2,193	12	2,211	100	2,311
3	2,119	200	2,319	13	2,127	200	2,327
4	1,967	100	2,067	14	1,953	300	2,253
5	1,997	200	2,197	15	2,141	200	2,341
6	2,050	300	2,350	16	1,836	100	1,936
7	2,035	200	2,235	17	2,103	200	2,303
8	2,088	100	2,188	18	2,058	300	2,358
9	2,023	200	2,223	19	2,119	200	2,319
10	2,144	300	2,444	20	2,032	100	2,132

An orthodox economist regresses C on Y, using the data in the table, and obtains (standard errors in parentheses):

$$\begin{aligned} \widehat{C} &= \underset{(252)}{512} + \underset{(0.11)}{0.68}Y \qquad R^2 = 0.67 \\ & \qquad\qquad\qquad\qquad\quad F = 36.49 \end{aligned}$$

Explain why this result was obtained, despite the fact that C does not depend on Y at all. In particular, comment on the t and F statistics.

A9.2 A small macroeconomic model of a closed economy consists of a consumption function, an investment function, and an income identity:

$$\begin{aligned} C_t &= \beta_1 + \beta_2 Y_t + u_t \\ I_t &= \alpha_1 + \alpha_2 r_t + v_t \\ Y_t &= C_t + I_t + G_t \end{aligned}$$

where C_t is aggregate consumer expenditure in year t, It is aggregate investment, G_t is aggregate current public expenditure, Y_t is aggregate output, and r_t is the rate of interest. State which variables in the model are endogenous and exogenous, and explain how you would fit the equations, if you could.

A9.3 The model is now expanded to include a demand for money equation and an equilibrium condition for the money market:

$$\begin{aligned} M_t^d &= \delta_1 + \delta_2 Y_t + \delta_3 r_t + w_t \\ M_t^d &= \overline{M}_t \end{aligned}$$

where M_t^d is the demand for money in year t and M_t is the supply of money, assumed exogenous. State which variables are endogenous and exogenous in the expanded model and explain how you would fit the equations, including those in Exercise A9.2, if you could.

A9.4 Table 9.2 reports a simulation comparing OLS and IV parameter estimates and standard errors for 10 samples. The reported R^2 (not shown in that table) for the OLS and IV regressions are shown in the table below.

Sample	OLS R^2	IV R^2
1	0.59	0.16
2	0.69	0.52
3	0.78	0.73
4	0.61	0.37
5	0.40	0.06
6	0.72	0.57
7	0.60	0.33
8	0.58	0.44
9	0.69	0.43
10	0.39	0.13

We know that, for large samples, the IV estimator is preferable to the OLS estimator because it is consistent, while the OLS estimator is inconsistent. However, do the smaller OLS standard errors in Table 9.2 and the larger OLS values of R^2 in the present table indicate that OLS is actually preferable for small samples ($n = 20$ in the simulation)?

A9.5 A researcher investigating the relationship between aggregate wages, W, aggregate profits, P, and aggregate income, Y, postulates the following model:

$$\begin{aligned} W &= \beta_1 + \beta_2 Y + u && (1) \\ P &= \alpha_1 + \alpha_2 Y + \alpha_3 K + v && (2) \\ Y &= W + P && (3) \end{aligned}$$

where K is aggregate stock of capital and u and v are disturbance terms that satisfy the usual regression model assumptions and may be assumed to be distributed independently of each other. The third equation is an identity, all forms of income being classified either as wages or as profits. The researcher intends to fit the model using data from a sample of industrialised countries, with the variables measured on a per capita basis in a common currency. K may be assumed to be exogenous.

- Explain why ordinary least squares (OLS) would yield inconsistent estimates if it were used to fit (1) and derive the large-sample bias in the slope coefficient.
- Explain what can be inferred about the finite-sample properties of OLS if used to fit (1).
- Demonstrate mathematically how one might obtain a consistent estimate of β_2 in (1).
- Explain why (2) is not identified (underidentified).
- Explain whether (3) is identified.
- At a seminar, one of the participants asserts that it is possible to obtain an estimate of α_2 even though equation (2) is underidentified. Any change in income that is not a change in wages must be a change in profits, by definition, and so one can estimate α_2 as $(1 - \widehat{\beta}_2)$, where $\widehat{\beta}_2$ is the consistent estimate of β_2 found in the third part of this question. The researcher does not think that this is right but is confused and says that he will look into it after the seminar. What should he have said?

A9.6 A researcher has data on e, the annual average rate of growth of employment, x the annual average rate of growth of output, and p, the annual average rate of growth of productivity, for a sample of 25 countries, the average rates being calculated for the period 1995–2005 and expressed as percentages. The researcher hypothesises that the variables are related by the following model:

$$\begin{aligned} e &= \beta_1 + \beta_2 x + u && (1) \\ x &= e + p. && (2) \end{aligned}$$

The second equation is an identity because p is defined as the difference between x and e. The researcher believes that p is exogenous. The correlation coefficient for x and p is 0.79.

- Explain why the OLS estimator of β_2 would be inconsistent, if the researcher's model is correctly specified. Derive analytically the large-sample bias, and state whether it is possible to determine its sign.
- Explain how the researcher might use p to construct an IV estimator of β_2, that is consistent if p is exogenous. Demonstrate analytically that the estimator is consistent.
- The OLS and IV regressions are summarised below (standard errors in parentheses). Comment on them, making use of your answers to the first two parts of this question.

$$\begin{array}{lcll} \text{OLS} & \widehat{e} = & -0.52 + 0.48x & (3) \\ & & (0.27) \quad (0.08) & \\ \text{IV} & \widehat{e} = & 0.37 + 0.17x & (4) \\ & & (0.42) \quad (0.14) & \end{array}$$

- A second researcher hypothesises that both x and p are exogenous and that equation (2) should be written:

$$e = x - p. \qquad (5)$$

  On the assumption that this is correct, explain why the slope coefficients in (3) and (4) are both biased and determine the direction of the bias in each case.
- Explain what would be the result of fitting (5), regressing e on x and p.

A9.7 A researcher has data from the World Bank *World Development Report 2000* on F, average fertility (average number of children born to each woman during her life), M, under-five mortality (number of children, per 100, dying before reaching the age of 5), and S, average years of female schooling, for a sample of 54 countries. She hypothesises that fertility is inversely related to schooling and positively related to mortality, and that mortality is inversely related to schooling:

$$\begin{aligned} F &= \beta_1 + \beta_2 S + \beta_3 M + u && (1) \\ M &= \alpha_1 + \alpha_2 S + v && (2) \end{aligned}$$

where u and v are disturbance terms that may be assumed to be distributed independently of each other. S may be assumed to be exogenous.

- Derive the reduced form equations for F and M.
- Explain what would be the most appropriate method to fit equation (1).
- Explain what would be the most appropriate method to fit equation (2).

The researcher decides to fit (1) using ordinary least squares, and she decides also to perform a simple regression of F on S, again using ordinary least squares, with the following results (standard errors in parentheses):

$$\begin{array}{lll} \widehat{F} = 4.08 - 0.17S + 0.015M & R^2 = 0.83 & (3) \\ \qquad (0.61) \; (0.04) \;\; (0.003) & & \\ \widehat{F} = 6.99 - 0.36S & R^2 = 0.71 & (4) \\ \qquad (0.39) \; (0.03) & & \end{array}$$

- Explain why the coefficient of S differs in the two equations.
- Explain whether one may validly perform t tests on the coefficients of (4).

At a seminar someone hypothesises that female schooling may be negatively influenced by fertility, especially in the poorer developing countries in the sample, and this would affect (4). To investigate this, the researcher adds the following equation to the model:

$$S = \delta_1 + \delta_2 F + \delta_3 G + w \qquad (5)$$

where G is GNP per capita and w is a disturbance term. She regresses F on S (1) instrumenting for S with G (column (b) in the output below), and (2) using ordinary least squares, as in equation (4) (column (B) in the output below). The correlation between S and G was 0.70. She performs a Durbin–Wu–Hausman test to compare the coefficients.

```
                 ---- Coefficients ----
             |      (b)          (B)            (b-B)     sqrt(diag(V_b-V_B))
             |      IV           OLS         Difference          S.E.
-------------+----------------------------------------------------------------
           S |  -.2965323    -.3637397         .0672074       .0347484
       _cons |   6.162605     6.992907        -.8303019       .4194891
------------------------------------------------------------------------------
     b = consistent under Ho and Ha; obtained from ivreg
     B = inconsistent under Ha, efficient under Ho; obtained from regress

    Test:  Ho:  difference in coefficients not systematic
                chi2(  1) = (b-B)'[(V_b-V_B)^(-1)](b-B)
                          =     3.31
                          Prob>chi2 =      0.1158
```

- Discuss whether G is likely to be a valid instrument.
- What should the researcher's conclusions be with regard to the test?

A9.8 Aggregate demand Q_D for a certain commodity is determined by its price, P, aggregate income, Y, and population, POP:

$$Q_D = \beta_1 + \beta_2 P + \beta_3 Y + \beta_4 POP + u_D$$

and aggregate supply is given by:

$$Q_S = \alpha_1 + \alpha_2 P + u_S$$

where u_D and u_S are independently distributed disturbance terms.

- Demonstrate that the estimator of α_2 will be inconsistent if ordinary least squares (OLS) is used to fit the supply equation, showing that the large-sample bias is likely to be negative.
- Demonstrate that a consistent estimator of α_2 will be obtained if the supply equation is fitted using instrumental variables (IV), using Y as an instrument.

The model is used for a Monte Carlo experiment, with α_2 set equal to 0.2 and suitable values chosen for the other parameters. The table shows the estimates of

α_2 obtained in 10 samples using OLS, using IV with Y as an instrument, using IV with POP as an instrument, and using two-stage least squares (TSLS) with Y and POP. s.e. is standard error. The correlation between P and Y averaged 0.50 across the samples. The correlation between P and POP averaged 0.63 across the samples. Discuss the results obtained.

	OLS		IV with Y		IV with POP		TSLS	
	coef.	s.e.	coef.	s.e.	coef.	s.e.	coef.	s.e.
1	0.15	0.03	0.22	0.05	0.21	0.05	0.21	0.03
2	0.08	0.04	0.24	0.11	0.19	0.08	0.21	0.06
3	0.11	0.02	0.18	0.06	0.19	0.05	0.19	0.04
4	0.16	0.02	0.20	0.04	0.19	0.03	0.19	0.02
5	0.15	0.02	0.27	0.09	0.18	0.04	0.20	0.03
6	0.14	0.03	0.24	0.08	0.18	0.05	0.20	0.04
7	0.20	0.03	0.22	0.05	0.26	0.04	0.25	0.03
8	0.15	0.03	0.21	0.06	0.24	0.05	0.23	0.04
9	0.11	0.02	0.17	0.05	0.14	0.03	0.15	0.03
10	0.17	0.03	0.16	0.05	0.24	0.05	0.20	0.03

A9.9 A researcher has the following data for a sample of 1,000 manufacturing enterprises on the following variables, each measured as an annual average for the period 2001–2005: G, average annual percentage rate of growth of sales; R, expenditure on research and development; and A, expenditure on advertising. R and A are measured as a proportion of sales revenue. He hypothesises the following model:

$$G = \beta_1 + \beta_2 R + \beta_3 A + u_G \qquad (1)$$

$$R = \alpha_1 + \alpha_2 G + u_R \qquad (2)$$

where u_G and u_R are disturbance terms distributed independently of each other.

A second researcher believes that expenditure on quality control, Q, measured as a proportion of sales revenue, also influences the growth of sales, and hence that the first equation should be written:

$$G = \beta_1 + \beta_2 R + \beta_3 A + \beta_4 Q + u_G. \qquad (1*)$$

A and Q may be assumed to be exogenous variables.

- Derive the reduced form equation for G for the first researcher.
- Explain why ordinary least squares (OLS) would be an inconsistent estimator of the parameters of equation (2).
- The first researcher uses instrumental variables (IV) to estimate α_2 in (2). Explain the procedure and demonstrate that the IV estimator of α_2 is consistent.
- The second researcher uses two stage least squares (TSLS) to estimate α_2 in (2). Explain the procedure and demonstrate that the TSLS estimator is consistent.

- Explain why the TSLS estimator used by the second researcher ought to produce 'better' results than the IV estimator used by the first researcher, if the growth equation is given by (1*). Be specific about what you mean by 'better'.

- Suppose that the first researcher is correct and the growth equation is actually given by (1), not (1*). Compare the properties of the two estimators in this case.

- Suppose that the second researcher is correct and the model is given by (1*) and (2), but A is not exogenous after all. Suppose that A is influenced by G:

$$A = \gamma_1 + \gamma_2 G + u_A \qquad (3)$$

where u_A is a disturbance term distributed independently of u_G and u_R. How would this affect the properties of the IV estimator of α_2 used by the first researcher?

A9.10 A researcher has data for 100 workers in a large organisation on hourly earnings, *EARNINGS*, skill level of the worker, *SKILL*, and a measure of the intelligence of the worker, *IQ*. She hypothesises that *LGEARN*, the natural logarithm of *EARNINGS*, depends on *SKILL*, and that *SKILL* depends on *IQ*.

$$\begin{aligned} LGEARN &= \beta_1 + \beta_2 SKILL + u \qquad (1) \\ SKILL &= \alpha_1 + \alpha_2 IQ + v \qquad (2) \end{aligned}$$

where u and v are disturbance terms. The researcher is not sure whether u and v are distributed independently of each other.

- State, with a brief explanation, whether each variable is endogenous or exogenous, and derive the reduced form equations for the endogenous variables.

- Explain why the researcher could use ordinary least squares (OLS) to fit equation (1) if u and v are distributed independently of each other.

- Show that the OLS estimator of β_2 is inconsistent if u and v are positively correlated and determine the direction of the large-sample bias.

- Demonstrate mathematically how the researcher could use instrumental variables (IV) estimation to obtain a consistent estimate of β_2.

- Explain the advantages and disadvantages of using IV, rather than OLS, to estimate β_2, given that the researcher is not sure whether u and v are distributed independently of each other.

- Describe in general terms a test that might help the researcher decide whether to use OLS or IV. What are the limitations of the test?

- Explain whether it is possible for the researcher to fit equation (2) and obtain consistent estimates.

A9.11 This exercise relates to the Further material section.

In general in an introductory econometrics course, issues and problems are treated separately, one at a time. In practice in empirical work, it is common for multiple problems to be encountered simultaneously. When this is the case, the one-at-a-time analysis may no longer be valid. In the case of the AJR study, both endogeneity and measurement error seem to be issues. This exercise looks at both together, within the context of that model.

Let S be the correct good governance variable and let R be the measured variable, with measurement error w. Thus the model may be written:

$$\begin{aligned} Y &= \beta_1 + \beta_2 S + u \\ S &= \alpha_1 + \alpha_2 Y + v \\ R &= S + w. \end{aligned}$$

It may be assumed that w has zero expectation and constant variance σ_w^2 across observations, and that it is distributed independently of S and the disturbance terms in the equations in the model. Investigate the likely direction of the bias in the OLS estimator of β_2 in large samples.

9.5 Answers to the starred exercises in the textbook

9.1 A simple macroeconomic model consists of a consumption function and an income identity:

$$\begin{aligned} C &= \beta_1 + \beta_2 Y + u \\ Y &= C + I \end{aligned}$$

where C is aggregate consumption, I is aggregate investment, Y is aggregate income, and u is a disturbance term. On the assumption that I is exogenous, derive the reduced form equations for C and Y.

Answer:

Substituting for Y in the first equation:

$$C = \beta_1 + \beta_2(C + I) + u.$$

Hence:

$$C = \frac{\beta_1}{1-\beta_2} + \frac{\beta_2 I}{1-\beta_2} + \frac{u}{1-\beta_2}$$

and:

$$Y = C + I = \frac{\beta_1}{1-\beta_2} + \frac{I}{1-\beta_2} + \frac{u}{1-\beta_2}.$$

9.2 It is common to write an earnings function with the logarithm of the hourly wage as the dependent variable and characteristics such as years of schooling, cognitive ability, years of work experience, etc as the explanatory variables. Explain whether such an equation should be regarded as a reduced form equation or a structural equation.

Answer:

In the conventional model of the labour market, the wage rate and the quantity of labour employed are both endogenous variables jointly determined by the interaction of demand and supply. According to this model, the wage equation is a reduced form equation.

9.3 In the simple macroeconomic model:

$$\begin{aligned} C &= \beta_1 + \beta_2 Y + u \\ Y &= C + I \end{aligned}$$

described in Exercise 9.1, demonstrate that OLS would yield inconsistent results if used to fit the consumption function, and investigate the direction of the bias in the slope coefficient.

Answer:

The first step in the analysis of the OLS slope coefficient is to break it down into the true value and error component in the usual way:

$$\widehat{\beta}_2^{\text{OLS}} = \frac{\sum\left(Y_i - \overline{Y}\right)\left(C_i - \overline{C}\right)}{\sum\left(Y_i - \overline{Y}\right)^2} = \beta_2 + \frac{\sum\left(Y_i - \overline{Y}\right)\left(u_i - \bar{u}\right)}{\sum\left(Y_i - \overline{Y}\right)^2}.$$

From the reduced form equation in Exercise 9.1 we see that Y depends on u and hence we will not be able to obtain a closed-form expression for the expectation of the error term. Instead we take plims, having first divided the numerator and the denominator of the error term by n so that they will possess limits as n goes to infinity.

$$\text{plim } \widehat{\beta}_2^{\text{OLS}} = \beta_2 + \frac{\text{plim } \frac{1}{n}\sum\left(Y_i - \overline{Y}\right)\left(u_i - \bar{u}\right)}{\text{plim } \frac{1}{n}\sum\left(Y_i - \overline{Y}\right)^2} = \beta_2 + \frac{\text{cov}(Y, u)}{\text{var(Y)}}.$$

We next substitute for Y since it is an endogenous variable. We have two choices: we could substitute from the structural equation, or we could substitute from the reduced form. If we substituted from the structural equation, in this case the income identity, we would introduce another endogenous variable, C, and we would find ourselves going round in circles. So we must choose the reduced form.

$$\begin{aligned} \text{plim } \widehat{\beta}_2^{\text{OLS}} &= \beta_2 + \frac{\text{cov}\left(\left[\frac{\beta_1}{1-\beta_2} + \frac{I}{1-\beta_2} + \frac{u}{1-\beta_2}\right], u\right)}{\text{var}\left(\frac{\beta_1}{1-\beta_2} + \frac{I}{1-\beta_2} + \frac{u}{1-\beta_2}\right)} \\ &= \beta_2 + \frac{\frac{1}{1-\beta_2}(\text{cov}(I, u) + \text{cov}(u, u))}{\left(\frac{1}{1-\beta_2}\right)^2 \text{var}(I + u)} \\ &= \beta_2 + (1 - \beta_2)\frac{\text{cov}(I, u) + \text{var}(u)}{\text{var}(I) + \text{var}(u) + 2\text{cov}(I, u)}. \end{aligned}$$

On the assumption that I is exogenous, it is distributed independently of u and $\text{cov}(I, u) = 0$. So:

$$\text{plim } \widehat{\beta}_2^{\text{OLS}} = \beta_2 + (1 - \beta_2)\frac{\sigma_u^2}{\sigma_I^2 + \sigma_u^2}$$

since the sample variances tend to the population variances as the sample becomes large. Since the variances are positive, the sign of the bias depends on the sign of $(1 - \beta_2)$. It is reasonable to assume that the marginal propensity to consume is positive and less than 1, in which case this term will be positive and the large-sample bias in $\widehat{\beta}_2^{\text{OLS}}$ will be upwards.

The OLS estimate of the intercept is also inconsistent:

$$\widehat{\beta}_1^{\text{OLS}} = \overline{C} - \widehat{\beta}_2^{\text{OLS}}\overline{Y} = \beta_1 + \beta_2\overline{Y} + \bar{u} - \widehat{\beta}_2^{\text{OLS}}\overline{Y}.$$

Hence:

$$\begin{aligned} \text{plim } \widehat{\beta}_1^{\text{OLS}} &= \beta_1 + (\beta_2 - \text{ plim } \widehat{\beta}_2^{\text{OLS}}) \text{ plim } \overline{Y} \\ &= \beta_1 - (1 - \beta_2)\frac{\sigma_u^2}{\sigma_I^2 + \sigma_u^2} \text{ plim } \overline{Y}. \end{aligned}$$

This is evidently biased downwards, as one might expect, given that the slope coefficient was biased upwards.

9.6 The table gives consumption per capita, C, gross fixed capital formation per capita, I, and gross domestic product per capita, Y, all measured in US\$, for 33 countries in 1998. The output from an OLS regression of C on Y, and an IV regression using I as an instrument for Y, are shown. Comment on the differences in the results.

	C	I	Y		C	I	Y
Australia	15,024	4,749	19,461	South Korea	4,596	1,448	6,829
Austria	19,813	6,787	26,104	Luxembourg	26,400	9,767	42,650
Belgium	18,367	5,174	24,522	Malaysia	1,683	873	3,268
Canada	15,786	4,017	20,085	Mexico	3,359	1,056	4,328
China–PR	446	293	768	Netherlands	17,558	4,865	24,086
China–HK	17,067	7,262	24,452	New Zealand	11,236	2,658	13,992
Denmark	25,199	6,947	32,769	Norway	23,415	9,221	32,933
Finland	17,991	4,741	24,952	Pakistan	389	79	463
France	19,178	4,622	24,587	Philippines	760	176	868
Germany	20,058	5,716	26,219	Portugal	8,579	2,644	9,976
Greece	9,991	2,460	11,551	Spain	11,255	3,415	14,052
Iceland	25,294	6,706	30,622	Sweden	20,687	4487	26,866
India	291	84	385	Switzerland	27,648	7,815	36,864
Indonesia	351	216	613	Thailand	1,226	479	1,997
Ireland	13,045	4,791	20,132	UK	19,743	4,316	23,844
Italy	16,134	4,075	20,580	USA	26,387	6,540	32,377
Japan	21,478	7,923	30,124				

```
. reg C Y
------------------------------------------------------------------------------
    Source |       SS       df       MS              Number of obs =      33
-----------+------------------------------           F(  1,    31) = 1331.29
     Model |  2.5686e+09     1  2.5686e+09           Prob> F       =  0.0000
  Residual |  59810749.2    31  1929379.01           R-squared     =  0.9772
-----------+------------------------------           Adj R-squared =  0.9765
     Total |  2.6284e+09    32  82136829.4           Root MSE      =    1389
------------------------------------------------------------------------------
         C |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-----------+------------------------------------------------------------------
         Y |   .7303066   .0200156    36.49   0.000     .6894845    .7711287
     _cons |   379.4871   443.6764     0.86   0.399     -525.397    1284.371
------------------------------------------------------------------------------
```

```
. ivregress 2sls C (Y=I)
-------------------------------------------------------------------------------
Instrumental variables (2SLS) regression             Number of obs =      33
                                                     Wald chi2(1)  = 1269.09
Prob> chi2    =  0.0000
                                                     R-squared     =  0.9770
                                                     Root MSE      =  1353.9
------------------------------------------------------------------------------
         C |      Coef.   Std. Err.      z    P>|z|     [95% Conf. Interval]
-----------+------------------------------------------------------------------
         Y |   .7183909   .0201658    35.62   0.000     .6788667    .7579151
     _cons |    600.946   442.7386     1.36   0.175    -266.8057    1468.698
------------------------------------------------------------------------------
Instrumented:  Y
Instruments:   I
------------------------------------------------------------------------------
```

Answer:

Assuming the simple macroeconomic model:

$$\begin{aligned} C &= \beta_1 + \beta_2 Y + u \\ Y &= C + I \end{aligned}$$

where C is consumption per capita, I is investment per capita, and Y is income per capita, and I is assumed exogenous, the OLS estimator of the marginal propensity to consume will be biased upwards. As was shown in Exercise 9.3:

$$\operatorname{plim} \widehat{\beta}_2^{\mathrm{OLS}} = \beta_2 + (1 - \beta_2)\frac{\sigma_u^2}{\sigma_I^2 + \sigma_u^2}.$$

Hence the IV estimate should be expected to be lower, but only by a small amount, given the data. With $\widehat{\beta}_2$ estimated at 0.72, $(1 - \widehat{\beta}_2)$ is 0.28. σ_u^2 is estimated at 1.95 million and σ_I^2 is 7.74 million. Hence, on the basis of these estimates, the bias should be about 0.06. The actual difference in the OLS and IV estimates is smaller still. However, the actual difference would depend on the purely random sampling error as well as the bias, and it is possible that in this case the sampling error happens to have offset the bias to some extent.

9.11 Consider the price inflation/wage inflation model given by equations (9.1) and (9.2):

$$\begin{aligned} p &= \beta_1 + \beta_2 w + u_p \\ w &= \alpha_1 + \alpha_2 p + \alpha_3 U + u_w . \end{aligned}$$

We have seen that the first equation is exactly identified, U being used as an instrument for w. Suppose that TSLS is applied to this model, despite the fact that it is exactly identified, rather than overidentified. How will the results differ?

Answer:

If we fit the reduced form, we obtain a fitted equation:

$$\widehat{w} = h_1 + h_2 U.$$

The TSLS estimator is then given by

$$\begin{aligned} \widehat{\beta}_2^{\mathrm{TSLS}} &= \frac{\sum \left(\widehat{w}_i - \overline{\widehat{w}}\right)(p_i - \bar{p})}{\sum \left(\widehat{w}_i - \overline{\widehat{w}}\right)(w_i - \bar{w})} = \frac{\sum \left(h_1 + h_2 U_i - h_1 - h_2 \overline{U}\right)(p_i - \bar{p})}{\sum \left(h_1 + h_2 U_i - h_1 - h_2 \overline{U}\right)(w_i - \bar{w})} \\ &= \frac{\sum h_2 \left(U_i - \overline{U}\right)(p_i - \bar{p})}{\sum h_2 \left(U_i - \overline{U}\right)(w_i - \bar{w})} = \widehat{\beta}_2^{\mathrm{IV}} \end{aligned}$$

where $\widehat{\beta}_2^{\mathrm{IV}}$ is the IV estimator using U. Hence the estimator is exactly the same. [Note: This is a special case of Exercise 8.18.]

9.15 Suppose the first equation in the model in Box 9.2 is fitted, with Q used as an instrument for Y. Describe the likely properties of the estimator of α_2.

Answer:

The first equation in Box 9.2 is:

$$X = \alpha_1 + \alpha_2 Y + u$$

The reduced form equation for Y is:

$$Y = \frac{1}{1 - \alpha_2 \beta_2}(\beta_1 + \alpha_1 \beta_2 + \beta_2 u + v).$$

Q is not a valid instrument for Y because it is not a determinant of Y. Mathematically, it can be shown that:

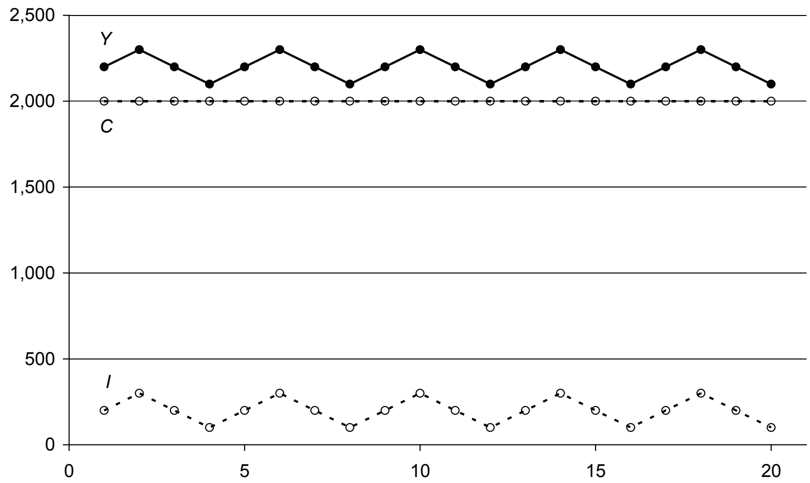
$$\operatorname{plim} \widehat{\alpha}_2^{\mathrm{IV}} = \alpha_2 + \frac{\operatorname{cov}(Q, u)}{\operatorname{cov}(Q, Y)}.$$

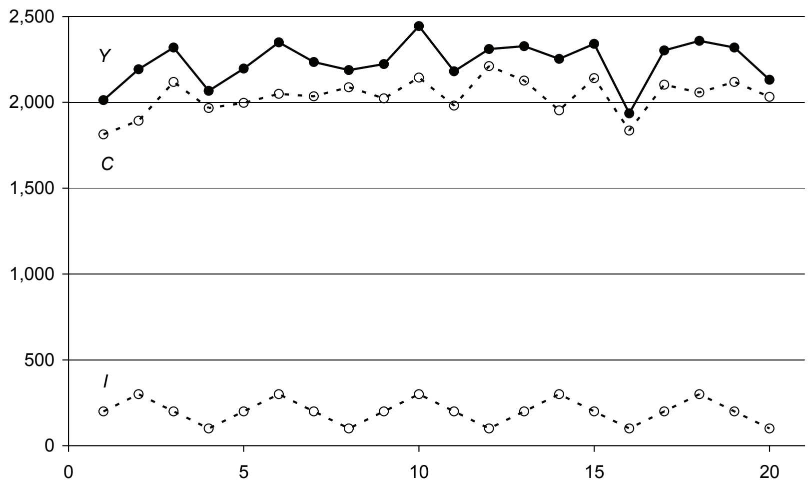
The numerator of the second term is zero, but so is its denominator and therefore the expression is undefined.

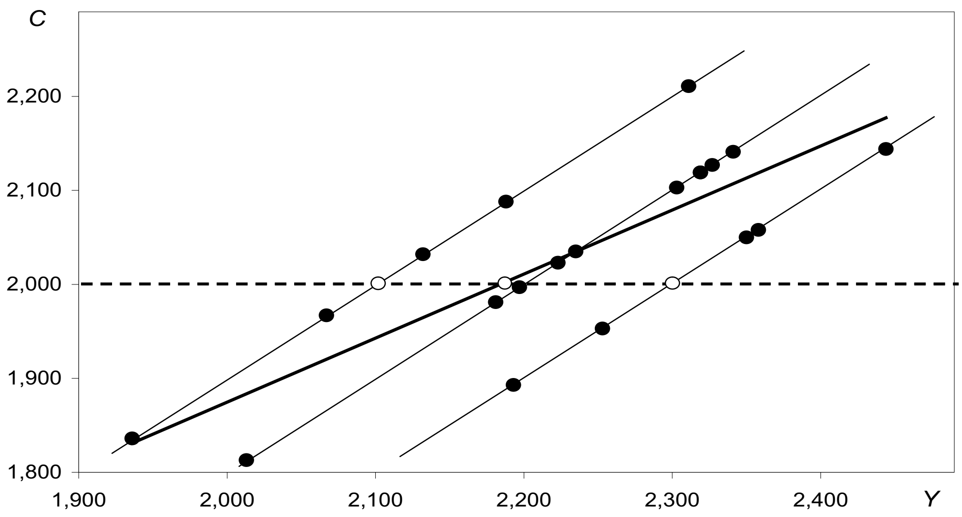
9.6 Answers to the additional exercises

A9.1 The positive coefficient of Y_t in the regression is attributable wholly to simultaneous equations bias. The three figures show this graphically.

The first diagram shows what the time series for C_t, I_t, and Y_t would look like if there were no random component of consumption. The series for C_t is constant at 2,000. That for I_t is a wave form, and that for Y_t is the same wave form shifted upward by 2,000. The second diagram shows the effect of adding the random component to consumption. Y_t still has a wave form, but there is a clear correlation between it and C_t.







In the third diagram, C_t is plotted against Y_t, with and without the random component. The three large circles represent the data when there is no random component. One circle represents the five data points [C = 2,000, Y = 2,100]; the middle circle represents the ten data points [C = 2,000, Y = 2,200]; and the other circle represents the five data points [C = 2,000, Y = 2,300]. A regression line based on these three points would be horizontal (the dashed line). The solid circles represent the 20 data points when the random component is affecting C_t and Y_t, and the solid line is the regression line for these points. Note that these 20 data points fall into three groups: five which lie on a 45 degree line through the left large circle, 10 which lie on the 45 degree line through the middle circle (actually, you can only see nine), and five on the 45 degree line through the right circle.

If OLS is used to fit the equation:

$$\widehat{\beta}_2^{\mathrm{OLS}} = \frac{\sum\left(Y_i - \overline{Y}\right)\left(C_i - \overline{C}\right)}{\sum\left(Y_i - \overline{Y}\right)^2} = \frac{\sum\left(Y_i - \overline{Y}\right)\left([2000 + z_i] - [2000 + \bar{z}]\right)}{\sum\left(Y_i - \overline{Y}\right)^2} = \frac{\sum\left(Y_i - \overline{Y}\right)(z_i - \bar{z})}{\sum\left(Y_i - \overline{Y}\right)^2}.$$

Note that at this stage we have broken down the slope coefficient into its true value plus an error term. The true value does not appear explicitly because it is zero, so we only have the error term. We cannot take expectations because both the numerator and the denominator are functions of z:

$$Y = C + I = 2000 + I + z.$$

z is a component of C and hence of Y. As a second-best procedure, we investigate the large-sample properties of the estimator by taking plims. We must first divide the numerator and denominator by n so that they tend to finite limits:

$$\operatorname{plim}\widehat{\beta}_2^{\mathrm{OLS}} = \frac{\operatorname{plim}\frac{1}{n}\sum\left(Y_i - \overline{Y}\right)(z_i - \bar{z})}{\operatorname{plim}\frac{1}{n}\sum\left(Y_i - \overline{Y}\right)^2} = \frac{\operatorname{cov}(Y, z)}{\operatorname{var}(Y)}.$$

Substituting for Y from its reduced form equation:

$$\operatorname{plim}\widehat{\beta}_2^{\mathrm{OLS}} = \frac{\operatorname{cov}([2000 + i + z], z)}{\operatorname{var}(2000 + I + z)} = \frac{\operatorname{cov}(I, z) + \operatorname{var}(z)}{\operatorname{var}(I) + \operatorname{var}(z) + 2\operatorname{cov}(I, z)} = \frac{\sigma_z^2}{\sigma_I^2 + \sigma_z^2}.$$

$\text{cov}(I, z) = 0$ because I is distributed independently of z. σ_z^2 is equal to 10,000 (since we are told that σ_z is equal to 100). Over a four-year cycle, the mean value of I is 200 and hence its population variance is given by:

$$\sigma_I^2 = \frac{1}{4}\left[0 + 100^2 + 0 + (-100)^2\right] = 5000.$$

Hence:

$$\text{plim } \widehat{\beta}_2^{\text{OLS}} = \frac{10000}{15000} = 0.67.$$

The actual coefficient in the 20-observation sample, 0.68, is very close to this (probably atypically close for such a model).

The estimator of the intercept, whose true value is 2,000, is biased downwards because $\widehat{\beta}_2^{\text{OLS}}$ is biased upwards. The standard errors of the coefficients are invalid because the regression model assumption B.7 is violated, and hence t tests would be invalid.

By virtue of the fact that $Y = C + I$, C is being regressed against a variable which is largely composed of itself. Hence the high R^2 is inevitable, despite the fact that there is no behavioural relationship between C and Y. Mathematically, R^2 is equal to the square of the sample correlation between the actual and fitted values of C. Since the fitted values of C are a linear function of the values of Y, R^2 is equal to the square of the sample correlation between C and Y. The population correlation coefficient is given by

$$\begin{aligned}
\rho_{C,Y} &= \frac{\text{cov}(C, Y)}{\sqrt{\text{var}(C)\text{var}(Y)}} = \frac{\text{cov}\left([2000 + z], [2000 + I + z]\right)}{\sqrt{\text{var}\left([2000 + z]\right)\text{var}\left([2000 + I + z]\right)}} \\
&= \frac{\text{var}(z)}{\sqrt{\text{var}(z)\text{var}[I + z]}} = \frac{\sigma_z^2}{\sqrt{\sigma_z^2\left(\sigma_I^2 + \sigma_z^2\right)}}.
\end{aligned}$$

Hence in large samples:

$$R^2 = \frac{10000^2}{10000[10000 + 5000]} = 0.67.$$

R^2 in the regression is exactly equal to this, the closeness probably being something of a coincidence.

Since regression model assumption B.7 is violated, the F statistic cannot be used to perform an F test of goodness of fit.

A9.2 C_t, I_t, and Y_t are endogenous, the first two being the dependent variables of the behavioural relationships and the third being defined by an identity. G_t and r_t are exogenous.

Either I_t or r_t could be used as an instrument for Y_t in the consumption function. If it can be assumed that u_t and v_t are distributed independently, It can also be regarded as exogenous as far as the determination of C_t and Y_t are concerned. It would be preferable to r_t since it is more highly correlated with Y_t. One's first thought, then, would be to use TSLS, with the first stage fitting the equation:

$$Y_t = \frac{\beta_1}{1 - \beta_2} + \frac{I_t}{1 - \beta_2} + \frac{G_t}{1 - \beta_2} + \frac{u_t}{1 - \beta_2}.$$

Note, however, that the equation implies the restriction that the coefficients of I_t and G_t are equal. Hence all one has to do is to define a variable:

$$Z_t = I_t + G_t$$

and use Z_t as an instrument for Y_t in the consumption function.

The investment function would be fitted using OLS since r_t is exogenous. The income identity does not need to be fitted.

A9.3 M_t^d is endogenous because it is determined by the second of the two new relationships. The addition of the first of these relationships makes r_t endogenous. To see this, substituting for C_t and I_t in the income identity, using the consumption function and the investment function, one obtains:

$$Y_t = \frac{(\alpha_1 + \beta_1) + \alpha_2 r_t + u_t + v_t}{1 - \beta_2}.$$

This is usually known as the IS curve. Substituting for M_t^d in the first of the two new relationships, using the second, one has:

$$\overline{M}_t = \delta_1 + \delta_2 Y_t + \delta_3 r_t + w_t.$$

This is usually known as the LM curve. The equilibrium values of both Y_t and r_t are determined by the intersection of these two curves and hence r_t is endogenous as well as Y_t. G_t remains exogenous, as before, and M_t is also exogenous.

The consumption and investment functions are overidentified and one would use TSLS to fit them, the exogenous variables being government expenditure and the supply of money. The demand for money equation is exactly identified, two of the explanatory variables, r_t and Y_t, being endogenous, and the two exogenous variables being available to act as instruments for them.

A9.4 The OLS standard errors are invalid so a comparison is illegitimate. They are not of any great interest anyway because the OLS estimator is biased. Figure 9.3 in the text shows that the variance of the OLS estimator is smaller than that of the IV estimator, but, using a criterion such as the mean square error, there is no doubt that the IV estimator should be preferred. The comment about R^2 is irrelevant. OLS has a better fit but we have had to abandon the least squares principle because it yields inconsistent estimates.

A9.5 *Explain why ordinary least squares (OLS) would yield inconsistent estimates if it were used to fit (1) and derive the large-sample bias in the slope coefficient.*

At some point we will need the reduced form equation for Y. Substituting into the third equation from the first two, and rearranging, it is:

$$Y = \frac{1}{1 - \alpha_2 - \beta_2}(\alpha_1 + \beta_1 + \alpha_3 K + u + v).$$

Since Y depends on u, the assumption that the disturbance term be distributed independently of the regressors is violated in (1).

$$\widehat{\beta}_2^{\text{OLS}} = \frac{\sum\left(Y_i - \overline{Y}\right)\left(W_i - \overline{W}\right)}{\sum\left(Y_i - \overline{Y}\right)^2} = \beta_2 + \frac{\sum\left(Y_i - \overline{Y}\right)(u_i - \overline{u})}{\sum\left(Y_i - \overline{Y}\right)^2}$$

after substituting for W from (1) and simplifying. We are not able to obtain a closed-form expression for the expectation of the error term because u influences both its numerator and denominator, directly and by virtue of being a component of Y, as seen in the reduced form. Dividing both the numerator and denominator by n, and noting that:

$$\text{plim}\ \frac{1}{n}\sum\left(Y_i-\overline{Y}\right)^2=\text{var}(Y)$$

as a consequence of a law of large numbers, and that it can also be shown that:

$$\text{plim}\ \frac{1}{n}\sum\left(Y_i-\overline{Y}\right)(u_i-\bar{u})=\text{cov}(Y,u)$$

we can write

$$\text{plim}\ \widehat{\beta}_2^{\text{OLS}}=\beta_2+\frac{\text{plim}\ \frac{1}{n}\sum\left(Y_i-\overline{Y}\right)(u_i-\bar{u})}{\text{plim}\ \frac{1}{n}\sum\left(Y_i-\overline{Y}\right)^2}=\beta_2+\frac{\text{cov}(Y,u)}{\text{var}(Y)}.$$

Now:

$$\begin{aligned}\text{cov}(Y,u)&=\text{cov}\left(\frac{1}{1-\alpha_2-\beta_2}(\alpha_1+\beta_1+\alpha_3K+u+v),u\right)\\&=\frac{1}{1-\alpha_2-\beta_2}(\alpha_3\text{cov}(K,u)+\text{var}(u)+\text{cov}(v,u))\end{aligned}$$

the covariance of u with the constants being zero. Since K is exogenous, $\text{cov}(K,u)=0$. We are told that u and v are distributed independently of each other, and so $\text{cov}(u,v)=0$. Hence:

$$\text{plim}\ \widehat{\beta}_2^{\text{OLS}}=\beta_2+\frac{1}{1-\alpha_2-\beta_2}\frac{\sigma_u^2}{\text{plim var}(Y)}.$$

From the reduced form equation for Y it is evident that $(1-\alpha_2-\beta_2)>0$, and so the large-sample bias will be positive.

Explain what can be inferred about the finite-sample properties of OLS if used to fit (1).

It is not possible for an estimator that is unbiased in a finite sample to develop a bias if the sample size increases. Therefore, since the estimator is biased in large samples, it must also be biased in finite ones. The plim may well be a guide to the mean of the estimator in a finite sample, but this is not guaranteed and it is unlikely to be exactly equal to the mean.

Demonstrate mathematically how one might obtain a consistent estimate of β_2 in (1).

Use K as an instrument for Y:

$$\widehat{\beta}_2^{\text{IV}}=\frac{\sum\left(K_i-\overline{K}\right)\left(W_i-\overline{W}\right)}{\sum\left(K_i-\overline{K}\right)\left(Y_i-\overline{Y}\right)}=\beta_2+\frac{\sum\left(K_i-\overline{K}\right)(u_i-\bar{u})}{\sum\left(K_i-\overline{K}\right)\left(Y_i-\overline{Y}\right)}$$

after substituting for W from (1) and simplifying. We are not able to obtain a closed-form expression for the expectation of the error term because u influences both its numerator and denominator, directly and by virtue of being a component of Y, as seen in the reduced form. Dividing both the numerator and denominator by n, and noting that it can be shown that:

$$\operatorname{plim} \frac{1}{n}\sum \left(K_i - \overline{K}\right)(u_i - \overline{u}) = \operatorname{cov}(K, u) = 0$$

since K is exogenous, and that:

$$\operatorname{plim} \frac{1}{n}\sum \left(K_i - \overline{K}\right)\left(Y_i - \overline{Y}\right) = \operatorname{cov}(K, Y)$$

we can write:

$$\operatorname{plim} \widehat{\beta}_2^{\mathrm{IV}} = \beta_2 + \frac{\operatorname{cov}(K, u)}{\operatorname{cov}(K, Y)} = \beta_2.$$

$\operatorname{cov}(K, Y)$ is non-zero since the reduced form equation for Y reveals that K is a determinant of Y. Hence the instrumental variable estimator is consistent.

Explain why (2) is not identified (underidentified).

(2) is underidentified because the endogenous variable Y is a regressor and there is no valid instrument to use with it. The only potential instrument is the exogenous variable K and it is already a regressor in its own right.

Explain whether (3) is identified.

(3) is an identity so the issue of identification does not arise.

At a seminar, one of the participants asserts that it is possible to obtain an estimate of α_2 even though equation (2) is underidentified. Any change in income that is not a change in wages must be a change in profits, by definition, and so one can estimate α_2 as $(1 - \widehat{\beta}_2)$, where $\widehat{\beta}_2$ is the consistent estimate of β_2 found in the third part of this question. The researcher does not think that this is right but is confused and says that he will look into it after the seminar. What should he have said?

The argument would be valid if Y were exogenous, in which case one could characterise β_2 and α_2 as being the effects of Y on W and P, holding other variables constant. But Y is endogenous, and so the coefficients represent only part of an adjustment process. Y cannot change autonomously, only in response to variations in K, u, or v.

The reduced form equations for W and P are:

$$\begin{aligned}
W &= \beta_1 + \frac{\beta_2}{1-\alpha_2-\beta_2}(\alpha_1 + \beta_1 + \alpha_3 K + u + v) + u \\
&= \frac{1}{1-\alpha_2-\beta_2}(\beta_1 + \alpha_1\beta_2 - \alpha_2\beta_1 + \alpha_3\beta_2 K + (1-\alpha_2)u + \beta_2 v) \\
P &= \alpha_1 + \frac{\alpha_2}{1-\alpha_2-\beta_2}(\alpha_1 + \beta_1 + \alpha_3 K + u + v) + \alpha_3 K + v \\
&= \frac{1}{1-\alpha_2-\beta_2}(\alpha_1 - \alpha_1\beta_2 + \alpha_2\beta_1 + \alpha_3(1-\beta_2)K + \alpha_2 u + (1-\beta_2)v).
\end{aligned}$$

Thus, for example, a change in K will lead to changes in W and P in the proportions $\beta_2 : (1 - \beta_2)$, not $\beta_2 : \alpha_2$. The same is true of changes caused by a variation in v. For a variation in u, the proportions would be $(1 - \alpha_2) : \alpha_2$.

A9.6 *Explain why the OLS estimator of β_2 would be inconsistent, if the researcher's model is correctly specified. Derive analytically the largesample bias, and state whether it is possible to determine its sign.*

The reduced form equation for x is:

$$x = \frac{\beta_1 + p + u}{1 - \beta_2}.$$

Thus:

$$\begin{aligned}
\widehat{\beta}_2^{\text{OLS}} &= \frac{\sum (x_i - \bar{x})(e_i - \bar{e})}{\sum (x_i - \bar{x})^2} = \frac{\sum (x_i - \bar{x})(\beta_1 + \beta_2 x_i + u_i - \beta_1 - \beta_2 \bar{x} - \bar{u})}{\sum (x_i - \bar{x})^2} \\
&= \beta_2 + \frac{\sum (x_i - \bar{x})(u_i - \bar{u})}{\sum (x_i - \bar{x})^2}.
\end{aligned}$$

It is not possible to obtain a closed-form expression for the expectation of the estimator because the error term is a nonlinear function of u. Instead we investigate whether the estimator is consistent, first dividing the numerator and the denominator of the error term by n so that they tend to limits as the sample size becomes large.

$$\begin{aligned}
\text{plim } \widehat{\beta}_2^{\text{OLS}} &= \beta_2 + \frac{\text{plim } \frac{1}{n} \sum \left(\frac{1}{1-\beta_2} [\beta_1 + p_i + u_i - \beta_1 - \bar{p} - \bar{u}] \right) (u_i - \bar{u})}{\text{plim } \frac{1}{n} \sum (x_i - \bar{x})^2} \\
&= \beta_2 + \frac{1}{1 - \beta_2} \frac{\text{plim } \frac{1}{n} \sum (p_i - \bar{p})(u_i - \bar{u}) + \text{ plim } \frac{1}{n} \sum (u_i - \bar{u})^2}{\text{plim } \frac{1}{n} \sum (x_i - \bar{x})^2} \\
&= \beta_2 + \frac{1}{1 - \beta_2} \frac{\text{cov}(p, u) + \text{var}(u)}{\text{var}(x)} = \beta_2 + \frac{1}{1 - \beta_2} \frac{\sigma_u^2}{\sigma_x^2}
\end{aligned}$$

since $\text{cov}(p, u) = 0$, p being exogenous. It is reasonable to assume that employment grows less rapidly than output, and hence β_2, and so $(1 - \beta_2)$, are less than 1. The bias is therefore likely to be positive.

Explain how the researcher might use p to construct an IV estimator of β_2 that is consistent if p is exogenous. Demonstrate analytically that the estimator is consistent.

p is available as an instrument, being exogenous, and therefore independent of u, being correlated with x, and not being in the equation in its own right.

$$\begin{aligned}
\widehat{\beta}_2^{\text{IV}} &= \frac{\sum (p_i - \bar{p})(e_i - \bar{e})}{\sum (p_i - \bar{p})(x_i - \bar{x})} = \frac{\sum (p_i - \bar{p})(\beta_1 + \beta_2 x_i + u_i - \beta_1 - \beta_2 \bar{x} - \bar{u})}{\sum (p_i - \bar{p})(x_i - \bar{x})} \\
&= \beta_2 + \frac{\sum (p_i - \bar{p})(u_i - \bar{u})}{\sum (p_i - \bar{p})(x_i - \bar{x})}.
\end{aligned}$$

Hence, dividing the numerator and the denominator of the error term by n so that they tend to limits as the sample size becomes large,

$$\text{plim } \widehat{\beta}_2^{\text{IV}} = \beta_2 + \frac{\text{plim } \frac{1}{n}\sum(p_i - \bar{p})(u_i - \bar{u})}{\text{plim } \frac{1}{n}\sum(p_i - \bar{p})(x_i - \bar{x})} = \beta_2 + \frac{\text{cov}(p, u)}{\text{cov}(p, x)} = \beta_2$$

since $\text{cov}(p, u) = 0$, p being exogenous, and $\text{cov}(p, x) \neq 0$, x being determined partly by p.

The OLS and IV regressions are summarised below (standard errors in parentheses). Comment on them, making use of your answers to the first two parts of this question.

$$\begin{array}{lcll} \text{OLS} \quad & \widehat{e} & = & -0.52 + 0.48x \qquad (3) \\ & & & (0.27) \quad (0.08) \\ \text{IV} \quad & \widehat{e} & = & 0.37 + 0.17x \qquad (4) \\ & & & (0.42) \quad (0.14) \end{array}$$

The IV estimate of the slope coefficient is lower than the OLS estimate, as expected. The standard errors are not comparable because the OLS ones are invalid.

A second researcher hypothesises that both x and p are exogenous and that equation (2) should be written:

$$e = x - p. \qquad (5)$$

On the assumption that this is correct, explain why the slope coefficients in (3) and (4) are both biased and determine the direction of the bias in each case.

If (5) is correct, (3) is a misspecification that omits p and includes a redundant intercept. From the identity, the true values of the coefficients of x and p are 1 and -1, respectively. For (3):

$$E(\widehat{\beta}_2^{\text{OLS}}) = 1 - 1 \times \frac{\sum(x_i - \bar{x})(p_i - \bar{p})}{\sum(x_i - \bar{x})^2}.$$

x and p are positively correlated, so the bias will be downwards.

For (4):

$$\begin{aligned} \widehat{\beta}_2^{\text{IV}} &= \frac{\sum(p_i - \bar{p})(e_i - \bar{e})}{\sum(p_i - \bar{p})(x_i - \bar{x})} = \frac{\sum(p_i - \bar{p})([x_i - p_i] - [\bar{x} - \bar{p}])}{\sum(p_i - \bar{p})(x_i - \bar{x})} \\ &= 1 - \frac{\sum(p_i - \bar{p})^2}{\sum(p_i - \bar{p})(x_i - \bar{x})} = 1 - \frac{\frac{1}{n}\sum(p_i - \bar{p})^2}{\frac{1}{n}\sum(p_i - \bar{p})(x_i - \bar{x})}. \end{aligned}$$

Hence:

$$\text{plim } \widehat{\beta}_2^{\text{IV}} = 1 - \frac{\text{var}(p)}{\text{cov}(x, p)}$$

and so again the bias is downwards.

Explain what would be the result of fitting (5), regressing e on x and p.

One would obtain a perfect fit with the coefficient of x equal to 1, the coefficient of p equal to -1, and $R^2 = 1$.

A9.7 *Derive the reduced form equations for F and M.*

(2) is the reduced form equation for M. Substituting for M in (1), we have:

$$F = (\beta_1 + \alpha_1\beta_3) + (\beta_2 + \alpha_2\beta_3)S + u + \beta_3 v.$$

Explain what would be the most appropriate method to fit equation (1).

Since M does not depend on u, OLS may be used to fit (1).

Explain what would be the most appropriate method to fit equation (2).

There are no endogenous explanatory variables in (2), so again OLS may be used.

Explain why the coefficient of S differs in the two equations.

In (3), the coefficient is an estimate of the direct effect of S on fertility, controlling for M. In (4), the reduced form equation, it is an estimate of the total effect, taking account of the indirect effect via M (female education reduces mortality, and a reduction in mortality leads to a reduction in fertility).

Explain whether one may validly perform t tests on the coefficients of (4).

It is legitimate to use OLS to fit (4), so the t tests are valid.

Discuss whether G is likely to be a valid instrument.

G should be a valid instrument since it is highly correlated with S, it may reasonably be considered to be exogenous and therefore uncorrelated with the disturbance term in (4), and it does not appear in the equation in its own right (though perhaps it should).

What should the researchers conclusions be with regard to the test?

With 1 degree of freedom as indicated by the output, the critical value of chi-squared at the 5 per cent significance level is 3.84. Therefore we do not reject the null hypothesis of no significant difference between the estimates of the coefficients and conclude that there is no need to instrument for S. (4) should be preferred because OLS is more efficient than IV, when both are consistent.

A9.8 *Demonstrate that the estimate of α_2 will be inconsistent if ordinary least squares (OLS) is used to fit the supply equation, showing that the large-sample bias is likely to be negative.*

The reduced form equation for P is:

$$P = \frac{1}{\alpha_2 - \beta_2}(\beta_1 - \alpha_1 + \beta_3 Y + \beta_4 POP + u_D - u_S).$$

The OLS estimator of α_2 is:

$$\begin{aligned}
\widehat{\alpha}_2^{\text{OLS}} &= \frac{\sum\left(P_i - \overline{P}\right)\left(Q_i - \overline{Q}\right)}{\sum\left(P_i - \overline{P}\right)^2} = \frac{\sum\left(P_i - \overline{P}\right)\left(\alpha_1 + \alpha_2 P_i + u_{Si} - \alpha_1 - \alpha_2\overline{P} - \bar{u}_S\right)}{\sum\left(P_i - \overline{P}\right)^2} \\
&= \alpha_2 + \frac{\sum\left(P_i - \overline{P}\right)(u_{Si} - \bar{u}_S)}{\sum\left(P_i - \overline{P}\right)^2}.
\end{aligned}$$

We cannot take expectations because u_S is a determinant of both the numerator and the denominator of the error term, in view of the reduced form equation for P. Instead, we take probability limits, after first dividing the numerator and the denominator of the error term by n to ensure that limits exist.

$$\operatorname{plim} \widehat{\alpha}_2^{\text{OLS}} = \alpha_2 + \frac{\operatorname{plim} \frac{1}{n}\sum\left(P_i - \overline{P}\right)(u_{Si} - \bar{u}_S)}{\operatorname{plim} \frac{1}{n}\sum\left(P_i - \overline{P}\right)^2} = \alpha_2 + \frac{\operatorname{cov}(P, u_S)}{\operatorname{var}(P)}.$$

Substituting from the reduced form equation for P:

$$\begin{aligned}\operatorname{plim} \widehat{\alpha}_2^{\text{OLS}} &= \alpha_2 + \frac{\operatorname{cov}\left(\frac{1}{\alpha_2-\beta_2}(\beta_1 - \alpha_1 + \beta_3 Y + \beta_4 POP + u_D - u_S), u_S\right)}{\operatorname{var}(P)} \\ &= \alpha_2 - \frac{\frac{1}{\alpha_2-\beta_2}\operatorname{var}(u_S)}{\operatorname{var}(P)} = \alpha_2 - \frac{1}{\alpha_2 - \beta_2}\frac{\sigma_{u_S}^2}{\sigma_P^2}\end{aligned}$$

assuming that Y and POP are exogenous and so $\operatorname{cov}(u_S, Y) = \operatorname{cov}(u_S, POP) = 0$. We are told that u_S and u_D are distributed independently, so $\operatorname{cov}(u_S, u_D) = 0$. Since it is reasonable to suppose that α_2 is positive and β_2 is negative, the large-sample bias will be negative.

Demonstrate that a consistent estimate of α_2 will be obtained if the supply equation is fitted using instrumental variables (IV), using Y as an instrument.

$$\begin{aligned}\widehat{\alpha}_2^{\text{IV}} &= \frac{\sum\left(Y_i - \overline{Y}\right)\left(Q_i - \overline{Q}\right)}{\sum\left(Y_i - \overline{Y}\right)\left(P_i - \overline{P}\right)} = \frac{\sum\left(Y_i - \overline{Y}\right)\left(\alpha_1 + \alpha_2 P_i + u_{Si} - \alpha_1 - \alpha_2\overline{P} - \bar{u}_S\right)}{\sum\left(Y_i - \overline{Y}\right)\left(P_i - \overline{P}\right)} \\ &= \alpha_2 + \frac{\sum\left(Y_i - \overline{Y}\right)(u_{Si} - \bar{u}_S)}{\sum\left(Y_i - \overline{Y}\right)\left(P_i - \overline{P}\right)}.\end{aligned}$$

We cannot take expectations because u_S is a determinant of both the numerator and the denominator of the error term, in view of the reduced form equation for P. Instead, we take probability limits, after first dividing the numerator and the denominator of the error term by n to ensure that limits exist.

$$\operatorname{plim} \widehat{\alpha}_2^{\text{IV}} = \alpha_2 + \frac{\operatorname{plim} \frac{1}{n}\sum\left(Y_i - \overline{Y}\right)(u_{Si} - \bar{u}_S)}{\operatorname{plim} \frac{1}{n}\sum\left(Y_i - \overline{Y}\right)\left(P_i - \overline{P}\right)} = \alpha_2 + \frac{\operatorname{cov}(Y, u)}{\operatorname{cov}(Y, P)} = \alpha_2$$

since $\operatorname{cov}(Y, u_s) = 0$ and $\operatorname{cov}(P, Y) \neq 0$, Y being a determinant of P.

The model is used for a Monte Carlo experiment ... Discuss the results obtained.

- The OLS estimates are clearly biased downwards.
- The IV and TSLS estimates appear to be distributed around the true value, although one would need a much larger number of samples to be sure of this.
- The IV estimates with POP appear to be slightly closer to the true value than those with Y, as should be expected given the higher correlation, and the TSLS estimates appear to be slightly closer than either, again as should be expected.

- The OLS standard errors should be ignored. The standard errors for the IV regressions using POP tend to be smaller than those using Y, reflecting the fact that POP is a better instrument. Those for the TSLS regressions are smallest of all, reflecting its greater efficiency.

A9.9 *Derive the reduced form equation for G for the first researcher.*

$$G = \frac{1}{1-\alpha_2\beta_2}(\beta_1 + \alpha_1\beta_2 + \beta_3 A + u_G + \beta_2 u_R).$$

Explain why ordinary least squares (OLS) would be an inconsistent estimator of the parameters of equation (2).

The reduced form equation for G demonstrates that G is not distributed independently of the disturbance term u_R, a requirement for the consistency of OLS when fitting (2).

The first researcher uses instrumental variables (IV) to estimate α_2 in (2). Explain the procedure and demonstrate that the IV estimator of α_2 is consistent.

The first researcher would use A as an instrument for G. It is exogenous, so independent of u_R; correlated with G; and not in the equation in its own right. The estimator of the slope coefficient is:

$$\begin{aligned}\widehat{\alpha}_2^{\mathrm{IV}} &= \frac{\sum\left(A_i-\overline{A}\right)\left(R_i-\overline{R}\right)}{\sum\left(A_i-\overline{A}\right)\left(G_i-\overline{G}\right)} = \frac{\sum\left(A_i-\overline{A}\right)\left([\alpha_1+\alpha_2 G_i+u_{Ri}]-[\alpha_1+\alpha_2\overline{G}=\bar{u}]\right)}{\sum\left(A_i-\overline{A}\right)\left(G_i-\overline{G}\right)} \\ &= \alpha_2 + \frac{\sum\left(A_i-\overline{A}\right)\left(u_{Ri}-\bar{u}_R\right)}{\sum\left(A_i-\overline{A}\right)\left(G_i-\overline{G}\right)}.\end{aligned}$$

Hence:

$$\operatorname{plim}\widehat{\alpha}_2^{\mathrm{IV}} = \alpha_2 + \operatorname{plim}\frac{\frac{1}{n}\sum\left(A_i-\overline{A}\right)\left(u_{Ri}-\bar{u}_R\right)}{\frac{1}{n}\sum\left(A_i-\overline{A}\right)\left(G_i-\overline{G}\right)} = \alpha_2 + \frac{\operatorname{cov}(A,u_R)}{\operatorname{cov}(A,G)} = \alpha_2$$

since $\operatorname{cov}(A,u_R)=0$, A being exogenous, and $\operatorname{cov}(A,G)\neq 0$, A being a determinant of G.

The second researcher uses two stage least squares (TSLS) to estimate α_2 in (2). Explain the procedure and demonstrate that the TSLS estimator is consistent.

The reduced form equation for G for the second researcher is:

$$G = \frac{1}{1-\alpha_2\beta_2}(\beta_1 + \alpha_1\beta_2 + \beta_3 A + \beta_4 Q + u_G + \beta_2 u_R).$$

It is fitted using TSLS. The fitted values of G are used as the instrument:

$$\widehat{\alpha}_2^{\mathrm{TSLS}} = \frac{\sum\left(\widehat{G}_i-\overline{\widehat{G}}\right)\left(R_i-\overline{R}\right)}{\sum\left(\widehat{G}_i-\overline{\widehat{G}}\right)\left(G_i-\overline{G}\right)}.$$

Following the same method as in the third part of the question:

$$\text{plim } \widehat{\alpha}_2^{\text{TSLS}} = \alpha_2 + \frac{\text{cov}(\widehat{G}, u_R)}{\text{cov}(\widehat{G}, G)} = \alpha_2$$

$\text{cov}(\widehat{G}, u_R)$ because $\widehat{G}$ is a linear combination of the exogenous variables, and $\text{cov}(\widehat{G}, G) \neq 0$.

Explain why the TSLS estimator used by the second researcher ought to produce 'better' results than the IV estimator used by the first researcher, if the growth equation is given by (1). Be specific about what you mean by 'better'.*

The TSLS estimator of α_2 should have a smaller variance. The variance of an IV estimator is inversely proportional to the square of the correlation of G and the instrument. $\widehat{G}$ is the linear combination of A and Q that has the highest correlation. It will therefore, in general, have a lower variance than the IV estimator using A.

Suppose that the first researcher is correct and the growth equation is actually given by (1), not (1). Compare the properties of the two estimators in this case.*

If the first researcher is correct, A is the optimal instrument because it will be more highly correlated with G (in the population) than the TSLS combination of A and Q and it will therefore be more efficient.

Suppose that the second researcher is correct and the model is given by (1) and (2), but A is not exogenous after all. Suppose that A is influenced by G:*

$$A = \gamma_1 + \gamma_2 G + u_A$$

where u_A is a disturbance term distributed independently of u_G and u_R. How would this affect the properties of the IV estimator of α_2 used by the first researcher?

$\text{cov}(A, u_R)$ would not be equal to 0 and so the estimator would be inconsistent.

A9.10 *State, with a brief explanation, whether each variable is endogenous or exogenous, and derive the reduced form equations for the endogenous variables.*

In this model *LGEARN* and *SKILL* are endogenous. *IQ* is exogenous. The reduced form equation for *LGEARN* is:

$$LGEARN = \beta_1 + \alpha_1\beta_2 + \alpha_2\beta_2 IQ + u + \beta_2 v.$$

The reduced form equation for *SKILL* is the structural equation.

Explain why the researcher could use ordinary least squares (OLS) to fit equation (1) if u and v are distributed independently of each other.

SKILL is not determined either directly or indirectly by u. Thus in equation (1) there is no violation of the requirement that the regressor be distributed independently of the disturbance term.

Show that the OLS estimator of β_2 is inconsistent if u and v are positively correlated and determine the direction of the large-sample bias.

Writing L for *LGEARN*, S for *SKILL*:

$$\begin{aligned}\widehat{\beta}_2^{\mathrm{OLS}} &= \frac{\sum\left(S_i-\overline{S}\right)\left(L_i-\overline{L}\right)}{\sum\left(S_i-\overline{S}\right)^2}=\frac{\sum\left(S_i-\overline{S}\right)\left([\beta_1+\beta_2 S_i+u_i]-[\beta_1+\beta_2\overline{S}+\overline{u}]\right)}{\sum\left(S_i-\overline{S}\right)^2}\\ &= \beta_2+\frac{\sum\left(S_i-\overline{S}\right)(u_i-\overline{u})}{\sum\left(S_i-\overline{S}\right)^2}.\end{aligned}$$

We cannot obtain a closed-form expression for the expectation of the error term since S depends on v and v is correlated with u. Hence instead we take plims, dividing the numerator and the denominator by n to ensure that the limits exist:

$$\operatorname{plim}\widehat{\beta}_2^{\mathrm{OLS}}=\beta_2+\frac{\operatorname{plim}\frac{1}{n}\sum\left(S_i-\overline{S}\right)(u_i-\overline{u})}{\operatorname{plim}\frac{1}{n}\sum\left(S_i-\overline{S}\right)^2}=\beta_2+\frac{\operatorname{cov}(S,u)}{\operatorname{var}(S)}.$$

Now:

$$\operatorname{cov}(S,u)=\operatorname{cov}([\alpha_1+\alpha_2 IQ+v],u)=\operatorname{cov}(v,u)$$

since α_1 is a constant and IQ is exogenous. Hence the numerator of the error term is positive in large samples. The denominator, being a variance, is also positive. So the large-sample bias is positive.

Demonstrate mathematically how the researcher could use instrumental variables (IV) estimation to obtain a consistent estimate of β_2.

The researcher could use IQ as an instrument for *SKILL*:

$$\begin{aligned}\widehat{\beta}_2^{\mathrm{IV}} &= \frac{\sum\left(I_i-\overline{I}\right)\left(L_i-\overline{L}\right)}{\sum\left(I_i-\overline{I}\right)\left(S_i-\overline{S}\right)}=\frac{\sum\left(I_i-\overline{I}\right)\left([\beta_1+\beta_2 S_i+u_i]-[\beta_1+\beta_2\overline{S}+\overline{u}]\right)}{\sum\left(I_i-\overline{I}\right)\left(S_i-\overline{S}\right)}\\ &= \beta_2+\frac{\sum\left(I_i-\overline{I}\right)(u_i-\overline{u})}{\sum\left(I_i-\overline{I}\right)\left(S_i-\overline{S}\right)}.\end{aligned}$$

We cannot obtain a closed-form expression for the expectation of the error term since S depends on v and v is correlated with u. Hence instead we take plims, dividing the numerator and the denominator by n to ensure that the limits exist:

$$\operatorname{plim}\widehat{\beta}_2^{\mathrm{IV}}=\beta_2+\frac{\operatorname{plim}\frac{1}{n}\sum\left(I_i-\overline{I}\right)(u_i-\overline{u})}{\operatorname{plim}\frac{1}{n}\sum\left(I_i-\overline{I}\right)\left(S_i-\overline{S}\right)}=\beta_2+\frac{\operatorname{cov}(I,u)}{\operatorname{cov}(I,s)}.$$

The numerator of the error term is zero because I is exogenous. The denominator is not zero because S is determined by I. Hence the IV estimator is consistent.

Explain the advantages and disadvantages of using IV, rather than OLS, to estimate β_2, given that the researcher is not sure whether u and v are distributed independently of each other.

The advantage of IV is that, being consistent, there will be no bias in large samples and hence one may hope that there is no serious bias in a finite sample. One disadvantage is that there is a loss of efficiency if u and v are independent. Even if they are not independent, the IV estimator may be inferior to the OLS estimator using some criterion such as the mean square error that allows a trade-off between the bias of an estimator and its variance.

Describe in general terms a test that might help the researcher decide whether to use OLS or IV. What are the limitations of the test?

Durbin–Wu–Hausman test. Also known as Hausman test. The test statistic is a chi-squared statistic based on the differences of all the coefficients in the regression. The null hypothesis is that *SKILL* is distributed independently of u and the differences in the coefficients are random. If the test statistic exceeds its critical value, given the significance level of the test, we reject the null hypothesis and conclude that we ought to use IV rather than OLS. The main limitation is lack of power if the instrument is weak.

Explain whether it is possible for the researcher to fit equation (2) and obtain consistent estimates.

There is no reason why the equation should not be fitted using OLS.

A9.11 Substituting for Y from the first equation into the second, and re-arranging, we have the reduced form equation for S:

$$S = \frac{\alpha_1 + \alpha_2\beta_1 + v + \alpha_2 u}{1 - \alpha_2\beta_2}.$$

Substituting from the third equation into the first, we have:

$$Y = \beta_1 + \beta_2 R + u - \beta_2 w.$$

If this equation is fitted using OLS, we have:

$$\begin{aligned}
\text{plim } \widehat{\beta}_2^{\text{OLS}} &= \beta_2 + \frac{\text{cov}(R, [u - \beta_2 w])}{\text{var}(R)} = \beta_2 + \frac{\text{cov}([S + w], [u - \beta_2 w])}{\text{var}(S + w)} \\
&= \beta_2 + \frac{\alpha_2\gamma\sigma_u^2 - \beta_2\sigma_w^2}{\sigma_S^2 + \sigma_w^2} = \beta_2 + \frac{\alpha_2\gamma\sigma_u^2 - \beta_2\sigma_w^2}{\gamma^2(\sigma_v^2 + \alpha_2^2\sigma_u^2) + \sigma_w^2}
\end{aligned}$$

where:

$$\gamma = \frac{1}{1 - \alpha_2\beta_2}.$$

The denominator of the bias term is positive. Hence the bias will be positive if (the component attributable to simultaneity) is greater than (the component attributable to measurement error), and negative if it is smaller.

Chapter 10

Binary choice and limited dependent variable models, and maximum likelihood estimation

10.1 Overview

The first part of this chapter describes the linear probability model, logit analysis, and probit analysis, three techniques for fitting regression models where the dependent variable is a qualitative characteristic. Next it discusses tobit analysis, a censored regression model fitted using a combination of linear regression analysis and probit analysis. This leads to sample selection models and heckman analysis. The second part of the chapter introduces maximum likelihood estimation, the method used to fit all of these models except the linear probability model.

10.2 Learning outcomes

After working through the corresponding chapter in the text, studying the corresponding slideshows, and doing the starred exercises in the text and the additional exercises in this subject guide, you should be able to:

- describe the linear probability model and explain its defects
- describe logit analysis, giving the mathematical specification
- describe probit analysis, including the mathematical specification
- calculate marginal effects in logit and probit analysis
- explain why OLS yields biased estimates when applied to a sample with censored observations, even when the censored observations are deleted
- explain the problem of sample selection bias and describe how the heckman procedure may provide a solution to it (in general terms, without mathematical detail)
- explain the principle underlying maximum likelihood estimation
- apply maximum likelihood estimation from first principles in simple models.

10.3 Further material

Limiting distributions and the properties of maximum likelihood estimators

Provided that weak regularity conditions involving the differentiability of the likelihood function are satisfied, maximum likelihood (ML) estimators have the following attractive properties in large samples:

(1) They are consistent.

(2) They are asymptotically normally distributed.

(3) They are asymptotically efficient.

The meaning of the first property is familiar. It implies that the probability density function of the estimator collapses to a spike at the true value. This being the case, what can the other assertions mean? If the distribution becomes degenerate as the sample size becomes very large, how can it be described as having a normal distribution? And how can it be described as being efficient, when its variance, and the variance of any other consistent estimator, tend to zero?

To discuss the last two properties, we consider what is known as the limiting distribution of an estimator. This is the distribution of the estimator when the divergence between it and its population mean is multiplied by $\sqrt{n}$. If we do this, the distribution of a typical estimator remains nondegenerate as n becomes large, and this enables us to say meaningful things about its shape and to make comparisons with the distributions of other estimators (also multiplied by $\sqrt{n}$).

To put this mathematically, suppose that there is one parameter of interest, θ, and that $\widehat{\theta}$ is its ML estimator. Then (2) says that:

$$\sqrt{n}\left(\widehat{\theta}-\theta\right) \sim N(0,\, \sigma^2)$$

for some variance σ^2. (3) says that, given any other consistent estimator $\tilde{\theta}$, $\sqrt{\tilde{\theta}-\theta}$ cannot have a smaller variance.

Test procedures for maximum likelihood estimation

This section on ML tests contains material that is a little advanced for an introductory econometrics course. It is provided because likelihood ratio tests are encountered in the sections on binary choice models and because a brief introduction may be of help to those who proceed to a more advanced course.

There are three main approaches to testing hypotheses in maximum likelihood estimation: likelihood ratio (LR) tests, Wald tests, and Lagrange multiplier (LM) tests. Since the theory behind Lagrange multiplier tests is relatively complex, the present discussion will be confined to the first two types. We will start by assuming that the probability density function of a random variable X is a known function of a single unknown parameter θ and that the likelihood function for θ given a sample of n observations on X, $L(\theta \mid X_1, \ldots, X_n)$, satisfies weak regularity conditions involving its differentiability. In particular, we assume that θ is determined by the first-order condition $\mathrm{d}L/\mathrm{d}\theta = 0$. (This rules out estimators such as that in Exercise A10.7) The null hypothesis is $H_0 : \theta = \theta_0$, the alternative hypothesis is $H_1 : \theta \neq \theta_0$, and the maximum likelihood estimate of θ is $\widehat{\theta}$.

Likelihood ratio tests

A likelihood ratio test compares the value of the likelihood function at $\theta = \widehat{\theta}$ with its value at $\theta = \theta_0$. In view of the definition of $\widehat{\theta}$, $L(\widehat{\theta}) \geq L(\theta_0)$ for all θ_0. However, if the null hypothesis is true, the ratio $L(\widehat{\theta})/L(\theta_0)$ should not be significantly greater than 1. As a consequence, the logarithm of the ratio:

$$\log\left(\frac{L(\widehat{\theta})}{L(\theta_0)}\right) = \log L(\widehat{\theta}) - \log L(\theta_0)$$

should not be significantly different from zero. In that it involves a comparison of the measures of goodness of fit for unrestricted and restricted versions of the model, the LR test is similar to an F test.

Under the null hypothesis, it can be shown that in large samples the test statistic:

$$LR = 2\left(\log L(\widehat{\theta}) - \log L(\theta_0)\right)$$

has a chi-squared distribution with one degree of freedom. If there are multiple parameters of interest, and multiple restrictions, the number of degrees of freedom is equal to the number of restrictions.

Examples

We will return to the example in Section 10.6 in the textbook, where we have a normally-distributed random variable X with unknown population mean μ and known standard deviation equal to 1. Given a sample of n observations, the likelihood function is:

$$L(\widehat{\mu} \mid X_1, \ldots, X_n) = \left(\frac{1}{\sqrt{2\pi}\mathrm{e}^{(X_1-\mu)^2/2}}\right) \times \cdots \times \left(\frac{1}{\sqrt{2\pi}\mathrm{e}^{(X_n-\mu)^2/2}}\right).$$

The log-likelihood is:

$$\log L(\widehat{\mu} \mid X_1, \ldots, X_n) = n \log\left(\frac{1}{\sqrt{2\pi}}\right) - \frac{1}{2}\sum (X_i - \widehat{\mu})^2$$

and the unrestricted ML estimate is $\widehat{\mu} = \overline{X}$. The LR statistic for the null hypothesis $H_0 : \mu = \mu_0$ is therefore:

$$\begin{aligned} LR &= 2\left(\left(n\log\left(\frac{1}{\sqrt{2\pi}}\right) - \frac{1}{2}\sum (X_i - \overline{X})^2\right) - \left(n\log\left(\frac{1}{\sqrt{2\pi}}\right) - \frac{1}{2}(X_i - \mu_0)^2\right)\right) \\ &= \sum (X_i - \mu_0)^2 - \sum (X_i - \overline{X})^2 = n(\overline{X} - \mu_0)^2. \end{aligned}$$

If we relaxed the assumption $\sigma = 1$, the unrestricted likelihood function is:

$$L(\widehat{\mu}, \widehat{\sigma} \mid X_1, \ldots, X_n) = \left(\frac{1}{\widehat{\sigma}\sqrt{2\pi}}\mathrm{e}^{-\frac{1}{2}\left(\frac{X_1-\widehat{\mu}}{\widehat{\sigma}}\right)^2}\right) \times \cdots \times \left(\frac{1}{\widehat{\sigma}\sqrt{2\pi}}\mathrm{e}^{-\frac{1}{2}\left(\frac{X_n-\widehat{\mu}}{\widehat{\sigma}}\right)^2}\right)$$

and the log-likelihood is:

$$\log L(\widehat{\mu}, \widehat{\sigma} \,|\, X_1, \ldots, X_n) = n \log\left(\frac{1}{\sqrt{2\pi}}\right) - n \log \widehat{\sigma} - \frac{1}{2\widehat{\sigma}^2} \sum (X_i - \widehat{\mu})^2.$$

The first-order condition obtained by differentiating by σ is:

$$\frac{\partial \log L}{\partial \sigma} = -\frac{n}{\sigma} + \frac{1}{\sigma^3} \sum (X_i - \mu)^2 = 0$$

from which we obtain:

$$\widehat{\sigma}^2 = \frac{1}{n} \sum (X_i - \widehat{\mu})^2.$$

Substituting back into the log-likelihood function, the latter now becomes a function of μ only (and is known as the concentrated log-likelihood function or, sometimes, the profile log-likelihood function):

$$\log L(\mu \,|\, X_1, \ldots, X_n) = n \log\left(\frac{1}{\sqrt{2\pi}}\right) - n \log\left(\frac{1}{n} \sum (X_i - \mu)^2\right)^{1/2} - \frac{n}{2}.$$

As before, the ML estimator of μ is $\bar{X}$. Hence the LR statistic is:

$$\begin{aligned} LR &= 2\left(\left(n \log\left(\frac{1}{\sqrt{2\pi}}\right) - n \log\left(\frac{1}{n} \sum (X_i - \overline{X})^2\right)^{1/2} - \frac{n}{2}\right)\right. \\ &\quad \left. - \left(n \log\left(\frac{1}{\sqrt{2\pi}}\right) - n \log\left(\frac{1}{n} \sum (X_i - \mu_0)^2\right)^{1/2} - \frac{n}{2}\right)\right) \\ &= n \left(\log \sum (X_i - \mu_0)^2 - \log \sum (X_i - \overline{X})^2\right). \end{aligned}$$

It is worth noting that this is closely related to the F statistic obtained when one fits the least squares model:

$$X_i = \mu + u_i.$$

The least squares estimator of μ is $\overline{X}$ and $RSS = \sum (X_i - \overline{X})^2$.

If one imposes the restriction $\mu = \mu_0$, we have $RSS_{\mathrm{R}} = \sum (X_i - \mu_0)^2$ and the F statistic:

$$F(1, n-1) = \frac{\sum (X_i - \mu_0)^2 - \sum (X_i - \overline{X})^2}{\left(\sum (X_i - \overline{X})^2\right) / (n-1)}.$$

Returning to the LR statistic, we have:

$$\begin{aligned} LR &= n \log \frac{\sum (X_i - \mu_0)^2}{\sum (X_i - \overline{X})^2} = n \log\left(1 + \frac{\sum (X_i - \mu_0)^2 - \sum (X_i - \overline{X})^2}{\sum (X_i - \overline{X})^2}\right) \\ &\cong n \frac{\sum (X_i - \mu_0)^2 - \sum (X_i - \overline{X})^2}{\sum (X_i - \overline{X})^2} = \frac{n}{n-1} F \cong F. \end{aligned}$$

Note that we have used the approximation $\log(1 + a) = a$ which is valid when a is small enough for higher powers to be neglected.

Wald tests

Wald tests are based on the same principle as t tests in that they evaluate whether the discrepancy between the maximum likelihood estimate θ and the hypothetical value θ_0 is significant, taking account of the variance in the estimate. The test statistic for the null hypothesis $H_0 : \widehat{\theta} - \theta_0 = 0$ is:

$$\frac{\left(\widehat{\theta} - \theta_0\right)^2}{\widehat{\sigma}^2_{\widehat{\theta}}}$$

where $\widehat{\sigma}^2_{\widehat{\theta}}$ is the estimate of the variance of θ evaluated at the maximum likelihood value. $\widehat{\sigma}^2_{\widehat{\theta}}$ can be estimated in various ways that are asymptotically equivalent if the likelihood function has been specified correctly. A common estimator is that obtained as minus the inverse of the second differential of the log-likelihood function evaluated at the maximum likelihood estimate. Under the null hypothesis that the restriction is valid, the test statistic has a chi-squared distribution with one degree of freedom. When there are multiple restrictions, the test statistic becomes more complex and the number of degrees of freedom is equal to the number of restrictions.

Examples

We will use the same examples as for the LR test, first, assuming that $\sigma = 1$ and then assuming that it has to be estimated along with μ. In the first case the log-likelihood function is:

$$\log L(\mu \,|\, X_1, \ldots, X_n) = n \log\left(\frac{1}{\sqrt{2\pi}}\right) - \frac{1}{2}\sum (X_i - \mu)^2.$$

The first differential is $\sum (X_i - \mu)$ and the second is $-n$, so the estimate of the variance is $1/n$. The Wald test statistic is therefore $n(\overline{X} - \mu_0)^2$.

In the second example, where σ was unknown, the concentrated log-likelihood function is:

$$\begin{aligned}
\log L(\mu \,|\, X_1, \ldots, X_n) &= n \log\left(\frac{1}{\sqrt{2\pi}}\right) - n \log\left(\frac{1}{n}\sum (X_i - \mu)^2\right)^{1/2} - \frac{n}{2} \\
&= n \log\left(\frac{1}{\sqrt{2\pi}}\right) - \frac{n}{2}\log\frac{1}{n} - \frac{n}{2}\log\left(\sum (X_i - \mu)^2\right) - \frac{n}{2}.
\end{aligned}$$

The first derivative with respect to μ is:

$$\frac{\mathrm{d}\log L}{\mathrm{d}\mu} = n\frac{\sum (X_i - \mu)}{\sum (X_i - \mu)^2}.$$

The second derivative is:

$$\frac{\mathrm{d}^2\log L}{\mathrm{d}\mu^2} = n\frac{(-n)\left(\sum (X_i - \mu)^2\right) - \left(\sum (X_i - \mu)\right)\left(-2\sum (X_i - \mu)\right)}{\left[\sum (X_i - \mu)^2\right]^2}.$$

Evaluated at the ML estimator $\widehat{\mu} = \overline{X}$, $\sum (X_i - \mu) = 0$ and hence:

$$\frac{\mathrm{d}^2\log L}{\mathrm{d}\mu^2} = -\frac{n^2}{\sum (X_i - \mu)^2}$$

giving an estimated variance $\widehat{\sigma}^2/n$, given:

$$\widehat{\sigma}^2 = \frac{1}{n}\sum(X_i - \overline{X})^2.$$

Hence the Wald test statistic is:

$$\frac{(\overline{X} - \mu_0)^2}{\widehat{\sigma}^2/n}.$$

Under the null hypothesis, this is distributed as a chi-squared statistic with one degree of freedom.

When there is just one restriction, as in the present case, the Wald statistic is the square of the corresponding asymptotic t statistic (asymptotic because the variance has been estimated asymptotically). The chi-squared test and the t test are equivalent, given that, when there is one degree of freedom, the critical value of the chi-squared statistic for any significance level is the square of the critical value of the normal distribution.

LR test of restrictions in a regression model

Given the regression model:

$$Y_i = \beta_1 + \sum_{j=2}^{k}\beta_j X_{ij} + u_i$$

with u assumed to be iid $N(0,\, \sigma^2)$, the log-likelihood function for the parameters is:

$$\log L(\beta_1, \ldots, \beta_k, \sigma \,|\, Y_i, X_i, i = 1, \ldots, n) = n \log\left(\frac{1}{\sigma\sqrt{2\pi}}\right) - \frac{1}{2\sigma^2}\sum\left(Y_i - \beta_1 - \sum_{j=2}^{k}\beta_j X_{ij}\right)^2.$$

This is a straightforward generalisation of the expression for a simple regression derived in Section 10.6 in the textbook. Hence

$$\log L(\beta_1, \ldots, \beta_k, \sigma \,|\, Y_i, X_i, i = 1, \ldots, n) = -n\log\sigma - \frac{n}{2}\log 2\pi - \frac{1}{2\sigma^2}Z$$

where:

$$Z = \sum\left(Y_i - \beta_1 - \sum_{j=2}^{k}\beta_j X_{ij}\right)^2.$$

The estimates of the β parameters affect only Z. To maximise the log-likelihood, they should be chosen so as to minimise Z, and of course this is exactly what one is doing when one is fitting a least squares regression. Hence $Z = RSS$. It remains to determine the ML estimate of σ. Taking the partial differential with respect to σ, we obtain one of the first-order conditions for a maximum:

$$\frac{\partial \log L(\beta_1, \ldots, \beta_k, \sigma)}{\partial\sigma} = -\frac{n}{\sigma} + \frac{1}{\sigma^3}RSS = 0.$$

From this we obtain:

$$\widehat{\sigma}^2 = \frac{RSS}{n}.$$

Hence the ML estimator is the sum of the squares of the residuals divided by n. This is different from the least squares estimator, which is the sum of the squares of the residuals divided by $n-k$, but the difference disappears as the sample size becomes large. Substituting for $\widehat{\sigma}^2$ in the log-likelihood function, we obtain the concentrated likelihood function:

$$\begin{aligned}
\log L(\beta_1,\ldots,\beta_k \,|\, Y_i, X_i, i=1,\ldots,n) &= -n\log\left(\frac{RSS}{n}\right)^{1/2} - \frac{n}{2}\log 2\pi - \frac{1}{2Z/n}RSS \\
&= -\frac{n}{2}\log\frac{RSS}{n} - \frac{n}{2}\log 2\pi - \frac{n}{2} \\
&= -\frac{n}{2}(\log RSS + 1 + \log 2\pi - \log n).
\end{aligned}$$

We will re-write this as:

$$\log L_{\mathrm{U}} = -\frac{n}{2}(\log RSS_{\mathrm{U}} + 1 + \log 2\pi - \log n)$$

the subscript U emphasising that this is the unrestricted log-likelihood. If we now impose a restriction on the parameters and maximise the loglikelihood function subject to the restriction, it will be:

$$\log L_{\mathrm{R}} = -\frac{n}{2}(\log RSS_{\mathrm{R}} + 1 + \log 2\pi - \log n)$$

where $RSS_{\mathrm{R}} \geq RSS_{\mathrm{U}}$ and hence $\log L_{\mathrm{R}} \leq \log L_{\mathrm{U}}$. The LR statistic for a test of the restriction is therefore:

$$2(\log L_{\mathrm{U}} - L_{\mathrm{R}}) = n(\log RSS_{\mathrm{R}} - \log RSS_{\mathrm{U}}) = n\log\frac{RSS_{\mathrm{R}}}{RSS_{\mathrm{U}}}.$$

It is distributed as a chi-squared statistic with one degree of freedom under the null hypothesis that the restriction is valid. If there is more than one restriction, the test statistic is the same but the number of degrees of freedom under the null hypothesis that all the restrictions are valid is equal to the number of restrictions.

An example of its use is the common factor test in Section 12.3 in the text. As with all maximum likelihood tests, it is valid only for large samples. Thus for testing linear restrictions we should prefer the F test approach because it is valid for finite samples.

10.4 Additional exercises

A10.1 *What factors affect the decision to make a purchase of your category of expenditure in the CES data set?*

Define a new variable *CATBUY* that is equal to 1 if the household makes any purchase of your category and 0 if it makes no purchase at all. Regress *CATBUY* on *EXPPC*, *SIZE*, *REFAGE*, and *COLLEGE* (as defined in Exercise A5.6) using: (1) the linear probability model, (2) the logit model, and (3) the probit model. Calculate the marginal effects at the mean of *EXPPC*, *SIZE*, *REFAGE*, and *COLLEGE* for the logit and probit models and compare them with the coefficients of the linear probability model.

A10.2 Logit analysis was used to relate the event of a respondent working (*WORKING*, defined to be 1 if the respondent was working, and 0 otherwise) to the respondent's educational attainment (S, defined as the highest grade completed) using 1994 data from the National Longitudinal Survey of Youth 1979–. In this year the respondents were aged 29–36 and a substantial number of females had given up work to raise a family. The analysis was undertaken for females and males separately, with the output shown below (first females, then males, with iteration messages deleted):

```
. logit WORKING S if MALE==0

Logit Estimates                                       Number of obs =   2726
                                                      chi2(1)       =  70.42
                                                      Prob > chi2   = 0.0000
Log Likelihood = -1586.5519                           Pseudo R2     = 0.0217

------------------------------------------------------------------------------
 WORKING |      Coef.   Std. Err.       z     P>|z|       [95% Conf. Interval]
---------+--------------------------------------------------------------------
       S |   .1511872   .0186177      8.121   0.000       .1146971    .1876773
   _cons |  -1.049543   .2448064     -4.287   0.000      -1.529355   -.5697314
------------------------------------------------------------------------------


. logit WORKING S if MALE==1

Logit Estimates                                       Number of obs =   2573
                                                      chi2(1)       =  75.03
                                                      Prob > chi2   = 0.0000
Log Likelihood = -802.65424                           Pseudo R2     = 0.0446

------------------------------------------------------------------------------
 WORKING |      Coef.   Std. Err.       z     P>|z|       [95% Conf. Interval]
---------+--------------------------------------------------------------------
       S |   .2499295   .0306482      8.155   0.000       .1898601    .3099989
   _cons |  -.9670268   .3775658     -2.561   0.010      -1.707042   -.2270113
------------------------------------------------------------------------------
```

95 per cent of the respondents had S in the range 9–18 years and the mean value of S was 13.3 and 13.2 years for females and males, respectively.

From the logit analysis, the marginal effect of S on the probability of working at the mean was estimated to be 0.030 and 0.020 for females and males, respectively. Ordinary least squares regressions of *WORKING* on S yielded slope coefficients of 0.029 and 0.020 for females and males, respectively.

As can be seen from the second figure below, the marginal effect of educational attainment was lower for males than for females over most of the range $S \geq 9$. Discuss the plausibility of this finding.

As can also be seen from the second figure, the marginal effect of educational attainment decreases with educational attainment for both males and females over the range $S \geq 9$. Discuss the plausibility of this finding.

Compare the estimates of the marginal effect of educational attainment using logit analysis with those obtained using ordinary least squares.

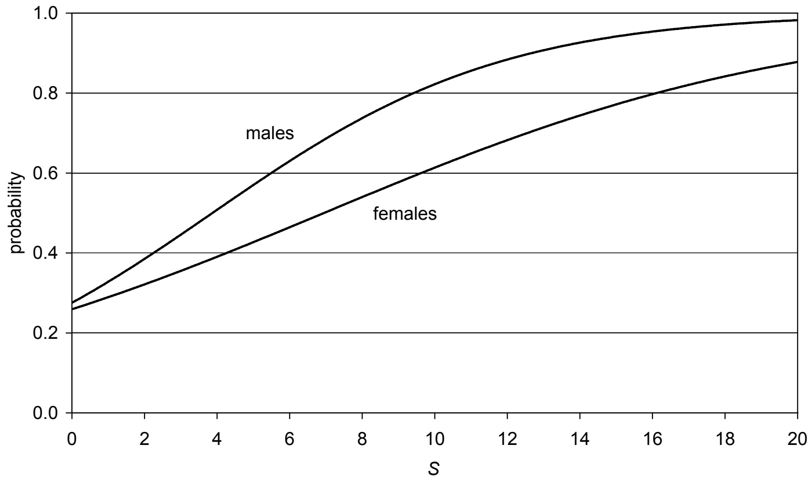


Figure 10.1: Probability of working, as a function of S.

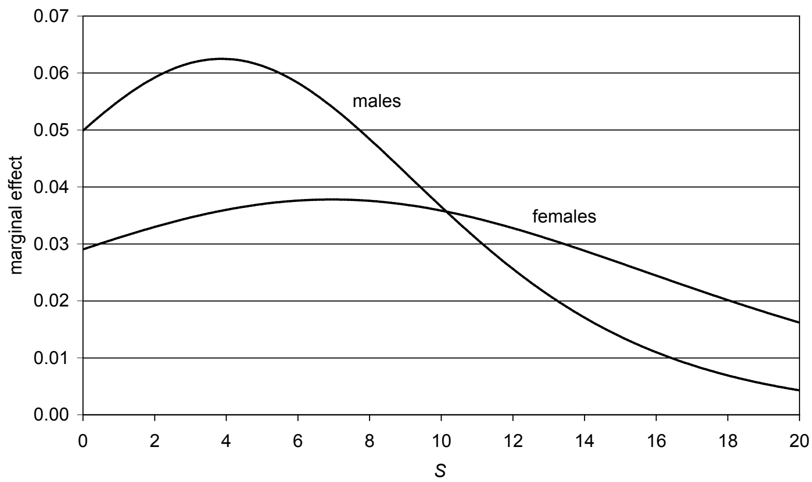


Figure 10.2: Marginal effect of S on the probability of working.

A10.3 A researcher has data on weight, height, and schooling for 540 respondents in the National Longitudinal Survey of Youth 1979– for the year 2002. Using the data on weight and height, he computes the body mass index for each individual. If the body mass index is 30 or greater, the individual is defined to be obese. He defines a binary variable, *OBESE*, that is equal to 1 for the 164 obese individuals and 0 for the other 376. He wishes to investigate whether obesity is related to schooling and fits an ordinary least squares (OLS) regression of *OBESE* on S, years of schooling, with the following result (t statistics in parentheses):

$$\widehat{OBESE} = \underset{(5.30)}{0.595} - \underset{(2.63)}{0.021}S \qquad (1)$$

This is described as the linear probability model (LPM). He also fits the logit model:

$$F(Z) = \frac{1}{1 + e^{-Z}}$$

where $F(Z)$ is the probability of being obese and $Z = \beta_1 + \beta_2 S$, with the following result (again, t statistics in parentheses):

$$\begin{aligned} \widehat{Z} &= \underset{(1.07)}{0.588} - \underset{(2.60)}{0.105}S \qquad (2) \end{aligned}$$

The figure below shows the probability of being obese and the marginal effect of schooling as a function of S, given the logit regression. Most (492 out of 540) of the individuals in the sample had 12 to 18 years of schooling.

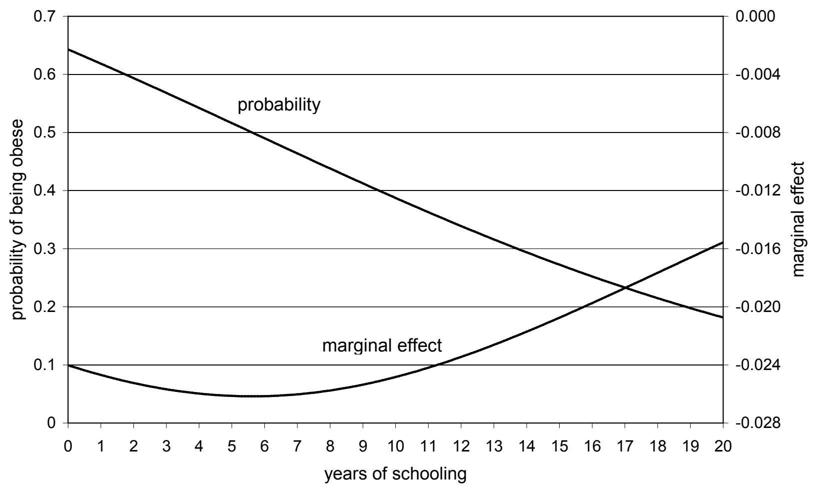


Figure 10.3: Scatter diagram of probability of being obese against years of schooling.

- Discuss whether the relationships indicated by the probability and marginal effect curves appear to be plausible.
- Add the probability function and the marginal effect function for the LPM to the diagram. Explain why you drew them the way you did.
- The logit model is considered to have several advantages over the LPM. Explain what these advantages are. Evaluate the importance of the advantages of the logit model in this particular case.
- The LPM is fitted using OLS. Explain how, instead, it might be fitted using maximum likelihood estimation:
  - Write down the probability of being obese for any obese individual, given S_i for that individual, and write down the probability of not being obese for any non-obese individual, again given S_i for that individual.
  - Write down the likelihood function for this sample of 164 obese individuals and 376 non-obese individuals.
  - Explain how one would use this function to estimate the parameters. [Note: You are not expected to attempt to derive the estimators of the parameters.]

- Explain whether your maximum likelihood estimators will be the same or different from those obtained using least squares.

A10.4 A researcher interested in the relationship between parenting, age and schooling has data for the year 2000 for a sample of 1,167 married males and 870 married females aged 35 to 42 in the National Longitudinal Survey of Youth 1979–. In particular, she is interested in how the presence of young children in the household is related to the age and education of the respondent. She defines *CHILDL6* to be 1 if there is a child less than 6 years old in the household and 0 otherwise and regresses it on AGE, age, and S, years of schooling, for males and females separately using probit analysis. Defining the probability of having a child less than 6 in the household to be $p = F(Z)$ where:

$$Z = \beta_1 + \beta_2 AGE + \beta_3 S$$

she obtains the results shown in the table below (asymptotic standard errors in parentheses).

	Males	Females
AGE	−0.137	−0.154
	(0.018)	(0.023)
S	0.132	0.094
	(0.015)	(0.020)
constant	0.194	0.547
	(0.358)	(0.492)
$\bar{Z}$	−0.399	−0.874
$f(\bar{Z})$	0.368	0.272

For males and females separately, she calculates:

$$\bar{Z} = \widehat{\beta}_1 + \widehat{\beta}_2 \overline{AGE} + \widehat{\beta}_3 \bar{S}$$

where $\overline{AGE}$ and $\bar{S}$ are the mean values of AGE and S and $\widehat{\beta}_1$, $\widehat{\beta}_2$, and $\widehat{\beta}_3$ are the probit coefficients in the corresponding regression, and she further calculates:

$$f(\bar{Z}) = \frac{1}{\sqrt{2\pi}} e^{-\bar{Z}^2/2}$$

where $f(Z) = \mathrm{d}F/\mathrm{d}Z$. The values of $\bar{Z}$ and $f(\bar{Z})$ are shown in the table.

- Explain how one may derive the marginal effects of the explanatory variables on the probability of having a child less than 6 in the household, and calculate for both males and females the marginal effects at the means of AGE and S.
- Explain whether the signs of the marginal effects are plausible. Explain whether you would expect the marginal effect of schooling to be higher for males or for females.

- At a seminar someone asks the researcher whether the marginal effect of S is significantly different for males and females. The researcher does not know how to test whether the difference is significant and asks you for advice. What would you say?

A10.5 A health economist investigating the relationship between smoking, schooling, and age, defines a dummy variable D to be equal to 1 for smokers and 0 for nonsmokers. She hypothesises that the effects of schooling and age are not independent of each other and defines an interactive term schooling*age. She includes this as an explanatory variable in the probit regression. Explain how this would affect the estimation of the marginal effects of schooling and age.

A10.6 A researcher has data on the following variables for 5,061 respondents in the National Longitudinal Survey of Youth 1979–:

- *MARRIED*, marital status in 1994, defined to be 1 if the respondent was married with spouse present and 0 otherwise;
- *MALE*, defined to be 1 if the respondent was male and 0 if female;
- *AGE* in 1994 (the range being 29–37);
- S, years of schooling, defined as highest grade completed, and
- *ASVABC*, score on a test of cognitive ability, scaled so as to have mean 50 and standard deviation 10.

She uses probit analysis to regress *MARRIED* on the other variables, with the output shown:

```
. probit MARRIED MALE AGE S ASVABC

Probit estimates                                  Number of obs   =       5061
                                                  LR chi2(4)      =     229.78
                                                  Prob > chi2     =     0.0000
Log likelihood = -3286.1289                       Pseudo R2       =     0.0338

------------------------------------------------------------------------------
     MARRIED |      Coef.   Std. Err.      z    P>|z|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
        MALE |  -.1215281    .036332    -3.34   0.001    -.1927375   -.0503188
         AGE |    .028571   .0081632     3.50   0.000     .0125715    .0445705
           S |  -.0017465     .00919    -0.19   0.849    -.0197587    .0162656
      ASVABC |   .0252911   .0022895    11.05   0.000     .0208038    .0297784
       _cons |  -1.816455   .2798724    -6.49   0.000    -2.364995   -1.267916
------------------------------------------------------------------------------
```

Variable	Mean	Marginal effect
MALE	0.4841	−0.0467
AGE	32.52	0.0110
S	13.31	−0.0007
ASVABC	48.94	0.0097

The means of the explanatory variables, and their marginal effects evaluated at the means, are shown in the table.

- Discuss the conclusions one may reach, given the probit output and the table, commenting on their plausibility.
- The researcher considers including *CHILD*, a dummy variable defined to be 1 if the respondent had children, and 0 otherwise, as an explanatory variable. When she does this, its z-statistic is 33.65 and its marginal effect 0.5685. Discuss these findings.

10.7 Suppose that the time, t, required to complete a certain process has probability density function:

$$f(t) = \alpha e^{-\alpha(t-\beta)} \quad \text{with } t > \beta > 0$$

and you have a sample of n observations with times $T_1, \ldots, T_n$.

Determine the maximum likelihood estimate of α, assuming that β is known.

A10.8 In Exercise 10.14 in the text, an event could occur with probability p. Given that the event occurred m times in a sample of n observations, the exercise required demonstrating that m/n was the ML estimator of p. Derive the LR statistic for the null hypothesis $p = p_0$. If $m = 40$ and $n = 100$, test the null hypothesis $p = 0.5$.

A10.9 For the variable in Exercise A10.8, derive the Wald statistic and test the null hypothesis $p = 0.5$.

10.5 Answers to the starred exercises in the textbook

10.1 [This exercise does not have a star in the text, but an answer to it is needed for comparison with the answer to Exercise 10.3.]

The output shows the result of an investigation of how the probability of a respondent obtaining a bachelor's degree from a four-year college is related to the score on *ASVABC*, using *EAWE* Data Set 21. *BACH* is a dummy variable equal to 1 for those with bachelor's degrees (years of schooling at least 16) and 0 otherwise. *ASVABC* is a measure of cognitive ability, scaled so that in the population it has mean 0 and standard deviation 1. Provide an interpretation of the coefficients. Explain why OLS is not a satisfactory estimation method for this kind of model.

```
. reg BACH ASVABC
----------------------------------------------------------------------------
    Source |       SS       df       MS              Number of obs =     500
-----------+------------------------------           F(  1,   498) =  123.14
     Model |  24.7674233     1  24.7674233           Prob > F      =  0.0000
  Residual |  100.160577   498  .201125656           R-squared     =  0.1983
-----------+------------------------------           Adj R-squared =  0.1966
     Total |     124.928   499  .250356713           Root MSE      =  .44847
----------------------------------------------------------------------------
      BACH |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-----------+----------------------------------------------------------------
    ASVABC |   .2479312   .0223421    11.10   0.000     .2040348    .2918277
     _cons |   .4206845   .0209535    20.08   0.000     .3795163    .4618526
----------------------------------------------------------------------------
```

Answer:

The slope coefficient indicates that the probability of earning a bachelor's degree rises by 25 per cent for every additional unit of the *ASVABC* score. *ASVABC* is scaled so that one unit is one standard deviation and it has mean zero. While this may be realistic for a range of values of *ASVABC*, it is not for very low ones. Very few of those with scores in the low end of the spectrum earned bachelors degrees and variations in the *ASVABC* score would be unlikely to have an effect on the probability. The intercept literally indicates that an individual with average score would have a 42 per cent probability of earning a bachelor's degree.

However, the linear probability model predicts nonsense negative probabilities for all those with scores less of -1.70 or less. It also suffers from the problem that the standard errors and t and F tests are invalid because the disturbance term does not have a normal distribution. Its distribution is not even continuous, consisting of only two possible values for each value of *ASVABC*.

10.3 The output shows the results of fitting a logit regression for *BACH*, as defined in Exercise 10.1, with the iteration messages deleted. 48.8 per cent of the respondents earned bachelor's degrees.

```
. logit BACH ASVABC
------------------------------------------------------------------------------
Logistic regression                               Number of obs   =        500
                                                  LR chi2(1)      =     110.38
                                                  Prob > chi2     =     0.0000
Log likelihood = -291.23809                       Pseudo R2       =     0.1593
------------------------------------------------------------------------------
        BACH |      Coef.   Std. Err.      z    P>|z|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
      ASVABC |   1.240198   .1377998     9.00   0.000     .9701151     1.51028
       _cons |  -.4077999   .1088093    -3.75   0.000    -.6210623   -.1945375
------------------------------------------------------------------------------
```

The diagram shows the probability of earning a bachelor's degree as a function of *ASVABC*. It also shows the marginal effect function.

- With reference to the diagram, discuss the variation of the marginal effect of the *ASVABC* score implicit in the logit regression.
- Sketch the probability and marginal effect diagrams for the OLS regression in Exercise 10.1 and compare them with those for the logit regression.

Answer:

ASVABC is scaled so that it has a mean of zero. From the curve for the cumulative probability in the figure it can be seen that, for a respondent with mean score, the probability of graduating from college is about 40 per cent. For those one standard deviation above the mean, it is nearly 70 per cent. For those one standard deviation below, a little lower than 20 percent. Looking at the curve for the marginal probability, it can be seen that the marginal effect is greatest for those of average cognitive ability, and still quite high a standard deviation either way. For those two standard deviations above the mean, the marginal effect is low because most are going to college anyway. For those two standard deviations below, the effect is gain low, for the opposite reason.

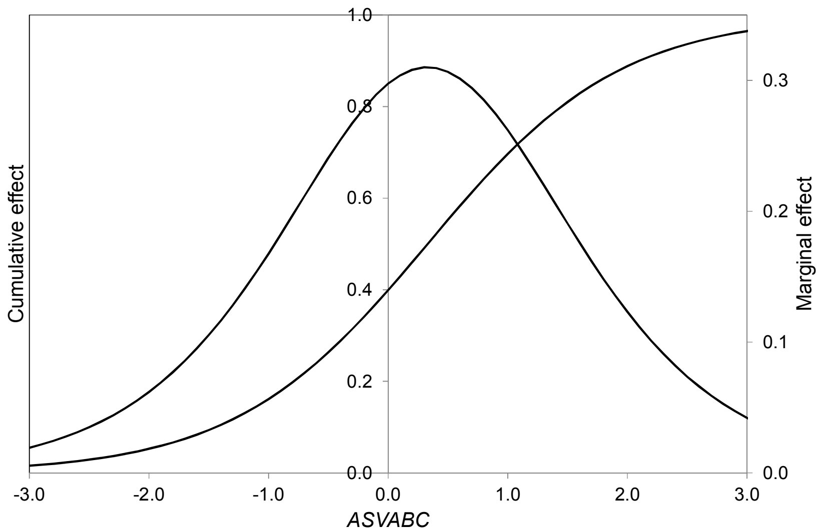


Figure 10.4: Scatter diagram of cumulative and marginal effects against *ASVABC*.

For the linear probability model in Exercise 10.1, the counterpart to the cumulative probability curve in the figure is a straight line using the regression result. In this example, the predictions of the linear probability model do not differ much from those of the logit model over the central range of the data. Its deficiencies become visible only at the extremes. The OLS counterpart to the marginal probability curve is a horizontal straight line at 0.25, showing that the marginal effect is somewhat underestimated in the central range and overestimated elsewhere.

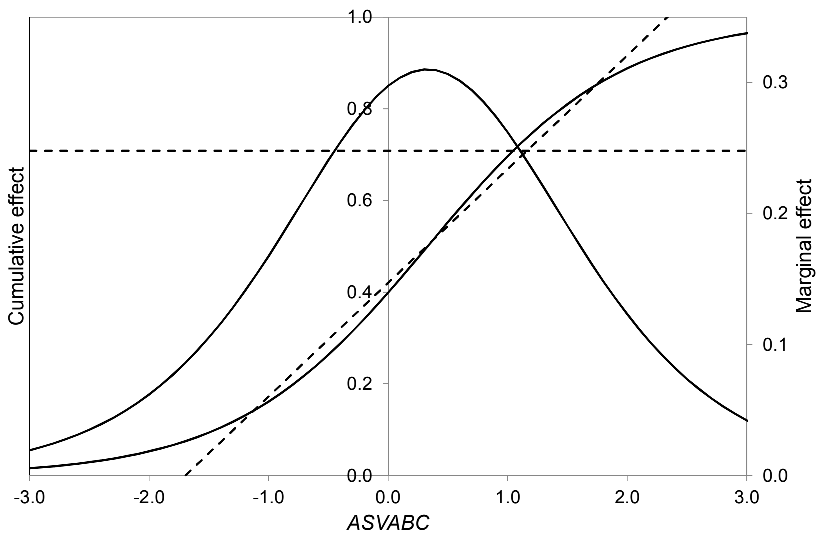


Figure 10.5: Scatter diagram of cumulative and marginal effects against *ASVABC*.

10.7 The following probit regression, with iteration messages deleted, was fitted using 2,108 observations on females in the National Longitudinal Survey of Youth using the *LFP2011* data set described in Exercise 10.2. The respondents were aged 27 to 31 and many of them were raising young families.

```
. probit WORKING S AGE CHILDL06 CHILDL16 MARRIED ETHBLACK ETHHISP if MALE==0
----------------------------------------------------------------------------
Probit regression                                Number of obs   =       2108
                                                 LR chi2(7)      =     170.55
                                                 Prob > chi2     =     0.0000
Log likelihood = -972.89229                      Pseudo R2       =     0.0806
----------------------------------------------------------------------------
   WORKING |      Coef.   Std. Err.      z    P>|z|     [95% Conf. Interval]
-----------+----------------------------------------------------------------
         S |   .1046085   .0127118     8.23   0.000     .0796939    .1295232
       AGE |  -.0029273   .0237761    -0.12   0.902    -.0495277     .043673
  CHILDL06 |  -.4490263     .08128    -5.52   0.000    -.6083322   -.2897204
  CHILDL16 |  -.3055774   .1060307    -2.88   0.004    -.5133938    -.097761
   MARRIED |  -.1286145   .0724189    -1.78   0.076    -.2705529    .0133239
  ETHBLACK |  -.1070784   .0861386    -1.24   0.214    -.2759069    .0617502
   ETHHISP |   .0364241   .0987625     0.37   0.712    -.1571468     .229995
     _cons |  -.1885982   .7046397    -0.27   0.789    -1.569667     1.19247
----------------------------------------------------------------------------
```

WORKING is a binary variable equal to 1 if the respondent was working in 2011, 0 otherwise. *CHILDL06* is a dummy variable equal to 1 if there was a child aged less than 6 in the household, 0 otherwise. *CHILDL16* is a dummy variable equal to 1 if there was a child aged less than 16, but no child less than 6, in the household, 0 otherwise. *MARRIED* is equal to 1 if the respondent was married with spouse present, 0 otherwise. The remaining variables are as described in Appendix B. The mean values of the variables are given in the output from the sum command:

```
. sum WORKING S AGE CHILDL06 CHILDL16 MARRIED ETHBLACK ETHHISP if MALE==0
--------------------------------------------------------------------
  Variable |       Obs        Mean    Std. Dev.       Min        Max
-----------+--------------------------------------------------------
   WORKING |      2108    .7988615    .4009465          0          1
         S |      2108    14.32922    2.882736          6         20
       AGE |      2108    28.99336    1.386405         27         31
  CHILDL06 |      2108    .4407021    .4965891          0          1
  CHILDL16 |      2108    .1465844    .3537751          0          1
   MARRIED |      2108     .420778    .4938011          0          1
  ETHBLACK |      2108    .1783681    .3829132          0          1
   ETHHISP |      2108    .1233397    .3289047          0          1
--------------------------------------------------------------------
```

Calculate the marginal effects and discuss whether they are plausible.

Answer:

The marginal effects are calculated in the table below. As might be expected, having a child aged less than 6 has a large adverse effect, very highly significant. Schooling also has a very significant effect, more educated mothers making use of their investment by tending to stay in the labour force. Age has a significant negative effect, the reason for which is not obvious (the respondents were aged 29–36 in 1994). Being black also has an adverse effect, the reason for which is likewise not obvious. (The *WORKING* variable is defined to be 1 if the individual has recorded hourly earnings of at least $3. If the definition is tightened to including also the requirement that the employment status is employed, the latter effect is smaller, but still significant at the 5 per cent level.)

Variable	Mean	$\widehat{\beta}_2$	Mean$\times\widehat{\beta}_2$	$f(Z)$	$\widehat{\beta}_2 \times f(Z)$
S	14.3292	0.1046	1.4990	0.2627	0.0275
AGE	28.9934	−0.0029	−0.0849	0.2627	−0.0008
$CHILD06$	0.4407	−0.4490	−0.1979	0.2627	−0.1180
$CHILDL16$	0.1466	−0.3056	−0.0448	0.2627	−0.0803
$MARRIED$	0.4208	−0.1286	−0.0541	0.2627	−0.0338
$ETHBLACK$	0.1784	−0.1071	−0.0191	0.2627	−0.0281
$ETHHISP$	0.1233	0.1233	0.0045	0.2627	0.0096
constant	1.0000	−0.1886	−0.1886		
Total			0.9141		

10.12 Show that the tobit model may be regarded as a special case of a selection bias model.

Answer:

The selection bias model may be written:

$$\begin{aligned} B_i^* &= \delta_1 + \sum_{j=2}^{m} \delta_j Q_{ji} + \varepsilon_i \\ Y_i^* &= \beta_1 \sum_{j=2}^{k} \beta_j X_{ji} + u_i \\ Y_i &= Y_i^* \qquad \text{for } B_i^* > 0 \\ & \quad Y_i \text{ is not observed for } B_i^* \leq 0 \end{aligned}$$

where the Q variables determine selection. The tobit model is the special case where the Q variables are identical to the X variables and B^* is the same as Y^*.

10.14 An event is hypothesised to occur with probability p. In a sample of n observations, it occurred m times. Demonstrate that the maximum likelihood estimator of p is m/n.

Answer:

In each observation where the event did occur, the probability was p. In each observation where it did not occur, the probability was $(1-p)$. Since there were m of the former and $n-m$ of the latter, the joint probability was $p^m(1-p)^{n-m}$. Reinterpreting this as a function of p, given m and n, the log-likelihood function for p is:

$$\log L(p) - m \log p + (n-m)\log(1-p).$$

Differentiating with respect to p, we obtain the first-order condition for a minimum:

$$\frac{\mathrm{d} \log L(p)}{\mathrm{d}p} = \frac{m}{p} - \frac{n-m}{1-p} = 0.$$

This yields $p = m/n$. We should check that the second differential is negative and that we have therefore found a maximum. The second differential is:

$$\frac{\mathrm{d}^2 \log L(p)}{\mathrm{d}p^2} = -\frac{m}{p^2} - \frac{n-m}{(1-p)^2}.$$

Evaluated at $p = m/n$:

$$\frac{\mathrm{d}^2 \log L(p)}{\mathrm{d}p^2} = -\frac{n^2}{m} - \frac{n-m}{\left(1-\frac{m}{n}\right)^2} = -n^2\left(\frac{1}{m} + \frac{1}{n-m}\right).$$

This is negative, so we have indeed chosen the value of p that maximises the probability of the outcome.

10.18 Returning to the example of the random variable X with unknown mean μ and variance σ^2, the log-likelihood for a sample of n observations was given by equation (10.36):

$$\log L = -\frac{n}{2}\log 2\pi - \frac{n}{2}\log\sigma^2 + \frac{1}{\sigma^2}\left(-\frac{1}{2}(X_1-\mu)^2 - \cdots - \frac{1}{2}(X_n-\mu)^2\right).$$

The first-order condition forμ produced the ML estimator of μ and the first order condition for σ then yielded the ML estimator for σ. Often, the variance is treated as the primary dispersion parameter, rather than the standard deviation. Show that such a treatment yields the same results in the present case. Treat σ^2 as a parameter, differentiate $\log L$ with respect to it, and solve.

Answer:

$$\frac{\partial \log L}{\partial \sigma^2} = -\frac{n}{2\sigma^2} - \frac{1}{\sigma^4}\left(-\frac{1}{2}(X_1-\mu)^2 - \cdots - \frac{1}{2}(X_n-\mu)^2\right).$$

Hence:

$$\widehat{\sigma}^2 = \frac{1}{n}\left((X_1-\mu)^2 + \cdots + (X_n-\mu)^2\right)$$

as before. The ML estimator of μ is $\overline{X}$ as before.

10.19 In Exercise 10.7, $\log L_0$ is -1058.17. Compute the pseudo-R^2 and confirm that it is equal to that reported in the output.

Answer:

As defined in equation (10.48):

$$\text{pseudo-}R^2 = 1 - \frac{\log L}{\log L_0} = 1 - \frac{-972.8923}{-1058.17} = 0.0806$$

as appears in the output.

10.20 In Exercise 10.7, compute the likelihood ratio statistic $2(\log L - \log L_0)$, confirm that it is equal to that reported in the output, and perform the likelihood ratio test.

Answer:

The likelihood ratio statistic is $2(-972.89 + 1058.17) = 170.56$, which is that reported in the output, apart from rounding error in the last digit. Under the null hypothesis that the coefficients of the explanatory variables are all jointly equal to 0, this is distributed as a chi-squared statistic with degrees of freedom equal to the number of explanatory variables, in this case 7. The critical value of chi-squared at the 0.1 per cent significance level with 7 degrees of freedom is 24.32, and so we reject the null hypothesis at that level.

10.6 Answers to the additional exercises

A10.1 In the case of *FDHO* there were no non-purchasing households and so it was not possible to undertake the analysis.

The results for the logit analysis and the probit analysis were very similar. The linear probability model also yielded similar results for most of the commodities, the coefficients being similar to the logit and probit marginal effects and the t statistics being of the same order of magnitude as the z statistics for the logit and probit.

Most of the effects seem plausible with simple explanations. The total expenditure of the household and the size of the household were both highly significant factors in the decision to make a purchase for most categories of expenditure. The main exception, *TOB*. was instead influenced (negatively: survival bias?) by the age of the reference individual and, unsurprisingly, by his or her education.

Linear probability model, dependent variable *CATBUY*

		$EXPPC \times 10^{-4}$		$SIZE \times 10^{-2}$		$REFAGE \times 10^{-2}$		$COLLEGE$		Cases with probability	
	n	$\widehat{\beta}_2$	t	$\widehat{\beta}_3$	t	$\widehat{\beta}_4$	t	$\widehat{\beta}_5$	t	< 0	> 1
ADM	2,815	0.38	20.41	4.00	9.54	−0.34	−9.92	0.22	17.74	0	44
CLOT	4,500	0.33	18.74	5.38	13.61	−0.35	−10.72	0.05	4.12	0	144
DOM	1,661	0.30	17.37	4.18	10.78	0.16	5.08	0.09	7.99	0	181
EDUC	561	0.13	11.83	3.13	12.38	−0.12	−5.80	0.05	6.01	612	0
ELEC	5,828	0.08	7.33	2.71	11.09	0.16	7.76	0.02	2.07	0	254
FDAW	5,102	0.23	14.57	2.23	6.41	−0.27	−9.56	0.11	10.85	0	223
*FDHO**	6,334										
FOOT	1,827	0.28	15.83	5.93	14.81	−0.22	−6.65	0.01	1.01	0	4
FURN	487	0.14	13.47	1.65	6.87	−0.07	−3.74	0.01	1.66	149	0
GASO	5,710	0.09	7.70	3.23	12.07	−0.00	−0.14	0.07	8.61	0	331
HEAL	4,802	0.21	12.82	3.18	8.77	0.82	27.46	0.11	9.82	0	406
HOUS	6,223	0.03	5.24	0.52	4.36	0.04	4.44	0.01	2.30	0	484
LIFE	1,253	0.35	15.82	3.91	11.02	0.19	8.36	0.04	3.49	0	1
LOCT	692	0.04	3.42	−0.23	−0.80	−0.15	−6.38	0.00	0.42	0	0
MAPP	399	0.10	10.34	1.59	7.23	−0.00	−0.01	−0.01	−1.54	0	0
PERS	3,817	0.30	15.56	4.55	10.53	0.29	8.19	0.12	9.28	0	66
READ	2,287	0.25	13.48	2.52	5.98	0.37	10.76	0.16	13.03	0	10
SAPP	1,037	0.20	13.80	2.86	8.61	−0.03	−1.12	0.03	3.30	0	0
TELE	5,788	0.07	6.29	3.52	14.09	0.31	15.12	0.01	1.65	0	455
TEXT	992	0.19	13.25	2.45	7.50	−0.03	−1.22	0.04	3.84	0	0
TOB	1,155	−0.01	−0.54	0.24	0.69	−0.17	−5.90	−0.10	−9.16	0	0
TOYS	2,504	0.24	12.14	6.26	14.36	−0.13	−3.58	0.06	4.70	0	4
TRIP	516	0.23	21.63	0.93	3.88	−0.03	−1.39	0.03	4.58	415	0

**FDHO* had no observations with zero expenditure.

Logit model, dependent variable *CATBUY*									
		$EXPPC\times10^{-4}$		$SIZE\times10^{-2}$		$REFAGE\times10^{-2}$		$COLLEGE$	
	n	$\widehat{\beta}_2$	z	$\widehat{\beta}_3$	z	$\widehat{\beta}_4$	z	$\widehat{\beta}_5$	z
ADM	2,815	2.06	18.34	20.02	10.04	−1.69	10.02	1.00	16.52
CLOT	4,500	2.51	17.22	32.00	13.44	−1.72	−9.92	0.18	2.98
DOM	1,661	1.50	15.28	22.50	10.55	0.91	4.99	0.54	8.01
EDUC	561	1.38	11.60	35.93	12.32	−2.22	−7.14	0.81	6.99
ELEC	5,828	1.63	7.28	44.17	10.57	2.03	7.48	0.19	1.89
FDAW	5,102	2.71	14.40	17.42	6.78	−1.79	−8.99	0.63	9.16
FDHO	6,334								
FOOT	1,827	1.39	14.69	29.17	14.24	−1.25	−7.00	0.08	1.23
FURN	487	1.43	12.00	21.16	6.66	−1.28	−4.17	0.28	2.46
GASO	5,710	1.50	7.50	47.81	11.71	0.16	0.66	0.71	7.87
HEAL	4,802	2.29	13.58	21.11	9.12	5.22	24.36	0.59	8.61
HOUS	6,223	4.31	5.78	37.81	4.81	2.42	4.27	0.35	1.76
LIFE	1,253	1.38	13.94	24.61	10.71	1.28	6.33	0.27	3.71
LOCT	692	0.41	3.50	−1.75	−0.60	−1.57	−6.35	0.05	0.51
MAPP	399	1.21	9.65	23.27	5.89	−0.05	−0.16	−0.13	−1.11
PERS	3,817	1.78	15.07	21.91	10.92	1.30	8.11	0.48	8.46
READ	2,287	1.18	12.35	11.97	5.97	1.77	10.61	0.77	12.64
SAPP	1,037	1.24	12.47	19.99	8.37	−0.29	−1.37	0.29	3.71
TELE	5,788	1.24	6.20	51.87	12.34	3.82	13.66	0.18	1.78
TEXT	992	1.20	11.97	17.77	7.28	−0.31	−1.44	0.34	4.27
TOB	1,155	−0.07	−0.64	1.28	0.55	−1.17	−5.85	−0.62	−8.95
TOYS	2,504	1.04	11.53	27.08	13.84	−0.59	−3.69	0.27	4.70
TRIP	516	1.92	15.76	9.60	2.62	−0.42	−1.41	0.75	5.92

Probit model, dependent variable *CATBUY*									
		$EXPPC\times10^{-4}$		$SIZE\times10^{-2}$		$REFAGE\times10^{-2}$		$COLLEGE$	
	n	$\widehat{\beta}_2$	z	$\widehat{\beta}_3$	z	$\widehat{\beta}_4$	z	$\widehat{\beta}_5$	z
ADM	2,815	1.17	19.26	11.97	9.93	−1.01	−10.03	0.61	16.96
CLOT	4,500	1.34	18.00	18.37	13.62	−1.03	−10.00	0.12	3.31
DOM	1,661	0.89	15.77	13.35	10.52	0.53	5.00	0.31	7.95
EDUC	561	0.78	11.88	19.78	12.61	−1.15	−7.36	0.40	7.02
ELEC	5,828	0.71	7.18	19.93	10.53	0.96	7.17	0.10	2.03
FDAW	5,102	1.37	14.87	9.53	6.72	−1.03	−9.08	0.37	9.50
FDHO	6,334								
FOOT	1,827	0.82	15.39	17.60	14.43	−0.74	−6.98	0.05	1.29
FURN	487	0.80	12.45	11.37	6.83	−0.63	−4.15	0.12	2.24
GASO	5,710	0.61	7.37	21.79	11.79	0.08	0.60	0.40	8.43
HEAL	4,802	1.18	13.94	11.97	9.11	3.05	25.25	0.34	8.56
HOUS	6,223	1.33	5.76	14.17	4.56	0.98	4.22	0.19	2.26
LIFE	1,253	0.81	14.78	14.40	10.74	0.76	6.56	0.15	3.69
LOCT	692	0.21	3.30	−0.80	−0.54	−0.79	−6.26	0.02	0.50
MAPP	399	0.67	9.94	12.10	7.00	−0.03	−0.17	−0.07	−1.32
PERS	3,817	0.97	15.47	12.93	10.79	0.80	8.15	0.31	8.81
READ	2,287	0.70	12.74	7.14	5.86	1.07	10.63	0.47	12.87
SAPP	1,037	0.73	12.95	11.49	8.42	−0.15	−1.28	0.15	3.63
TELE	5,788	0.55	6.11	24.85	12.54	1.91	13.66	0.10	2.01
TEXT	992	0.71	12.53	10.21	7.33	−0.18	−1.46	0.18	4.16
TOB	1,155	−0.05	−0.79	0.84	0.63	−0.67	−5.86	−0.35	−8.89
TOYS	2,504	0.62	11.91	16.57	14.04	−0.37	−3.72	0.17	4.77
TRIP	516	1.06	16.91	4.84	2.66	−0.21	−1.42	0.35	5.93

Marginal effects, linear probability model, logit and probit						
	$EXPPC4 \times 10^{-4}$			$SIZE \times 10^{-2}$		
	LPM	logit	probit	LPM	logit	probit
ADM	0.38	0.51	0.46	4.00	4.93	4.72
CLOT	0.33	0.48	0.44	5.38	6.14	6.04
DOM	0.30	0.28	0.28	4.18	4.21	4.25
EDUC	0.13	0.09	0.10	3.13	2.24	2.57
ELEC	0.08	0.10	0.09	2.71	2.73	2.66
FDAW	0.23	0.36	0.34	2.23	2.32	2.37
FDHO						
FOOT	0.28	0.28	0.28	5.93	5.82	5.89
FURN	0.14	0.09	0.10	1.65	1.32	1.48
GASO	0.09	0.11	0.09	3.23	3.47	3.35
HEAL	0.21	0.35	0.33	3.18	3.23	3.34
HOUS	0.03	0.04	0.04	−0.23	−0.17	−0.15
LIFE	0.35	0.21	0.22	3.91	3.72	3.86
LOCT	0.04	0.04	0.04	−0.23	−0.17	−0.15
MAPP	0.10	0.07	0.08	1.59	1.27	1.39
PERS	0.30	0.42	0.37	4.55	5.18	4.96
READ	0.25	0.27	0.26	2.52	2.73	2.65
SAPP	0.20	0.16	0.17	2.86	2.60	2.74
TELE	0.07	0.08	0.07	3.52	3.14	3.29
TEXT	0.19	0.15	0.16	2.45	2.23	2.36
TOB	−0.01	−0.01	−0.01	0.24	0.19	0.22
TOYS	0.24	0.25	0.24	6.26	6.45	6.36
TRIP	0.23	0.11	0.13	0.93	0.58	0.61

Marginal effects, linear probability model, logit and probit						
	REFAGE$\times 10^{-2}$			*COLLEGE*		
	LPM	logit	probit	LPM	logit	probit
ADM	−0.34	−0.42	−0.40	0.22	0.24	0.24
CLOT	−0.35	−0.33	−0.34	0.05	0.04	0.04
DOM	0.16	0.17	0.17	0.09	0.10	0.10
EDUC	−0.12	−0.14	−0.15	0.05	0.05	0.05
ELEC	0.16	0.13	0.13	0.02	0.01	0.01
FDAW	−0.27	−0.24	−0.26	0.11	0.08	0.09
FDHO						
FOOT	−0.22	−0.25	−0.25	0.01	0.02	0.02
FURN	−0.07	−0.08	−0.08	0.01	0.02	0.02
GASO	−0.00	0.01	0.01	0.07	0.05	0.06
HEAL	0.82	0.80	0.85	0.11	0.09	0.09
HOUS	0.04	0.02	0.03	0.01	0.00	0.01
LIFE	0.19	0.19	0.20	0.04	0.04	0.04
LOCT	−0.15	−0.15	−0.15	0.00	0.00	0.00
MAPP	−0.00	0.00	0.00	−0.01	−0.01	−0.01
PERS	0.29	0.31	0.31	0.12	0.11	0.12
READ	0.37	0.40	0.40	0.16	0.18	0.17
SAPP	−0.03	−0.04	−0.04	0.03	0.04	0.04
TELE	0.31	0.23	0.25	0.01	0.01	0.01
TEXT	−0.03	−0.04	−0.04	0.04	0.04	0.04
TOB	−0.17	−0.17	−0.17	−0.10	−0.09	−0.09
TOYS	−0.13	−0.14	−0.14	0.06	0.06	0.06
TRIP	−0.03	−0.03	−0.03	0.03	0.04	0.04

A10.2 The finding that the marginal effect of educational attainment was lower for males than for females over most of the range $S \geq 9$ is plausible because the probability of working is much closer to 1 for males than for females for $S \geq 9$, and hence the possible sensitivity of the participation rate to S is smaller.

The explanation of the finding that the marginal effect of educational attainment decreases with educational attainment for both males and females over the range $S \geq 9$ is similar. For both sexes, the greater is S, the greater is the participation rate, and hence the smaller is the scope for it being increased by further education.

The OLS estimates of the marginal effect of educational attainment are given by the slope coefficients and they are very similar to the logit estimates at the mean, the reason being that most of the observations on S are confined to the middle part of the sigmoid curve where it is relatively linear.

A10.3 *Discuss whether the relationships indicated by the probability and marginal effect curves appear to be plausible.*

The probability curve indicates an inverse relationship between schooling and the probability of being obese. This seems entirely plausible. The more educated tend to have healthier lifestyles, including eating habits. Over the relevant range, the marginal effect falls a little in absolute terms (is less negative) as schooling increases. This is in keeping with the idea that further schooling may have less effect on the highly educated than on the less educated (but the difference is not large).

Add the probability function and the marginal effect function for the LPM to the diagram. Explain why you drew them the way you did.

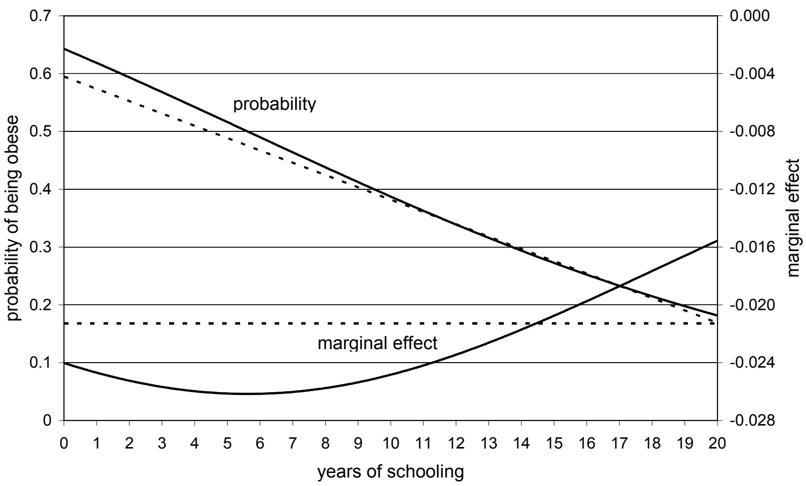


Figure 10.6: Scatter diagram of probability of being obese and marginal effect against years of schooling.

The estimated probability function for the LPM is just the regression equation and the marginal effect is the coefficient of S. They are shown as the dashed lines in the diagram.

The logit model is considered to have several advantages over the LPM. Explain what these advantages are. Evaluate the importance of the advantages of the logit model in this particular case.

The disadvantages of the LPM are (1) that it can give nonsense fitted values (predicted probabilities greater than 1 or less than 0); (2) the disturbance term in observation i must be equal to either $-1 - F(Z_i)$ (if the dependent variable is equal to 1) or $-F(Z_i)$ (if the dependent variable is equal to 0) and so it violates the usual assumption that the disturbance term is normally distributed, although this may not matter asymptotically; (3) the disturbance term will be heteroskedastic because Z_i is different for different observations; (4) the LPM implicitly assumes that the marginal effect of each explanatory variable is constant over its entire range, which is often intuitively unappealing.

In this case, nonsense predictions are clearly not an issue. The assumption of a constant marginal effect does not seem to be a problem either, given the approximate linearity of the logit $F(Z)$.

The LPM is fitted using OLS. Explain how, instead, it might be fitted using maximum likelihood estimation:

Write down the probability of being obese for any obese individual, given S_i for that individual, and write down the probability of not being obese for any non-obese *individual, again given S_i for that individual.*

Obese: $p_i^O = \beta_1 + \beta_2 S_i$; not obese: $p_i^{NO} = 1 - \beta_1 - \beta_2 S_i$.

Write down the likelihood function for this sample of 164 obese individuals and 376 non-obese individuals.

$$L(\beta_1, \beta_2 \,|\, \text{data}) = \prod_{\text{OBESE}} p_i^O \prod_{\text{NOT OBESE}} p_i^{NO} = \prod_{\text{OBESE}} (\beta_1 + \beta_2 S_i) \prod_{\text{NOT OBESE}} (1 - \beta_1 - \beta_2 S_i).$$

*Explain how one would use this function to estimate the parameters. [***Note:** *You are not expected to attempt to derive the estimators of the parameters.]*

You would use some algorithm to find the values of β_1 and β_2 that maximises the function.

Explain whether your maximum likelihood estimators will be the same or different from those obtained using least squares.

Least squares involves finding the extremum of a completely different expression and will therefore lead to different estimators.

10.4 *Explain how one may derive the marginal effects of the explanatory variables on the probability of having a child less than 6 in the household, and calculate for both males and females the marginal effects at the means of AGE and S.*

Since p is a function of Z, and Z is a linear function of the X variables, the marginal effect of X_j is:

$$\frac{\partial p}{\partial X_j} = \frac{\mathrm{d}p}{\mathrm{d}Z} \frac{\partial Z}{\partial X_j} = \frac{\mathrm{d}p}{\mathrm{d}Z} \beta_j$$

where β_j is the coefficient of X_j in the expression for Z. In the case of probit analysis, $p = F(Z)$ is the cumulative standardised normal distribution. Hence $\mathrm{d}p/\mathrm{d}Z$ is just the standardised normal distribution.

For males, this is 0.368 when evaluated at the means. Hence the marginal effect of AGE is $0.368 \times -0.137 = -0.050$ and that of S is $0.368 \times 0.132 = 0.049$. For females the corresponding figures are $0.272 \times -0.154 = -0.042$ and $0.272 \times 0.094 = 0.026$, respectively. So for every extra year of age, the probability is reduced by 5.0 per cent for males and 4.2 per cent for females. For every extra year of schooling, the probability increases by 4.9 per cent for males and 2.6 per cent for females.

Explain whether the signs of the marginal effects are plausible. Explain whether you would expect the marginal effect of schooling to be higher for males or for females.

Yes. Given that the cohort is aged 35–42, the respondents have passed the age at which most adults start families, and the older they are, the less likely they are to have small children in the household. At the same time, the more educated the respondent, the more likely he or she is to have started having a family relatively late, so the positive effect of schooling is also plausible. However, given the age of the cohort, it is likely to be weaker for females than for males, given that most females intending to have families will have started them by this time, irrespective of their education.

At a seminar someone asks the researcher whether the marginal effect of S is significantly different for males and females. The researcher does not know how to test whether the difference is significant and asks you for advice. What would you say?

Fit a probit regression for the combined sample, adding a male intercept dummy and male slope dummies for AGE and S. Test the coefficient of the slope dummy for S.

10.5 The Z function will be of the form:

$$Z = \beta_1 + \beta_2 A + \beta_3 S + \beta_4 AS$$

so the marginal effects are:

$$\frac{\partial p}{\partial A} = \frac{\mathrm{d}p}{\mathrm{d}Z}\frac{\partial Z}{\partial A} = f(Z)(\beta_2 + \beta_4 S)$$

and:

$$\frac{\partial p}{\partial S} = \frac{\mathrm{d}p}{\mathrm{d}Z}\frac{\partial Z}{\partial S} = f(Z)(\beta_3 + \beta_4 A).$$

Both factors depend on the values of A and/or S, but the marginal effects could be evaluated for a representative individual using the mean values of A and S in the sample.

A10.6 *Discuss the conclusions one may reach, given the probit output and the table, commenting on their plausibility.*

Being male has a small but highly significant negative effect. This is plausible because males tend to marry later than females and the cohort is still relatively young.

Age has a highly significant positive effect, again plausible because older people are more likely to have married than younger people.

Schooling has no apparent effect at all. It is not obvious whether this is plausible.

Cognitive ability has a highly significant positive effect. Again, it is not obvious whether this is plausible.

The researcher considers including CHILD, a dummy variable defined to be 1 if the respondent had children, and 0 otherwise, as an explanatory variable. When she does this, its z-statistic is 33.65 and its marginal effect 0.5685. Discuss these findings.

Obviously one would expect a high positive correlation between being married and having children and this would account for the huge and highly significant coefficient. However getting married and having children are often a joint decision, and accordingly it is simplistic to suppose that one characteristic is a determinant of the other. The finding should not be taken at face value.

A10.7 *Determine the maximum likelihood estimate of α, assuming that β is known.*

The log-likelihood function is:

$$\log L(\alpha \mid \beta, T_1, \ldots, T_n) = n \log \alpha - \alpha \sum (T_i - \beta).$$

Setting the first derivative with respect to α equal to zero, we have:

$$\frac{n}{\widehat{\alpha}} - \sum(T_i - \beta) = 0$$

and hence:

$$\widehat{\alpha} = \frac{1}{\overline{T} - \beta}.$$

The second derivative is $-n/\widehat{\alpha}^2$, which is negative, confirming we have maximised the loglikelihood function.

A10.8 From the solution to Exercise 10.14, the log-likelihood function for p is:

$$\log L(p) = m \log p + (n - m)\log(1 - p).$$

Thus the LR statistic is:

$$\begin{aligned} LR &= 2\left(\left(m \log \frac{m}{n} + (n-m)\log\left(1 - \frac{m}{n}\right)\right) - (m \log p_0 + (n-m)\log(1 - p_0))\right) \\ &= 2\left(m \log\left(\frac{m/n}{p_0}\right) + (n-m)\log\left(\frac{1 - m/n}{1 - p_0}\right)\right). \end{aligned}$$

If $m = 40$ and $n = 100$, the LR statistic for $H_0 : p = 0.5$ is:

$$\text{LR} = 2\left(40 \log\left(\frac{0.4}{0.5}\right) + 60 \log\left(\frac{0.6}{0.5}\right)\right) = 4.03.$$

We would reject the null hypothesis at the 5 per cent level (critical value of chi-squared with one degree of freedom 3.84) but not at the 1 per cent level (critical value 6.64).

A10.9 The first derivative of the log-likelihood function is:

$$\frac{\mathrm{d} \log L(p)}{\mathrm{d}p} = \frac{m}{p} - \frac{n-m}{1-p} = 0$$

and the second differential is:

$$\frac{\mathrm{d} \log L(p)}{\mathrm{d}p^2} = -\frac{m}{p^2} - \frac{n-m}{(1-p)^2}.$$

Evaluated at $p = m/n$:

$$\frac{\mathrm{d} \log L(p)}{\mathrm{d}p^2} = -\frac{n^2}{m} - \frac{n-m}{\left(1 - \frac{m}{n}\right)^2} = -n^2\left(\frac{1}{m} + \frac{1}{n-m}\right) = -\frac{n^3}{m(n-m)}.$$

The variance of the ML estimate is given by:

$$\left(-\frac{\mathrm{d} \log L(p)}{\mathrm{d}p^2}\right)^{-1} = \left(\frac{n^3}{m(n-m)}\right)^{-1} = \frac{m(n-m)}{n^3}.$$

The Wald statistic is therefore:

$$\frac{\left(\frac{m}{n} - p_0\right)^2}{\frac{m(n-m)}{n^3}} = \frac{\left(\frac{m}{n} - p_0\right)^2}{\frac{1}{n}\frac{m}{n}\frac{n-m}{n}}.$$

Given the data, this is equal to:

$$\frac{(0.4 - 0.5)^2}{\frac{1}{100} \times 0.4 \times 0.6} = 4.17.$$

Under the null hypothesis this has a chi-squared distribution with one degree of freedom, and so the conclusion is the same as in Exercise A.8.

Chapter 11
Models using time series data

11.1 Overview

This chapter introduces the application of regression analysis to time series data, beginning with static models and then proceeding to dynamic models with lagged variables used as explanatory variables. It is shown that multicollinearity is likely to be a problem in models with unrestricted lag structures and that this provides an incentive to use a parsimonious lag structure, such as the Koyck distribution. Two models using the Koyck distribution, the adaptive expectations model and the partial adjustment model, are described, together with well-known applications to aggregate consumption theory, Friedman's permanent income hypothesis in the case of the former and Brown's habit persistence consumption function in the case of the latter. The chapter concludes with a discussion of prediction and stability tests in time series models.

11.2 Learning outcomes

After working through the corresponding chapter in the text, studying the corresponding slideshows, and doing the starred exercises in the text and the additional exercises in this subject guide, you should be able to:

- explain why multicollinearity is a common problem in time series models, especially dynamic ones with lagged explanatory variables
- describe the properties of a model with a lagged dependent variable (ADL(1,0) model)
- describe the assumptions underlying the adaptive expectations and partial adjustment models
- explain the properties of OLS estimators of the parameters of ADL(1,0) models
- explain how predetermined variables may be used as instruments in the fitting of models using time series data
- explain in general terms the objectives of time series analysts and those constructing VAR models.

11.3 Additional exercises

A11.1 The output below shows the result of linear and logarithmic regressions of expenditure on food on income, relative price, and population (measured in thousands) using the Demand Functions data set, together with the correlations among the variables. Provide an interpretation of the regression coefficients and perform appropriate statistical tests.

```
============================================================
Dependent Variable: FOOD
Method: Least Squares
Sample: 1959 2003
Included observations: 45
============================================================
     Variable      Coefficient Std. Error t-Statistic  Prob.
============================================================
         C         -19.49285   88.86914  -0.219343   0.8275
        DPI         0.031713   0.010658   2.975401   0.0049
     PRELFOOD       0.403356   0.365133   1.104681   0.2757
        POP         0.001140   0.000563   2.024017   0.0495
============================================================
R-squared           0.988529    Mean dependent var 422.0374
Adjusted R-squared  0.987690    S.D. dependent var 91.58053
S.E. of regression  10.16104    Akaike info criteri7.559685
Sum squared resid   4233.113    Schwarz criterion  7.720278
Log likelihood     -166.0929    F-statistic       1177.745
Durbin-Watson stat  0.404076    Prob(F-statistic)  0.000000
============================================================
```

```
============================================================
Dependent Variable: LGFOOD
Method: Least Squares
Sample: 1959 2003
Included observations: 45
============================================================
       Variable    Coefficient Std. Error t-Statistic  Prob.
============================================================
           C        5.293654   2.762757   1.916077   0.0623
         LGDPI      0.589239   0.080158   7.351014   0.0000
       LGPRFOOD    -0.122598   0.084355  -1.453361   0.1537
         LGPOP     -0.289219   0.258762  -1.117706   0.2702
============================================================
R-squared           0.992245    Mean dependent var 6.021331
Adjusted R-squared  0.991678    S.D. dependent var 0.222787
S.E. of regression  0.020324    Akaike info criter-4.869317
Sum squared resid   0.016936    Schwarz criterion -4.708725
Log likelihood      113.5596    F-statistic       1748.637
Durbin-Watson stat  0.488502    Prob(F-statistic)  0.000000
============================================================
```

```
                    Correlation Matrix
============================================================
              LGFOOD      LGDPI      LGPRFOOD     LGPOP
============================================================
   LGFOOD     1.000000    0.995896   -0.613437    0.990566
   LGDPI      0.995896    1.000000   -0.604658    0.995241
  LGPRFOOD   -0.613437   -0.604658    1.000000   -0.641226
   LGPOP      0.990566    0.995241   -0.641226    1.000000
============================================================
```

A11.2 Perform regressions parallel to those in Exercise A11.1 using your category of expenditure and provide an interpretation of the coefficients.

A11.3 The output shows the result of a logarithmic regression of expenditure on food per capita, on income per capita, both measured in US$ million, and the relative price index for food. Provide an interpretation of the coefficients, demonstrate that the specification is a restricted version of the logarithmic regression in Exercise A11.1, and perform an F test of the restriction.

```
============================================================
Dependent Variable: LGFOODPC
Method: Least Squares
Sample: 1959 2003
Included observations: 45
============================================================
     Variable      Coefficient Std. Error t-Statistic  Prob.
============================================================
         C          -5.425877   0.353655  -15.34231   0.0000
      LGDPIPC        0.280229   0.014641   19.14024   0.0000
     LGPRFOOD        0.052952   0.082588   0.641160   0.5249
============================================================
R-squared             0.927348    Mean dependent var-6.321984
Adjusted R-squared    0.923889    S.D. dependent var 0.085249
S.E. of regression    0.023519    Akaike info criter-4.597688
Sum squared resid     0.023232    Schwarz criterion -4.477244
Log likelihood        106.4480    F-statistic        268.0504
Durbin-Watson stat    0.417197    Prob(F-statistic)  0.000000
============================================================
```

A11.4 Perform a regression parallel to that in Exercise A11.3 using your category of expenditure. Provide an interpretation of the coefficients, and perform an F test of the restriction.

A11.5 The output shows the result of a logarithmic regression of expenditure on food per capita, on income per capita, the relative price index for food, and population. Provide an interpretation of the coefficients, demonstrate that the specification is equivalent to that for the logarithmic regression in Exercise A11.1, and use it to perform a t test of the restriction in Exercise A11.3.

```
============================================================
Dependent Variable: LGFOODPC
Method: Least Squares
Sample: 1959 2003
```

```
Included observations: 45
============================================================
     Variable      Coefficient Std. Error t-Statistic  Prob.
============================================================
        C            5.293654   2.762757   1.916077   0.0623
      LGDPIPC        0.589239   0.080158   7.351014   0.0000
     LGPRFOOD       -0.122598   0.084355  -1.453361   0.1537
       LGPOP        -0.699980   0.179299  -3.903973   0.0003
============================================================
R-squared            0.947037    Mean dependent var-6.321984
Adjusted R-squared   0.943161    S.D. dependent var 0.085249
S.E. of regression   0.020324    Akaike info criter-4.869317
Sum squared resid    0.016936    Schwarz criterion -4.708725
Log likelihood       113.5596    F-statistic        244.3727
Durbin-Watson stat   0.488502    Prob(F-statistic)  0.000000
============================================================
```

A11.6 Perform a regression parallel to that in Exercise A11.5 using your category of expenditure, and perform a t test of the restriction implicit in the specification in Exercise A11.4.

A11.7 In Exercise 11.9 you fitted the model:

$$LGCAT = \beta_1 + \beta_2 LGDPI + \beta_3 LGDPI(-1) + \beta_4 LGPRCAT + \beta_5 LGPRCAT(-1) + u$$

where CAT stands for your category of expenditure.

- Show that $(\beta_2 + \beta_3)$ and $(\beta_4 + \beta_5)$ are theoretically the long-run (equilibrium) income and price elasticities.
- Reparameterise the model and fit it to obtain direct estimates of these long-run elasticities and their standard errors.
- Confirm that the estimates are equal to the sum of the individual shortrun elasticities found in Exercise 11.9.
- Compare the standard errors with those found in Exercise 11.9 and state your conclusions.

A11.8 In a certain bond market, the demand for bonds, B_t, in period t is negatively related to the expected interest rate, i^e_{t+1}, in period $t+1$:

$$B_t = \beta_1 + \beta_2 i^e_{t+1} + u_t \qquad (1)$$

where u_t is a disturbance term not subject to autocorrelation. The expected interest rate is determined by an adaptive expectations process:

$$i^e_{t+1} - i^e_t = \lambda(i_t - i^e_t) \qquad (2)$$

where i_t is the actual rate of interest in period t. A researcher uses the following model to fit the relationship:

$$B_t = \gamma_1 + \gamma_2 i_t + \gamma_3 B_{t-1} + v_t \qquad (3)$$

where v_t is a disturbance term.

- Show how this model may be derived from the demand function and the adaptive expectations process.
- Explain why inconsistent estimates of the parameters will be obtained if equation (3) is fitted using ordinary least squares (OLS). (A mathematical proof is not required. Do not attempt to derive expressions for the bias.)
- Describe a method for fitting the model that would yield consistent estimates.
- Suppose that u_t was subject to the first-order autoregressive process:

$$u_t = \rho u_{t-1} + \varepsilon_t$$

  where ε_t is not subject to autocorrelation. How would this affect your answer to the second part of this question?
- Suppose that the true relationship was actually:

$$B_t = \beta_1 + \beta_2 i_t + u_t \qquad (1*)$$

  with u_t not subject to autocorrelation, and the model is fitted by regressing B_t on i_t and B_{t-1}, as in equation (3), using OLS. How would this affect the regression results?
- How plausible do you think an adaptive expectations process is for modelling expectations in a bond market?

A11.9 The output shows the result of a logarithmic regression of expenditure on food on income, relative price, population, and lagged expenditure on food using the Demand Functions data set. Provide an interpretation of the regression coefficients, paying attention to both short-run and long-run dynamics, and perform appropriate statistical tests.

```
============================================================
Dependent Variable: LGFOOD
Method: Least Squares
Sample(adjusted): 1960 2003
Included observations: 44 after adjusting endpoints
============================================================
     Variable     Coefficient Std. Error t-Statistic  Prob.
============================================================
         C          1.487645   2.072156   0.717921   0.4771
       LGDPI        0.143829   0.090334   1.592194   0.1194
     LGPRFOOD      -0.095749   0.061118  -1.566613   0.1253
       LGPOP       -0.046515   0.189453  -0.245524   0.8073
    LGFOOD(-1)      0.727290   0.113831   6.389195   0.0000
============================================================
R-squared           0.995886    Mean dependent var 6.030691
Adjusted R-squared  0.995464    S.D. dependent var 0.216227
S.E. of regression  0.014564    Akaike info criter-5.513937
Sum squared resid   0.008272    Schwarz criterion -5.311188
Log likelihood      126.3066    F-statistic        2359.938
Durbin-Watson stat  1.103102    Prob(F-statistic)  0.000000
============================================================
```

A11.10 Perform a regression parallel to that in Exercise A11.9 using your category of expenditure. Provide an interpretation of the coefficients, and perform appropriate statistical tests.

A11.11 In his classic study *Distributed Lags and Investment Analysis* (1954), Koyck investigated the relationship between investment in railcars and the volume of freight carried on the US railroads using data for the period 1884–1939. Assuming that the desired stock of railcars in year t depended on the volume of freight in year $t-1$ and year $t-2$ and a time trend, and assuming that investment in railcars was subject to a partial adjustment process, he fitted the following regression equation using OLS (standard errors and constant term not reported):

$$\widehat{I}_t = 0.077F_{t-1} + 0.017F_{t-2} - 0.0033t - 0.110K_{t-1} \qquad R^2 = 0.85$$

where $I_t = K_t - K_{t-1}$ is investment in railcars in year t (thousands), K_t is the stock of railcars at the end of year t (thousands), and F_t is the volume of freight handled in year t (ton-miles).

Provide an interpretation of the equation and describe the dynamic process implied by it. (Note: It is best to substitute $K_t - K_{t-1}$ for I_t in the regression and treat it as a dynamic relationship determining K_t.)

A11.12 Two researchers agree that a model consists of the following relationships:

$$\begin{aligned} Y_t &= \alpha_1 + \alpha_2 X_t + u_t && (1)\\ X_t &= \beta_1 + \beta_2 Y_{t-1} + v_t && (2)\\ Z_t &= \gamma_1 + \gamma_2 Y_t + \gamma_3 X_t + \gamma_4 Q_t + w_t && (3) \end{aligned}$$

where u_t, v_t, and w_t, are disturbance terms that are drawn from fixed distributions with zero mean. It may be assumed that they are distributed independently of Q_t and of each other and that they are not subject to autocorrelation. All the parameters may be assumed to be positive and it may be assumed that $\alpha_2\beta_2 < 1$.

- One researcher asserts that consistent estimates will be obtained if (2) is fitted using OLS and (1) is fitted using IV, with Y_{t-1} as an instrument for X_t. Determine whether this is true.
- The other researcher asserts that consistent estimates will be obtained if both (1) and (2) are fitted using OLS, and that the estimate of β_2 will be more efficient than that obtained using IV. Determine whether this is true.

11.4 Answers to the starred exercises in the textbook

11.6

Year	Y	K	L	Year	Y	K	L
1899	100	100	100	1911	153	216	145
1900	101	107	105	1912	177	226	152
1901	112	114	110	1913	184	236	154
1902	122	122	118	1914	169	244	149
1903	124	131	123	1915	189	266	154
1904	122	138	116	1916	225	298	182
1905	143	149	125	1917	227	335	196
1906	152	163	133	1918	223	366	200
1907	151	176	138	1919	218	387	193
1908	126	185	121	1920	231	407	193
1909	155	198	140	1921	179	417	147
1910	159	208	144	1922	240	431	161

Source: Cobb and Douglas (1928)

The table gives the data used by Cobb and Douglas (1928) to fit the original Cobb–Douglas production function:

$$Y_t = \beta_1 K_t^{\beta_2} L_t^{\beta_3} v_t$$

Y_t, K_t, and L_t, being index number series for real output, real capital input, and real labour input, respectively, for the manufacturing sector of the United States for the period 1899–1922 (1899 = 100). The model was linearised by taking logarithms of both sides and the following regressions was run (standard errors in parentheses):

$$\begin{array}{rcll} \widehat{\log Y} & = & \underset{(0.43)}{-0.18} + \underset{(0.06)}{0.23} \log K + \underset{(0.15)}{0.81} \log L & \qquad R^2 = 0.96 \end{array}$$

Provide an interpretation of the regression coefficients.

Answer:

The elasticities of output with respect to capital and labour are 0.23 and 0.81, respectively, both coefficients being significantly different from zero at very high significance levels. The fact that the sum of the elasticities is close to one suggests that there may be constant returns to scale. Regressing output per worker on capital per worker, one has:

$$\widehat{\log \frac{Y}{L}} = \underset{(0.02)}{0.01} + \underset{(0.04)}{0.25} \log \frac{K}{L} \qquad R^2 = 0.63$$

The smaller standard error of the slope coefficient suggests a gain in efficiency. Fitting a reparameterised version of the unrestricted model:

$$\widehat{\log \frac{Y}{L}} = \underset{(0.43)}{-0.18} + \underset{(0.06)}{0.23} \log \frac{K}{L} + \underset{(0.09)}{0.04} \log L \qquad R^2 = 0.64$$

we find that the restriction is not rejected.

11.7 The Cobb–Douglas model in Exercise 11.6 makes no allowance for the possibility that output may be increasing as a consequence of technical progress, independently of K and L. Technical progress is difficult to quantify and a common way of allowing for it in a model is to include an exponential time trend:

$$Y_t = \beta_1 K_t^{\beta_2} L_t^{\beta_3} \mathrm{e}^{\rho t} v_t$$

where ρ is the rate of technical progress and t is a time trend defined to be 1 in the first year, 2 in the second, etc. The correlations between $\log K$, $\log L$ and t are shown in the table. Comment on the regression results.

$$\begin{array}{rcll} \widehat{\log Y} & = & 2.81 - 0.53 \log K + 0.91 \log L + 0.047t & \quad R^2 = 0.97 \\ & & (1.38)\ (0.34) \qquad\quad (0.14) \qquad\quad (0.021) & \end{array}$$

Correlation

	LGK	LGL	TIME
LGK	1.000000	0.909562	0.996834
LGL	0.909562	1.000000	0.896344
TIME	0.996834	0.896344	1.000000

Answer:

The elasticity of output with respect to labour is higher than before, now implausibly high given that, under constant returns to scale, it should measure the share of wages in output. The elasticity with respect to capital is negative and nonsensical. The coefficient of time indicates an annual exponential growth rate of 4.7 per cent, holding K and L constant. This is unrealistically high for the period in question. The implausibility of the results, especially those relating to capital and time (correlation 0.997), may be attributed to multicollinearity.

11.16 Demonstrate that the dynamic process (11.18) implies the long-run relationship given by (11.15).

Answer:

Equations (11.15) and (11.18) are:

$$\tilde{Y} = \frac{\beta_1}{1-\beta_3} + \frac{\beta_2}{1-\beta_3}\tilde{X} \qquad (11.15)$$

$$\begin{aligned} Y_t &= \beta_1(1+\beta_3+\beta_3^2+\cdots) + \beta_2 X_t + \beta_2\beta_3 X_{t-1} + \beta_2\beta_3^2 X_{t-2} + \cdots \qquad (11.18) \\ &\quad + u_t + \beta_3 u_{t-1} + \beta_3^2 u_{t-2} + \cdots . \end{aligned}$$

Putting $X = \tilde{X}$ for all X in (11.18), and ignoring the disturbance terms, the long-run relationship between Y and X is given by:

$$\begin{aligned} \tilde{Y} &= \beta_1(1+\beta_3+\beta_3^2+\cdots) + \beta_2\tilde{X} + \beta_2\beta_3\tilde{X} + \beta_2\beta_3^2\tilde{X} + \cdots \\ &= \frac{\beta_1}{1-\beta_3} + (1+\beta_3+\beta_3^2+\cdots)\beta_2\tilde{X} \\ &= \frac{\beta_1}{1-\beta_3} + \frac{\beta_2}{1-\beta_3}\tilde{X}. \end{aligned}$$

11.17 The compound disturbance term in the adaptive expectations model (11.37) does potentially give rise to a problem that will be discussed in Chapter 12 when we come to the topic of autocorrelation. It can be sidestepped by representing the model in the alternative form.

$$Y_t = \beta_1 + \beta_2\lambda X_t + \beta_2\lambda(1-\lambda)X_{t-1} + \cdots + \beta_2\lambda(1-\lambda)^s X_{t-s} + \beta_2(1-\lambda)^{s+1}X^e_{t-s} + u_t.$$

Show how this form might be obtained, and discuss how it might be fitted.

Answer:

We start by reprising equations (11.31) – (11.34) in the text. We assume that the dependent variable Y_t is related to X^e_{t+1}, the value of X anticipated in the next time period:

$$Y_t = \gamma_1 + \gamma_2 X^e_{t+1} + u_t. \qquad (11.31)$$

To make the model operational, we hypothesise that expectations are updated in response to the discrepancy between what had been anticipated for the current time period, X^e_{t+1}, and the actual outcome, X_t:

$$X^e_{t+1} - X^e_t = \lambda(X_t - X^e_t) \qquad (11.32)$$

where λ may be interpreted as a speed of adjustment. We can rewrite this as (11.33):

$$X^e_{t+1} = \lambda X_t + (1-\lambda)X^e_t. \qquad (11.33)$$

Hence we obtain (11.34):

$$Y_t = \gamma_1 + \gamma_2\lambda X_t + \gamma_2(1-\lambda)X^e_t + u_t. \qquad (11.34)$$

This includes the unobservable X^e_t on the right side. However, lagging (11.33), we have:

$$X^e_t = \lambda X_{t-1} + (1-\lambda)X^e_{t-1}.$$

Hence:

$$Y_t = \gamma_1 + \gamma_2\lambda X_t + \gamma_2\lambda(1-\lambda)X_{t-1} + \gamma_2(1-\lambda)^2 X^e_{t-1} + u_t.$$

This includes the unobservable X^e_{t-1} on the right side. However, continuing to lag and substitute, we have:

$$Y_t = \gamma_1 + \gamma_2\lambda X_t + \gamma_2\lambda(1-\lambda)X_{t-1} + \cdots + \gamma_2\lambda(1-\lambda)^s X_{t-s} + \gamma_2(1-\lambda)^{s+1}X^e_{t-s} + u_t.$$

Provided that s is large enough for $\gamma_2(1-\lambda)^{s+1}$ to be very small, this may be fitted, omitting the unobservable final term, with negligible omitted variable bias.

We would fit it with a nonlinear regression technique that respected the constraints implicit in the theoretical structure of the coefficients.

11.19 The output below shows the result of fitting the model:

$$\begin{aligned} LGFOOD &= \beta_1 + \beta_2\lambda LGDPI + \beta_2\lambda(1-\lambda)LGDPI(-1) + \beta_2\lambda(1-\lambda)^2 LGDPI(-2) \\ &\quad + \beta_2\lambda(1-\lambda)^3 LGDPI(-3) + \beta_3 LGPRFOOD + u \end{aligned}$$

using the data on expenditure on food in the Demand Functions data set. *LGFOOD* and *LGPRFOOD* are the logarithms of expenditure on food and the relative price index series for food. C(1), C(2), C(3), and C(4) are estimates of β_1, β_2, λ and β_3, respectively. Explain how the regression equation could be interpreted as an adaptive expectations model and discuss the dynamics implicit in it, both short-run and long-run. Should the specification have included further lagged values of *LGDPI*?

```
============================================================
Dependent Variable: LGFOOD
Method: Least Squares
Sample(adjusted): 1962 2003
Included observations: 42 after adjusting endpoints
Convergence achieved after 25 iterations
LGFOOD=C(1)+C(2)*C(3)*LGDPI + C(2)*C(3)*(1-C(3))*LGDPI(-1) + C(2)
        *C(3)*(1-C(3))^2*LGDPI(-2) + C(2)*C(3)*(1-C(3))^3*LGDPI(-3) +
        C(4)*LGPRFOOD
============================================================
                   Coefficient Std. Error t-Statistic  Prob.
============================================================
       C(1)          2.339513   0.468550   4.993091   0.0000
       C(2)          0.496425   0.012264   40.47818   0.0000
       C(3)          0.915046   0.442851   2.066264   0.0457
       C(4)         -0.089681   0.083250  -1.077247   0.2882
============================================================
R-squared            0.989621    Mean dependent var 6.049936
Adjusted R-squared   0.988802    S.D. dependent var 0.201706
S.E. of regression   0.021345    Akaike info criter-4.765636
Sum squared resid    0.017313    Schwarz criterion -4.600143
Log likelihood       104.0784    Durbin-Watson stat 0.449978
============================================================
```

Answer:

Suppose that the model is:

$$LGFOOD_t = \gamma_1 + \gamma_2 LGDPI^e_{t+1} + \gamma_3 LGPRFOOD_t + u_t$$

where $LGDPI^e_{t+1}$ is expected *LGDPI* at time $t+1$, and that expectations for income are subject to the adaptive expectations process:

$$LGDPI^e_{t+1} - LGDPI^e_t = \lambda(LGDPI_t - LGDPI^e_t).$$

The adaptive expectations process may be rewritten:

$$LGDPI^e_{t+1} = \lambda LGDPI_t + (1-\lambda) LGDPI^e_t.$$

Lagging this equation one period and substituting, one has:

$$LGDPI^e_{t+1} = \lambda LGDPI_t + \lambda(1-\lambda) LGDPI_{t-1} + (1-\lambda)^2 LGDPI^e_{t-1}.$$

Lagging a second time and substituting, one has:

$$LGDPI^e_{t+1} = \lambda LGDPI_t + \lambda(1-\lambda) LGDPI_{t-1} + \lambda(1-\lambda)^2 LGDPI_{t-2} + (1-\lambda)^3 LGDPI^e_{t-2}.$$

Lagging a third time and substituting, one has:

$$\begin{aligned} LGDPI^e_{t+1} &= \lambda LGDPI_t + \lambda(1-\lambda) LGDPI_{t-1} + \lambda(1-\lambda)^2 LGDPI_{t-2} \\ &\quad + \lambda(1-\lambda)^3 LGDPI^e_{t-3} + (1-\lambda)^4 LGDPI^e_{t-3}. \end{aligned}$$

Substituting this into the model, dropping the final unobservable term, one has the regression specification as stated in the question.

The estimates imply that the short-run income elasticity is 0.50. The speed of adjustment of expectations is 0.92. Hence the long-run income elasticity is 0.50/0.92 = 0.54. The price side of the model has been assumed to be static. The estimate of the price elasticity is −0.09. The coefficient of the dropped unobservable term is $\gamma_2(1-\lambda)^4$. Given our estimates of γ_2 and λ, its estimate is 0.0003. Hence we are justified in neglecting it.

11.22 A researcher is fitting the following supply and demand model for a certain commodity, using a sample of time series observations:

$$\begin{aligned} Q_{dt} &= \beta_1 + \beta_2 P_t + u_{dt} \\ Q_{st} &= \alpha_1 + \alpha_2 P_t + u_{st} \end{aligned}$$

where Q_{dt} is the amount demanded at time t, Q_{st} is the amount supplied, P_t is the market clearing price, and u_{dt} and u_{st} are disturbance terms that are not necessarily independent of each other. It may be assumed that the market clears and so $Q_{dt} = Q_{st}$.

- What can be said about the identification of (a) the demand equation, (b) the supply equation?
- What difference would it make if supply at time t was determined instead by price at time $t-1$? That is:

$$Q_{st} = \alpha_1 + \alpha_2 P_{t-1} + u_{st}.$$

- What difference would it make if it could be assumed that u_{dt} is distributed independently of u_{st}?

Answer:

The reduced form equation for P_t is:

$$P_t = \frac{1}{\alpha_2 - \beta_2}(\beta_1 - \alpha_1 + u_{dt} - u_{st}).$$

P_t is not independent of the disturbance term in either equation and so OLS would yield inconsistent estimates. There is no instrument available, so both equations are underidentified.

Provided that u_{dt} is not subject to autocorrelation, P_{t-1} could be used as an instrument in the demand equation. Provided that u_{st} is not subject to autocorrelation, OLS could be used to fit the second equation. It makes no difference whether or not u_{dt} is distributed independently of u_{st}.

The first equation could, alternatively, be fitted using OLS, with the variables switched. From the second equation, P_{t-1} determines Q_t, and then, given Q_t, the demand equation determines P_t:

$$P_t = \frac{1}{\beta_2}(Q_t - \beta_1 - u_{dt}).$$

The reciprocal of the slope coefficient provides a consistent estimator of β_2.

11.24 Consider the following simple macroeconomic model:

$$\begin{aligned} C_t &= \beta_1 + \beta_2 Y_t + u_{C_t} \\ I_t &= \alpha_1 + \alpha_2 (Y_t - Y_{t-1}) + u_{I_t} \\ Y_t &= C_t + I_t \end{aligned}$$

where C_t, I_t, and Y_t are aggregate consumption, investment, and income and u_{C_t} and u_{I_t} are disturbance terms. The first relationship is a conventional consumption function. The second relates investment to the change of output from the previous year. (This is known as an 'accelerator' model.) The third is an income identity. What can be said about the identification of the relationships in the model?

Answer:

The restriction on the coefficients of Y_t and Y_{t-1} in the investment equation complicates matters. A simple way of handling it is to define:

$$\Delta Y_t = Y_t - Y_{t-1}$$

and to rewrite the investment equation as:

$$I_t = \alpha_1 + \alpha_2 \Delta Y_t + u_{I_t}.$$

We now have four endogenous variables and four equations, and one exogenous variable. The consumption and investment equations are exactly identified. We would fit them using Y_{t-1} as an instrument for Y_t and ΔY_t, respectively. The other two equations are identities and do not need to be fitted.

11.5 Answers to the additional exercises

A11.1 The linear regression indicates that expenditure on food increases by \$0.032 billion for every extra \$ billion of disposable personal income (in other words, by 3.2 cents out of the marginal dollar), that it increases by \$0.403 billion for every point increase in the price index, and that it increases by \$0.001 billion for every additional thousand population. The income coefficient is significant at the 1 per cent level (ignoring problems to be discussed in Chapter 12). The positive price coefficient makes no sense (remember that the dependent variable is measured in real terms). The intercept has no plausible interpretation.

The logarithmic regression indicates that the income elasticity is 0.59 and highly significant, and the price elasticity is -0.12, not significant. The negative elasticity for population is not plausible. One would expect expenditure on food to increase in line with population, controlling for other factors, and hence, as a first approximation, the elasticity should be equal to 1. However, an increase in population, keeping income constant, would lead to a reduction in income per capita and hence to a negative income effect. Given that the income elasticity is less than 1, one would still expect a positive elasticity overall for population. At least the estimate is not significantly different from zero. In view of the high correlation, 0.995, between *LGDPI* and *LGPOP*, the negative estimate may well be a result of multicollinearity.

A11.2

OLS logarithmic regressions							
	LGDPI		*LGP*		*LGPOP*		R^2
	coef.	s.e.	coef.	s.e.	coef.	s.e.	
ADM	−1.43	0.20	−0.28	0.10	6.88	0.61	0.975
BOOK	−0.29	0.28	−1.18	0.21	4.94	0.82	0.977
BUSI	0.36	0.19	−0.11	0.27	2.79	0.51	0.993
CLOT	0.71	0.10	−0.70	0.05	0.15	0.36	0.998
DENT	1.23	0.14	−0.95	0.09	0.26	0.54	0.995
DOC	0.97	0.14	0.26	0.13	−0.27	0.52	0.993
FLOW	0.46	0.32	0.16	0.33	3.07	1.21	0.987
FOOD	0.59	0.08	−0.12	0.08	−0.29	0.26	0.992
FURN	0.36	0.28	−0.48	0.26	1.66	1.12	0.985
GAS	1.27	0.24	−0.24	0.06	−2.81	0.74	0.788
GASO	1.46	0.16	−0.10	0.04	−2.35	0.49	0.982
HOUS	0.91	0.08	−0.54	0.06	0.38	0.25	0.999
LEGL	1.17	0.16	−0.08	0.13	−1.50	0.54	0.976
MAGS	1.05	0.22	−0.73	0.44	−0.82	0.54	0.970
MASS	−1.92	0.22	−0.57	0.14	6.14	0.65	0.785
OPHT	0.30	0.45	0.28	0.59	3.68	1.40	0.965
RELG	0.56	0.13	−0.99	0.23	2.72	0.41	0.996
TELE	0.91	0.13	−0.61	0.11	1.79	0.49	0.998
TOB	0.54	0.17	−0.42	0.04	−1.21	0.57	0.883
TOYS	0.59	0.10	−0.54	0.06	2.57	0.39	0.999

The price elasticities mostly lie in the range 0 to −1, as they should, and therefore seem plausible. However the very high correlation between income and population, 0.995, has given rise to a problem of multicollinearity and as a consequence the estimates of their elasticities are very erratic. Some of the income elasticities look plausible, but that may be pure chance, for many are unrealistically high, or negative when obviously they should be positive. The population elasticities are even less convincing.

Correlations between prices, income and population					
	LGP, *LGDPI*	*LGP*, *LGPOP*		*LGP*, *LGDPI*	*LGP*, *LGPOP*
ADM	0.61	0.61	*GASO*	0.05	0.03
BOOK	0.88	0.87	*HOUS*	0.49	0.55
BUSI	0.98	0.97	*LEGL*	0.99	0.99
CLOT	−0.94	−0.96	*MAGS*	0.99	0.98
DENT	0.94	0.96	*MASS*	0.90	0.89
DOC	0.98	0.98	*OPHT*	−0.68	−0.67
FLOW	−0.93	−0.95	*RELG*	0.92	0.92
FOOD	−0.60	−0.64	*TELE*	−0.98	−0.99
FURN	−0.95	−0.97	*TOB*	0.83	0.86
GAS	0.77	0.76	*TOYS*	−0.97	−0.98

A11.3 The regression indicates that the income elasticity is 0.40 and the price elasticity 0.21, the former very highly significant, the latter significant at the 1 per cent level using a one-sided test. If the specification is:

$$\log \frac{FOOD}{POP} = \beta_1 + \beta_2 \log \frac{DPI}{POP} + \beta_3 \log PRELFOOD + u$$

it may be rewritten:

$$\log FOOD = \beta_1 + \beta_2 \log DPI + \beta_3 \log PRELFOOD + (1 - \beta_2) \log POP + u.$$

This is a restricted form of the specification in Exercise A11.2:

$$\log FOOD = \beta_1 + \beta_2 \log DPI + \beta_3 \log PRELFOOD + \beta_4 \log POP + u$$

with $\beta_4 = 1 - \beta_2$. We can test the restriction by comparing RSS for the two regressions:

$$F(1, 41) = \frac{(0.023232 - 0.016936)/1}{0.016936/41} = 15.24.$$

The critical value of $F(1, 40)$ at the 0.1 per cent level is 12.61. The critical value for $F(1, 41)$ must be slightly lower. Thus we reject the restriction. Since the restricted version is misspecified, our interpretation of the coefficients of this regression and the t tests are invalidated.

A11.4 Given that the critical values of $F(1, 41)$ at the 5 and 1 per cent levels are 4.08 and 7.31 respectively, the results of the F test may be summarised as follows:

- Restriction not rejected: *CLOT*, *DENT*, *DOC*, *FURN*, *HOUS*.
- Restriction rejected at the 5 per cent level: *MAGS*.
- Restriction rejected at the 1 per cent level: *ADM*, *BOOK*, *BUSI*, *FLOW*, *FOOD*, *GAS*, *GASO*, *LEGL*, *MASS*, *OPHT*, *RELG*, *TELE*, *TOB*, *TOYS*.

However, for reasons that will become apparent in the next chapter, these findings must be regarded as provisional.

Tests of a restriction				
	RSS_U	RSS_R	F	t
ADM	0.125375	0.480709	116.20	10.78
BOOK	0.223664	0.461853	43.66	6.61
BUSI	0.084516	0.167580	40.30	6.35
CLOT	0.021326	0.021454	0.25	−0.50
DENT	0.033275	0.034481	1.49	1.22
DOC	0.068759	0.069726	0.58	−0.76
FLOW	0.220256	0.262910	7.94	2.82
FOOD	0.016936	0.023232	15.24	−3.90
FURN	0.157153	0.162677	1.44	1.20
GAS	0.185578	0.300890	25.48	−5.05
GASO	0.078334	0.139278	31.90	−5.65
HOUS	0.011270	0.012106	3.04	1.74
LEGL	0.082628	0.102698	9.96	−3.16
MAGS	0.096620	0.106906	4.36	−2.09
MASS	0.143775	0.330813	53.34	7.30
OPHT	0.663413	0.822672	9.84	3.14
RELG	0.053785	0.135532	62.32	7.89
TELE	0.054519	0.080728	19.71	4.44
TOB	0.062452	0.087652	16.54	−4.07
TOYS	0.031269	0.071656	52.96	7.28

A11.5 If the specification is:

$$\log \frac{FOOD}{POP} = \beta_1 + \beta_2 \log \frac{DPI}{POP} + \beta_3 \log PRELFOOD + \gamma_1 POP + u$$

it may be rewritten:

$$\log FOOD = \beta_1 + \beta_2 \log DPI + \beta_3 \log PRELFOOD + (1 - \beta_2 + \gamma_1) \log POP + u.$$

This is equivalent to the specification in Exercise A11.1:

$$\log FOOD = \beta_1 + \beta_2 \log DPI + \beta_3 \log PRELFOOD + \beta_4 \log POP + u$$

with $\beta_4 = 1 - \beta_2 + \gamma_1$. Note that this is not a restriction. (1) – (3) are just different ways of writing the unrestricted model.

A t test of $H_0 : \gamma_1 = 0$ is equivalent to a t test of $H_0 : \beta_4 = 1 - \beta_2$, that is, that the restriction in Exercise A11.3 is valid. The t statistic for $LGPOP$ in the regression is -3.90, and hence again we reject the restriction. Note that the test is equivalent to the F test. -3.90 is the square root of 15.24, the F statistic, and it can be shown that the critical value of t is the square root of the critical value of F.

A11.6 The t statistics for all the categories of expenditure are supplied in the table in the answer to Exercise A11.4. Of course they are equal to the square root of the F statistic, and their critical values are the square roots of the critical values of F, so the conclusions are identical and, like those of the F test, should be treated as provisional.

A11.7 *Show that $\beta_2 + \beta_3$ and $(\beta_4 + \beta_5)$ are theoretically the long-run (equilibrium) income and price elasticities.*

In equilibrium, $LGCAT = \overline{LGCAT}$, $LGDPI = LGDPI(-1) = \overline{LGDPI}$ and $LGPRCAT = LGPRCAT(-1) = \overline{LGPRCAT}$. Hence, ignoring the transient effect of the disturbance term:

$$\begin{aligned}\overline{LGCAT} &= \beta_1 + \beta_2\overline{LGDPI} + \beta_3\overline{LGDPI} + \beta_4\overline{LGPRCAT} + \beta_5\overline{LGPRCAT} \\ &= \beta_1 + (\beta_2 + \beta_3)\overline{LGDPI} + (\beta_4 + \beta_5)\overline{LGPRCAT}.\end{aligned}$$

Thus the long-run equilibrium income and price elasticities are $\theta = \beta_2 + \beta_3$ and $\phi = \beta_4 + \beta_5$, respectively.

Reparameterise the model and fit it to obtain direct estimates of these long-run elasticities and their standard errors.

We will reparameterise the model to obtain direct estimates of θ and ϕ and their standard errors. Write $\beta_3 = \theta - \beta_2$ and $\phi = \beta_4 + \beta_5$ and substitute for β_3 and β_5 in the model. We obtain:

$$\begin{aligned}LGCAT &= \beta_1 + \beta_2 LGDPI + (\theta - \beta_2)LGDPI(-1) + \beta_4 LGPRCAT + (\phi - \beta_4)LGPRCAT(-1) + u \\ &= \beta_1 + \beta_2(LGDPI - LGDPI(-1)) + \theta LGDPI(-1) \\ &\quad + \beta_4(LGPRCAT - LGPRCAT(-1)) + \phi LGPRCAT(-1) + u \\ &= \beta_1 + \beta_2 DLGDPI + \theta LGDPI(-1) + \beta_4 DLGPRCAT + \phi LGPRCAT(-1) + u\end{aligned}$$

where $DLGDPI = LGDPI - LGDPI(-1)$ and $DLGPRCAT = LGPRCAT - LGPRCAT(-1)$.

The output for *HOUS* is shown below. *DLGPRCAT* has been abbreviated as *DLGP*.

```
============================================================
Dependent Variable: LGHOUS
Method: Least Squares
Sample(adjusted): 1960 2003
Included observations: 44 after adjusting endpoints
============================================================
     Variable      Coefficient Std. Error t-Statistic  Prob.
============================================================
         C           0.020785   0.144497   0.143844   0.8864
      DLGDPI         0.329571   0.150397   2.191340   0.0345
     LGDPI(-1)       1.013147   0.006815   148.6735   0.0000
       DLGP         -0.088813   0.165651  -0.536144   0.5949
   LGPRHOUS(-1)     -0.447176   0.035927  -12.44689   0.0000
============================================================
R-squared            0.999039    Mean dependent var 6.379059
Adjusted R-squared   0.998940    S.D. dependent var 0.421861
S.E. of regression   0.013735    Akaike info criter-5.631127
Sum squared resid    0.007357    Schwarz criterion -5.428379
Log likelihood       128.8848    F-statistic        10131.80
Durbin-Watson stat   0.536957    Prob(F-statistic)  0.000000
============================================================
```

Confirm that the estimates are equal to the sum of the individual shortrun elasticities found in Exercise 11.9.

The estimates of the long-run income and price elasticities are 1.01 and −0.45, respectively. The output below is for the model in its original form, where the coefficients are all short-run elasticities. It may be seen that, for both income and price, the sum of the estimates of the shortrun elasticities is indeed equal to the estimate of the long-run elasticity in the reparameterised specification.

```
============================================================
Dependent Variable: LGHOUS
Method: Least Squares
Sample(adjusted): 1960 2003
Included observations: 44 after adjusting endpoints
============================================================
     Variable      Coefficient Std. Error t-Statistic  Prob.
============================================================
         C           0.020785   0.144497   0.143844   0.8864
       LGDPI         0.329571   0.150397   2.191340   0.0345
     LGDPI(-1)       0.683575   0.147111   4.646648   0.0000
     LGPRHOUS       -0.088813   0.165651  -0.536144   0.5949
   LGPRHOUS(-1)     -0.358363   0.165782  -2.161660   0.0368
============================================================
R-squared            0.999039    Mean dependent var 6.379059
Adjusted R-squared   0.998940    S.D. dependent var 0.421861
S.E. of regression   0.013735    Akaike info criter-5.631127
Sum squared resid    0.007357    Schwarz criterion -5.428379
Log likelihood       128.8848    F-statistic        10131.80
Durbin-Watson stat   0.536957    Prob(F-statistic)  0.000000
============================================================
```

Compare the standard errors with those found in Exercise 11.9 and state your conclusions.

The standard errors of the long-run elasticities in the reparameterised version are much smaller than those of the short-run elasticities in the original specification, and the t statistics accordingly much greater. Our conclusion is that it is possible to obtain relatively precise estimates of the long-run impact of income and price, even though multicollinearity prevents us from deriving precise short-run estimates.

A11.8 *Show how this model may be derived from the demand function and the adaptive expectations process.*

The adaptive expectations process may be rewritten:

$$i_{t+1}^e = \lambda i_t + (1 - \lambda) i_t^e.$$

Substituting this into (1), one obtains:

$$B_t = \beta_1 + \beta_2 \lambda i_t + \beta_2 (1 - \lambda) i_t^e + u_t.$$

We note that if we lag (1) by one time period:

$$B_{t-1} = \beta_1 + \beta_2 i_t^e + u_{t-1}.$$

Hence:

$$\beta_2 i_t^e = B_{t-1} - \beta_1 - u_{t-1}.$$

Substituting this into the second equation above, one has:

$$B_t = \beta_1 \lambda + \beta_2 \lambda i_t + (1 - \lambda) B_{t-1} + u_t - (1 - \lambda) u_{t-1}.$$

This is equation (3) in the question, with $\gamma_1 = \beta_1\lambda$, $\gamma_2 = \beta_2\lambda$, $\gamma_3 = 1 - \lambda$, and $v_t = u_t - (1-\lambda)u_{t-1}$.

Explain why inconsistent estimates of the parameters will be obtained if equation (3) is fitted using ordinary least squares (OLS). (A mathematical proof is not required. Do not attempt to derive expressions for the bias.)

In equation (3), the regressor B_{t-1} is partly determined by u_{t-1}. The disturbance term v_t also has a component u_{t-1}. Hence the requirement that the regressors and the disturbance term be distributed independently of each other is violated. The violation will lead to inconsistent estimates because the regressor and the disturbance term are contemporaneously correlated.

Describe a method for fitting the model that would yield consistent estimates.

If the first equation in this exercise is true for time period $t+1$, it is true for time period t:

$$i_t^e = \lambda i_{t-1} + (1-\lambda)i_{t-1}^e.$$

Substituting into the second equation in (a), we now have:

$$B_t = \beta_1 + \beta_2\lambda i_t + \beta_2\lambda(1-\lambda)i_{t-1} + (1-\lambda)^2 i_{t-1}^e + u_t.$$

Continuing to lag and substitute, we have:

$$B_t = \beta_1 + \beta_2\lambda i_t + \beta_2\lambda(1-\lambda)i_{t-1} + \cdots + \beta_2\lambda(1-\lambda)^{s-1}i_{t-s+1} + (1-\lambda)^s i_{t-s+1}^e + u_t.$$

For s large enough, $(1-\lambda)^s$ will be so small that we can drop the unobservable term i_{t-s+1}^e with negligible omitted variable bias. The disturbance term is distributed independently of the regressors and hence we obtain consistent estimates of the parameters. The model should be fitted using a nonlinear estimation technique that takes account of the restrictions implicit in the specification.

Suppose that u_t were subject to the first-order autoregressive process:

$$u_t = \rho u_{t-1} + \varepsilon_t$$

where ε_t is not subject to autocorrelation. How would this affect your answer to the second part of this question?

v_t is now given by:

$$v_t = u_t - (1-\lambda)u_{t-1} = \rho u_{t-1} + \varepsilon_t - (1-\lambda)u_{t-1} = \varepsilon_t - (1-\rho-\lambda)u_{t-1}.$$

Since ρ and λ may reasonably be assumed to lie between 0 and 1, it is possible that their sum is approximately equal to 1, in which case v_t is approximately equal to the innovation ϵ_t. If this is the case, there would be no violation of the regression assumption described in the second part of this question and one could use OLS to fit (3) after all.

Suppose that the true relationship was actually:

$$B_t = \beta_1 + \beta_2 i_t + u_t \qquad (1*)$$

with u_t not subject to autocorrelation, and the model is fitted by regressing B_t on i_t and B_{t-1}, as in equation (3), using OLS. How would this affect the regression results?

The estimators of the coefficients will be inefficient in that B_{t-1} is a redundant variable. The inclusion of B_{t-1} will also give rise to finite sample bias that would disappear in large samples.

How plausible do you think an adaptive expectations process is for modelling expectations in a bond market?

The adaptive expectations model is implausible since the expectations process would change as soon as those traders taking advantage of their knowledge of it started earning profits.

A11.9 The regression indicates that the short-run income, price, and population elasticities for expenditure on food are 0.14, −0.10, and −0.05, respectively, and that the speed of adjustment is $(1 - 0.73) = 0.27$. Dividing by 0.27, the long-run elasticities are 0.52, −0.37, and −0.19, respectively. The income and price elasticities seem plausible. The negative population elasticity makes no sense, but it is small and insignificant. The estimates of the short-run income and price elasticities are likewise not significant, but this is not surprising given that the point estimates are so small.

A11.10 The table gives the result of the specification with a lagged dependent variable for all the categories of expenditure.

	OLS logarithmic regression									
	$LGDPI$		LGP		$LGPOP$		$LGCAT(-1)$		Long-run effects	
	coef.	s.e.	coef.	s.e.	coef.	s.e.	coef.	s.e.	DPI	P
ADM	−0.38	0.18	−0.10	0.06	2.03	0.74	0.68	0.09	−1.18	−0.33
$BOOK$	−0.36	0.20	−0.21	0.22	2.07	0.74	0.75	0.12	−1.46	−1.05
$BUSI$	0.10	0.13	0.03	0.18	0.78	0.45	0.72	0.11	0.33	0.09
$CLOT$	0.44	0.10	−0.40	0.07	0.01	0.32	0.43	0.09	0.77	−0.70
$DENT$	0.71	0.18	−0.46	0.16	−0.13	0.51	0.47	0.13	1.34	−0.87
DOC	0.23	0.14	−0.11	0.10	0.21	0.35	0.78	0.10	1.04	−0.52
$FLOW$	0.20	0.24	−0.31	0.27	0.07	0.98	0.75	0.11	0.81	−1.25
$FOOD$	0.14	0.09	−0.10	0.06	−0.05	0.19	0.73	0.11	0.53	−0.35
$FURN$	0.07	0.22	−0.07	0.22	0.82	0.91	0.68	0.12	0.21	−0.23
GAS	0.10	0.17	−0.06	0.03	−0.13	0.45	0.76	0.08	0.42	−0.26
$GASO$	0.32	0.11	−0.10	0.02	−0.59	0.25	0.80	0.06	1.56	−0.47
$HOUS$	0.30	0.05	−0.09	0.04	−0.13	0.10	0.73	0.05	1.11	−0.32
$LEGL$	0.40	0.14	0.10	0.09	−0.90	0.36	0.68	0.09	1.23	0.30
$MAGS$	0.57	0.21	−0.48	0.37	−0.56	0.44	0.55	0.12	1.27	−1.08
$MASS$	−0.28	0.29	−0.23	0.11	1.08	0.89	0.75	0.12	−1.14	−0.93
$OPHT$	0.30	0.24	−0.28	0.33	−0.45	0.85	0.88	0.09	2.48	−2.25
$RELG$	0.34	0.09	−0.71	0.17	1.25	0.38	0.51	0.09	0.68	−1.44
$TELE$	0.15	0.14	0.00	0.12	0.68	0.37	0.81	0.12	0.77	0.02
TOB	0.12	0.14	−0.12	0.05	−0.31	0.43	0.71	0.11	0.43	−0.43
$TOYS$	0.31	0.11	−0.27	0.08	1.44	0.47	0.47	0.12	0.58	−0.51

A11.11 *In his classic study Distributed Lags and Investment Analysis (1954), Koyck investigated the relationship between investment in railcars and the volume of freight carried on the US railroads using data for the period 1884–1939. Assuming that the desired stock of railcars in year t depended on the volume of freight in year $t-1$ and year $t-2$ and a time trend, and assuming that investment in railcars was subject to a partial adjustment process, he fitted the following regression equation using OLS (standard errors and constant term not reported):*

$$\widehat{I}_t = 0.077F_{t-1} + 0.017F_{t-2} - 0.0033t - 0.110K_{t-1} \qquad R^2 = 0.85$$

where $I_t = K_t - K_{t-1}$ is investment in railcars in year t (thousands), K_t is the stock of railcars at the end of year t (thousands), and F_t is the volume of freight handled in year t (ton-miles).

Provide an interpretation of the equation and describe the dynamic process implied by it. (Note: It is best to substitute $K_t - K_{t-1}$ for I_t in the regression and treat it as a dynamic relationship determining K_t).

Given the information in the question, the model may be written:

$$\begin{aligned} K_t^* &= \beta_1 + \beta_2 F_{t-1} + \beta_3 F_{t-2} + \beta_4 t + u_t \\ K_t - K_{t-1} &= I_t = \lambda(K_t^* - K_{t-1}). \end{aligned}$$

Hence:

$$I_t = \lambda\beta_1 + \lambda\beta_2 F_{t-1} + \lambda\beta_3 F_{t-2} + \lambda\beta_4 t - \lambda K_{t-1} + \lambda u_t.$$

From the fitted equation:

$$\begin{aligned} \widehat{\lambda} &= 0.110 \\ \widehat{\beta}_2 &= \frac{0.077}{0.110} = 0.70 \\ \widehat{\beta}_3 &= \frac{0.017}{0.110} = 0.15 \\ \widehat{\beta}_4 &= \frac{-0.0033}{0.110} = -0.030. \end{aligned}$$

Hence the short-run effect of an increase of 1 million ton-miles of freight is to increase investment in railcars by 7,000 one year later and 1,500 two years later. It does not make much sense to talk of a short-run effect of a time trend.

In the long-run equilibrium, neglecting the effects of the disturbance term, K_t and K_t^* are both equal to the equilibrium value $\overline{K}$ and F_{t-1} and F_{t-2} are both equal to their equilibrium value $\overline{F}$. Hence, using the first equation:

$$\overline{K} = \beta_1 + (\beta_2 + \beta_3)\overline{F} + \beta_4 t.$$

Thus an increase of one million ton-miles of freight will increase the stock of railcars by 940 and the time trend will be responsible for a secular decline of 33 railcars per year.

A11.12 *One researcher asserts that consistent estimates will be obtained if (2) is fitted using OLS and (1) is fitted using IV, with Y_{t-1} as an instrument for X_t. Determine whether this is true.*

(2) may indeed be fitted using OLS. Strictly speaking, there may be an element of bias in finite samples because of noncontemporaneous correlation between v_t and future values of Y_{t-1}.

We could indeed use Y_{t-1} as an instrument for X_t in (1) because Y_{t-1} is a determinant of X_t but is not (contemporaneously) correlated with u_t.

The other researcher asserts that consistent estimates will be obtained if both (1) and (2) are fitted using OLS, and that the estimate of β_2 will be more efficient than that obtained using IV. Determine whether this is true.

This assertion is also correct. X_t is not correlated with u_t, and OLS estimators are more efficient than IV estimators when both are consistent. Strictly speaking, there may be an element of bias in finite samples because of noncontemporaneous correlation between u_t and future values of X_t.

Chapter 12

Properties of regression models with time series data

12.1 Overview

This chapter begins with a statement of the regression model assumptions for regressions using time series data, paying particular attention to the assumption that the disturbance term in any time period be distributed independently of the regressors in all time periods. There follows a general discussion of autocorrelation: the meaning of the term, the reasons why the disturbance term may be subject to it, and the consequences of it for OLS estimators. The chapter continues by presenting the Durbin–Watson test for AR(1) autocorrelation and showing how the problem may be eliminated. Next it is shown why OLS yields inconsistent estimates when the disturbance term is subject to autocorrelation and the regression model includes a lagged dependent variable as an explanatory variable. Then the chapter shows how the restrictions implicit in the AR(1) specification may be tested using the common factor test, and this leads to a more general discussion of how apparent autocorrelation may be caused by model misspecification. This in turn leads to a general discussion of the issues involved in model selection and, in particular, to the general-to-specific methodology.

12.2 Learning outcomes

After working through the corresponding chapter in the text, studying the corresponding slideshows, and doing the starred exercises in the text and the additional exercises in this subject guide, you should be able to:

- explain the concept of autocorrelation and the difference between positive and negative autocorrelation
- describe how the problem of autocorrelation may arise
- describe the consequences of autocorrelation for OLS estimators, their standard errors, and t and F tests, and how the consequences change if the model includes a lagged dependent variable
- perform the Breusch–Godfrey and Durbin–Watson d tests for autocorrelation
- explain how the problem of AR(1) autocorrelation may be eliminated
- describe the restrictions implicit in the AR(1) specification

- perform the common factor test
- explain how apparent autocorrelation may arise as a consequence of the omission of an important variable or the mathematical misspecification of the regression model
- demonstrate that the static, AR(1), and ADL(1,0) specifications are special cases of the ADL(1,1) model
- explain the principles of the general-to-specific approach to model selection and the defects of the specific-to-general approach.

12.3 Additional exercises

A12.1 The output shows the result of a logarithmic regression of expenditure on food on income, relative price, and population, using an AR(1) specification. Compare the results with those in Exercise A11.1.

```
============================================================
Dependent Variable: LGFOOD
Method: Least Squares
Sample(adjusted): 1960 2003
Included observations: 44 after adjusting endpoints
Convergence achieved after 14 iterations
============================================================
     Variable      Coefficient Std. Error t-Statistic  Prob.
============================================================
         C           2.945983   3.943913   0.746969   0.4596
       LGDPI         0.469216   0.118230   3.968687   0.0003
     LGPRFOOD       -0.361862   0.122069  -2.964413   0.0052
       LGPOP         0.072193   0.379563   0.190200   0.8501
       AR(1)         0.880631   0.092512   9.519085   0.0000
============================================================
R-squared            0.996695    Mean dependent var 6.030691
Adjusted R-squared   0.996356    S.D. dependent var 0.216227
S.E. of regression   0.013053    Akaike info criter-5.732970
Sum squared resid    0.006645    Schwarz criterion -5.530221
Log likelihood       131.1253    F-statistic        2940.208
Durbin--Watson stat   1.556480    Prob(F-statistic)  0.000000
============================================================
Inverted AR Roots         .88
============================================================
```

A12.2 Perform Breusch–Godfrey and Durbin–Watson tests for autocorrelation for the logarithmic regression in Exercise A11.2. If you reject the null hypothesis of no autocorrelation, run the regression again using an AR(1) specification, and compare the results with those in Exercise A11.2.

A12.3 Perform an OLS ADL(1,1) logarithmic regression of expenditure on your category on current income, price, and population and lagged expenditure, income, price, and population. Use the results to perform a common factor test of the validity of the AR(1) specification in Exercise A12.2.

A12.4 A researcher has annual data on *LIFE*, aggregate consumer expenditure on life insurance, *DPI*, aggregate disposable personal income, and *PRELLIFE*, a price index for the cost of life insurance relative to general inflation, for the United States for the period 1959–1994. *LIFE* and *DPI* are measured in US$ billion. *PRELLIFE* is an index number series with 1992 = 100. She defines *LGLIFE*, *LGDPI*, and *LGPRLIFE* as the natural logarithms of *LIFE*, *DPI*, and *PRELLIFE*, respectively. She fits the regressions shown in columns (1) – (4) of the table, each with *LGLIFE* as the dependent variable. (Standard errors in parentheses; OLS = ordinary least squares; AR(1) is a specification appropriate when the disturbance term follows a first-order autoregressive process; *B–G* is the Breusch–Godfrey test statistic for AR(1) autocorrelation; d = Durbin–Watson d statistic; $\widehat{\rho}$ is the estimate of the autoregressive parameter in a first-order autoregressive process.)

	(1)	(2)	(3)	(4)	(5)
	OLS	AR(1)	OLS	OLS	OLS
LGDPI	1.37	1.41	0.42	0.28	—
	(0.10)	(0.25)	(0.60)	(0.17)	
LGPRLIFE	−0.67	−0.78	−0.59	−0.26	—
	(0.35)	(0.50)	(0.51)	(0.21)	
LGLIFE(−1)	—	—	0.82	0.79	0.98
			(0.10)	(0.09)	(0.02)
LGDPI(−1)	—	—	−0.15	—	—
			(0.61)		
LGPRLIFE(−1)	—	—	0.38	—	—
			(0.53)		
constant	−4.39	−4.20	−0.50	−0.51	0.12
	(0.88)	(1.69)	(0.72)	(0.70)	(0.08)
R^2	0.958	0.985	0.986	0.986	0.984
RSS	0.2417	0.0799	0.0719	0.0732	0.0843
B–G	23.48	—	0.61	0.34	0.10
d	0.36	1.85	2.02	1.92	2.05
$\widehat{\rho}$	—	0.82	—	—	—
		(0.11)			

- Discuss whether specification (1) is an adequate representation of the data.
- Discuss whether specification (3) is an adequate representation of the data.
- Discuss whether specification (2) is an adequate representation of the data.
- Discuss whether specification (4) is an adequate representation of the data.
- If you were presenting these results at a seminar, what would you say were your conclusions concerning the most appropriate of specifications (1) – (4)?
- At the seminar a commentator points out that in specification (4) neither *LGDPI* nor *LGPRLIFE* have significant coefficients and so these variables should be dropped. As it happens, the researcher has considered this specification, and the results are shown as specification (5) in the table. What would be your answer to the commentator?

A12.5 A researcher has annual data on the yearly rate of change of the consumer price index, p, and the yearly rate of change of the nominal money supply, m, for a certain country for the 51-year period 1958–2008. He fits the following regressions, each with p as the dependent variable. The first four regressions are fitted using OLS. The fifth is fitted using a specification appropriate when the disturbance term is assumed to follow an AR(1) process. $p(-1)$ indicates p lagged one year. $m(-1)$, $m(-2)$, and $m(-3)$ indicate m lagged 1, 2, and 3 years, respectively.

(1) explanatory variable m.

(2) explanatory variables m, $m(-1)$, $m(-2)$, and $m(-3)$.

(3) explanatory variables m, $p(-1)$, and $m(-1)$.

(4) explanatory variables m and $p(-1)$.

(5) explanatory variable m.

The results are shown in the table. Standard errors are shown in parentheses. RSS is the residual sum of squares. $B-G$ is the Breusch–Godfrey test statistic for AR(1) autocorrelation. d is the Durbin–Watson d statistic.

	1	2	3	4	5
	OLS	OLS	OLS	OLS	AR(1)
m	0.95	0.50	0.40	0.18	0.90
	(0.05)	(0.30)	(0.12)	(0.09)	(0.08)
$m(-1)$	—	0.30	−0.30	—	—
		(0.30)	(0.10)		
$m(-2)$	—	−0.15	—	—	—
		(0.30)			
$m(-3)$	—	0.30	—	—	—
		(0.30)			
$p(-1)$	—	—	0.90	0.80	—
			(0.20)	(0.20)	
constant	0.05	0.04	0.06	0.05	0.06
	(0.04)	(0.04)	(0.04)	(0.04)	(0.03)
RSS	0.0200	0.0150	0.0100	0.0120	0.0105
B–G	35.1	27.4	0.39	0.26	0.57
d	0.10	0.21	2.00	2.00	1.90

- Looking at all five regressions together, evaluate the adequacy of:
  - specification 1.
  - specification 2.
  - specification 3.
  - specification 4.
- Explain why specification 5 is a restricted version of one of the other specifications, stating the restriction, and explaining the objective of the manipulations that lead to specification 5.
- Perform a test of the restriction embodied in specification 5.
- Explain which would be your preferred specification.

A12.6 Derive the short-run (current year) and long-run (equilibrium) effect of m on p for each of the five specifications in Exercise A12.5, using the estimated coefficients.

A12.7 A researcher has annual data on aggregate consumer expenditure on taxis, $TAXI$, and aggregate disposable personal income, DPI, both measured in \$ billion at 2000 constant prices, and a relative price index for taxis, P, equal to 100 in 2000, for the United States for the period 1981–2005.

Defining $LGTAXI$, $LGDPI$, and LGP as the natural logarithms of $TAXI$, DPI, and P, respectively, he fits regressions (1) – (4) shown in the table. OLS = ordinary least squares; AR(1) indicates that the equation was fitted using a specification appropriate for first-order autoregressive autocorrelation; $\widehat{\rho}$ is an estimate of the parameter in the AR(1) process; B–G is the Breusch–Godfrey statistic for AR(1) autocorrelation; d is the Durbin–Watson d statistic; standard errors are given in parentheses.

	(1)	(2)	(3)	(4)
	OLS	AR(1)	OLS	AR(1)
$LGDPI$	2.06	1.28	2.28	2.24
	(0.10)	(0.84)	(0.05)	(0.07)
LGP	—	—	−0.99	−0.97
			(0.09)	(0.11)
constant	−12.75	−7.45	−9.58	−9.45
	(0.68)	(5.89)	(0.40)	(0.54)
$\widehat{\rho}$	—	0.88	—	0.26
		(0.09)		(0.22)
B–G	17.84	—	1.47	—
d	0.31	1.40	1.46	1.88
R^2	0.95	0.98	0.99	0.99

Figure 12.1 shows the actual values of $LGTAXI$ and the fitted values from regression (1). Figure 12.2 shows the residuals from regression (1) and the values of LGP.

- Evaluate regression (1).
- Evaluate regression (2). Explain mathematically what assumptions were being made by the researcher when he used the AR(1) specification and why he hoped the results would be better than those obtained with regression (1).
- Evaluate regression (3).
- Evaluate regression (4). In particular, discuss the possible reasons for the differences in the standard errors in regressions (3) and (4).
- At a seminar one of the participants says that the researcher should consider adding lagged values of $LGTAXI$, $LGDPI$, and LGP to the specification. What would be your view?

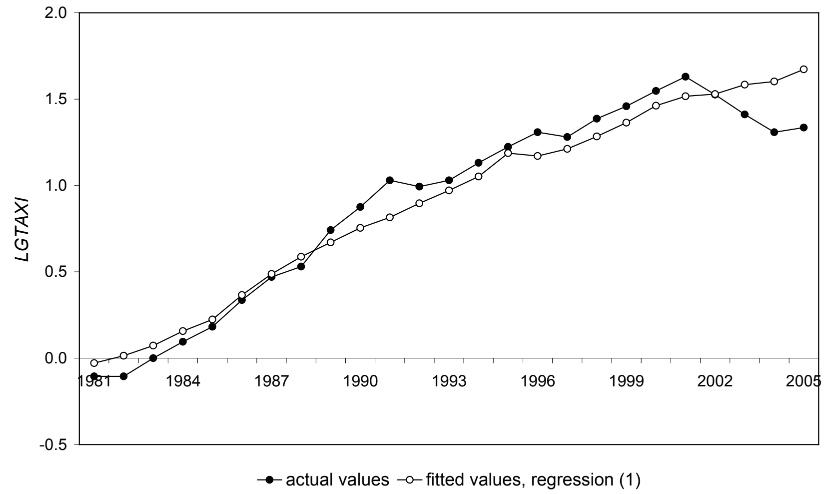


Figure 12.1: Actual values of $LGTAXI$ and the fitted values from regression (1).

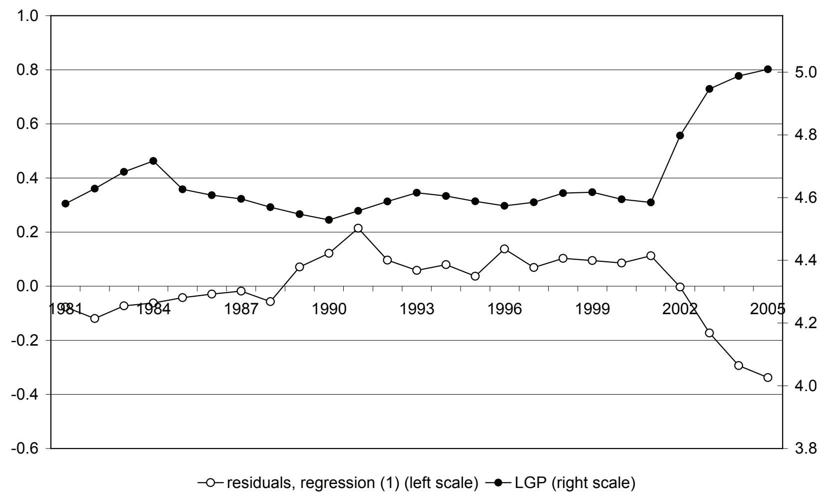


Figure 12.2: Residuals from regression (1) and the values of LGP.

A12.8 A researcher has annual data on I, investment as a percentage of gross domestic product, and r, the real long-term rate of interest for a certain economy for the period 1981–2010. He regresses I on r, (1) using ordinary least squares (OLS), (2) using an estimator appropriate for AR(1) residual autocorrelation, and (3) using OLS but adding $I(-1)$ and $r(-1)$ (I and r lagged one time period) as explanatory variables. The results are shown in columns (1), (2), and (3) of the table below. The residuals from regression (1) are shown in Figure 12.3.

He then obtains annual data on g, the rate of growth of gross domestic product of the economy, for the same period, and repeats the regressions, adding g (and, where appropriate, $g(-1)$) to the specifications as an explanatory variable. The results are shown in columns (4), (5), and (6) of the table. r and g are measured as per cent per year. The data for g are plotted in the figure.

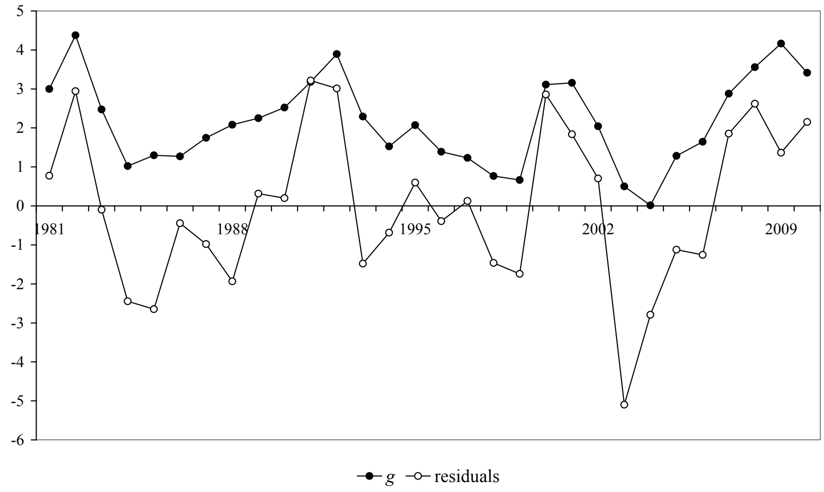


Figure 12.3: Residuals from regression (1).

	OLS	AR(1)	OLS	OLS	AR(1)	OLS
	(1)	(2)	(3)	(4)	(5)	(6)
r	−0.87	−0.83	−0.87	−1.81	−1.88	−1.71
	(0.98)	(1.05)	(1.08)	(0.49)	(0.50)	(0.52)
$I(-1)$	—	—	0.37	—	—	−0.22
			(0.16)			(0.18)
$r(-1)$	—	—	0.64	—	—	−0.98
			(1.08)			(0.64)
g	—	—	—	1.61	1.61	1.92
				(0.17)	(0.18)	(0.20)
$g(-1)$	—	—	—	—	—	−0.02
						(0.33)
$\widehat{\rho}$	—	0.37	—	—	−0.16	—
		(0.18)			(0.20)	
Constant	9.31	9.21	4.72	9.26	9.54	13.24
	(3.64)	(3.90)	(4.48)	(1.77)	(1.64)	(2.69)
B–G	4.42	—	4.24	0.70	—	0.98
d	0.99	1.36	1.33	2.30	2.05	2.09
RSS	120.5	103.9	103.5	27.4	26.8	23.5

Note: standard errors are given in parentheses. $\widehat{\rho}$ is the estimate of the autocorrelation parameter in the AR(1) specification. B–G is the Breusch–Godfrey statistic for AR(1) autocorrelation. d is the Durbin–Watson d statistic.

- Explain why the researcher was not satisfied with regression (1).
- Evaluate regression (2). Explain why the coefficients of $I(-1)$ and $r(-1)$ are not reported, despite the fact that they are part of the regression specification.
- Evaluate regression (3).

- Evaluate regression (4).
- Evaluate regression (5).
- Evaluate regression (6).
- Summarise your conclusions concerning the evaluation of the different regressions. Explain whether an examination of the figure supports your conclusions

A12.9 In Exercise A11.5 you performed a test of a restriction. The result of this test will have been invalidated if you found that the specification was subject to autocorrelation. How should the test be performed, assuming the correct specification is ADL(1,1)?

A12.10 Given data on a univariate process:

$$Y_t = \beta_1 + \beta_2 y_{t-1} + u_t$$

where $|\beta_2| < 1$ and u_t is iid, the usual OLS estimators will be consistent but subject to finite-sample bias. How should the model be fitted if u_t is subject to an AR(1) process?

A12.11 Explain what is correct, incorrect, confused or incomplete in the following statements, giving a brief explanation if not correct.

- The disturbance term in a regression model is said to be autocorrelated if its values in a sample of observations are not distributed independently of each other.
- When the disturbance term is subject to autocorrelation, the ordinary least squares estimators are inefficient and inconsistent, but they are not biased, and the t tests are invalid.
- It is a common problem in time series models because it always occurs when the dependent variable is correlated with its previous values.
- If this is the case, it could be eliminated by including the lagged value of the dependent variable as an explanatory variable.
- However, if the model is correctly specified and the disturbance term satisfies the regression model assumptions, adding the lagged value of the dependent variable as an explanatory variable will have the opposite effect and cause the disturbance term to be autocorrelated.
- A second way of dealing with the problem of autocorrelation is to use an instrumental variable.
- If the autocorrelation is of the AR(1) type, randomising the order of the observations will cause the Breusch–Godfrey statistic to be near zero, and the Durbin–Watson statistic to be near 2, thereby eliminating the problem.

12.4 Answers to the starred exercises in the textbook

12.7 Prove that σ_u^2 is related to σ_ε^2 as shown in (12.31), and show that weighting the first observation by $\sqrt{1-\rho^2}$ eliminates the heteroskedasticity.

Answer:

(12.31) is:

$$\sigma_u^2 = \frac{1}{1-\rho^2}\sigma_\varepsilon^2$$

and it assumes the first order AR(1) process (12.26): $u_t = \rho u_{t-1} + \varepsilon_t$. From the AR(1) process, neglecting transitory effects, $\sigma_{u_t} = \sigma_{u_{t-1}} = \sigma_u$ and so:

$$\sigma_u^2 = \rho^2\sigma_u^2 + \sigma_\varepsilon^2 = \frac{1}{1-\rho^2}\sigma_\varepsilon^2.$$

(Note that the covariance between u_{t-1} and ε_t is zero.) If the first observation is weighted by $\sqrt{1-\rho^2}$, the variance of the disturbance term will be:

$$\left(\sqrt{1-\rho^2}\right)^2\sigma_u^2 = (1-\rho^2)\frac{1}{1-\rho^2}\sigma_\varepsilon^2 = \sigma_\varepsilon^2$$

and it will therefore be the same as in the other observations in the sample.

12.10 The table gives the results of three logarithmic regressions using the Cobb–Douglas data for Y_t, K_t, and L_t, index number series for real output, real capital input, and real labor input, respectively, for the manufacturing sector of the United States for the period 1899–1922, reproduced in Exercise 11.6 (method of estimation as indicated; standard errors in parentheses; d = Durbin–Watson d statistic; B–G = Breusch–Godfrey test statistic for first-order autocorrelation):

	1: OLS	2: AR(1)	3: OLS
$\log K$	0.23	0.22	0.18
	(0.06)	(0.07)	(0.56)
$\log L$	0.81	0.86	1.03
	(0.15)	(0.16)	(0.15)
$\log Y(-1)$	—	—	0.40
			(0.21)
$\log K(-1)$	—	—	0.17
			(0.51)
$\log L(-1)$	—	—	−1.01
			(0.25)
constant	−0.18	−0.35	1.04
	(0.43)	(0.51)	(0.41)
$\widehat{\rho}$	—	0.19	—
		(0.25)	
R^2	0.96	0.96	0.98
RSS	0.0710	0.0697	0.0259
d	1.52	1.54	1.46
B–G	0.36	—	1.54

The first regression is that performed by Cobb and Douglas. The second fits the same specification, allowing for AR(1) autocorrelation. The third specification uses OLS with lagged variables. Evaluate the three regression specifications.

Answer:

For the first specification, the Breusch–Godfrey LM test for autocorrelation yields statistics of 0.36 (first order) and 1.39 (second order), both satisfactory. For the Durbin–Watson test, d_L and d_U are 1.19 and 1.55 at the 5 per cent level and 0.96 and 1.30 at the 1 per cent level, with 24 observations and two explanatory variables. Hence the specification appears more or less satisfactory. Fitting the model with an AR(1) specification makes very little difference, the estimate of ρ being low. However, when we fit the general ADL(1,1) model, neither of the first two specifications appears to be an acceptable simplification. The F statistic for dropping all the lagged variables is:

$$F(3, 18) = \frac{(0.0710 - 0.0259)/3}{0.0259/18} = 10.45.$$

The critical value of $F(3, 18)$ at the 0.1 per cent level is 8.49. The common factor test statistic is:

$$23 \log \frac{0.0697}{0.0259} = 22.77$$

and the critical value of chi-squared with two degrees of freedom is 13.82 at the 0.1 per cent level. The Breusch–Godfrey statistic for first-order autocorrelation is 1.54.

We come to the conclusion that Cobb and Douglas, who actually fitted a restricted version of the first specification, imposing constant returns to scale, were a little fortunate to obtain the plausible results they did.

12.11 Derive the final equation in Box 12.2 from the first two equations in the box. What assumptions need to be made when fitting the model?

Answer:

This exercise overlaps Exercise 11.17. The first two equations in the box are:

$$\begin{aligned} Y_t &= \beta_1 + \beta_2 X^e_{t+1} + u_t \\ X^e_{t+1} - X^e_t &= \lambda(X_t - X^e_t). \end{aligned}$$

We can rewrite the second equation as:

$$X^e_{t+1} = \lambda X_t + (1 - \lambda) X^e_t.$$

Substituting this into the first equation, we have:

$$Y_t = \beta_1 + \beta_2 \lambda X_t + \beta_2 (1 - \lambda) X^e_t + u_t.$$

This includes the unobservable X^e_t on the right side. However, lagging the second equation, we have:

$$X^e_t = \lambda X_{t-1} + (1 - \lambda) X^e_{t-1}.$$

Hence:

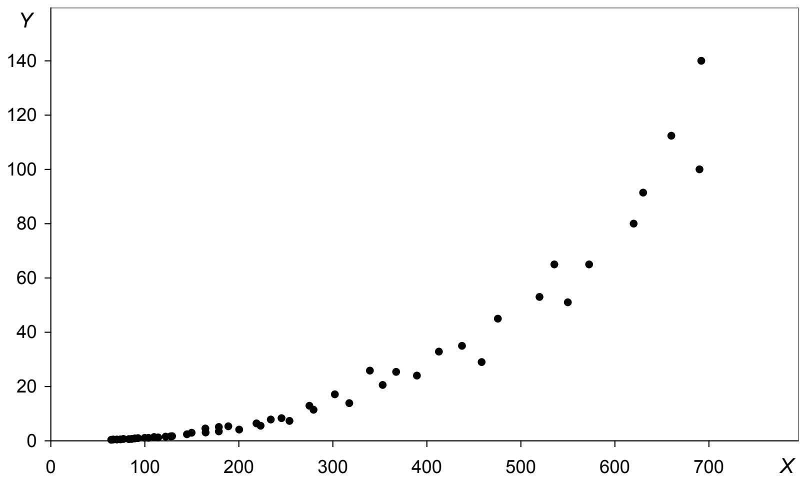
$$Y_t = \beta_1 + \beta_2 \lambda X_t + \beta_2 \lambda (1 - \lambda) X_{t-1} + \beta_2 (1 - \lambda)^2 X^e_{t-1} + u_t.$$

This includes the unobservable X^e_{t-1} on the right side. However, continuing to lag and substitute, we have:

$$Y_t = \beta_1 + \beta_2\lambda X_t + \beta_2\lambda(1-\lambda)X_{t-1} + \cdots + \beta_2\lambda(1-\lambda)^s X_{t-s} + \beta_2(1-\lambda)^{s+1}X^e_{t-s} + u_t.$$

Provided that s is large enough for $\beta_2(1-\lambda)^{s+1}$ to be very small, this may be fitted, omitting the unobservable final term, with negligible omitted variable bias. We would fit it with a nonlinear regression technique that respected the constraints implicit in the theoretical structure of the coefficients. The disturbance term is unaffected by the manipulations. Hence it is sufficient to assume that it is well-behaved in the original specification.

12.14 Using the 50 observations on two variables Y and X shown in the diagram below, an investigator runs the following five regressions (estimation method as indicated; standard errors in parentheses; all variables as logarithms in the logarithmic regressions; d = Durbin–Watson d statistic; B–G = Breusch–Godfrey test statistic):



	1	2	3	4	5
	Linear		Logarithmic		
	OLS	AR(1)	OLS	AR(1)	OLS
X	0.16	0.03	2.39	2.39	1.35
	(0.01)	(0.05)	(0.03)	(0.03)	(0.70)
$Y(-1)$	—	—	—	—	−0.11
					(0.15)
$X(-1)$	—	—	—	—	1.30
					(0.75)
$\widehat{\rho}$	—	1.16	—	−0.14	—
		(0.06)		(0.15)	
constant	−21.88	−2.52	−11.00	−10.99	−12.15
	(3.17)	(8.03)	(0.15)	(0.14)	(1.67)
R^2	0.858	0.974	0.993	0.993	0.993
RSS	7663	1366	1.011	0.993	0.946
d	0.26	2.75	2.17	1.86	21.95
B–G	39.54	—	0.85	—	1.03

Discuss each of the five regressions, explaining which is your preferred specification.

Answer:

The scatter diagram reveals that the relationship is nonlinear. If it is fitted with a linear regression, the residuals must be positive for the largest and smallest values of X and negative for the middle ones. As a consequence it is no surprise to find a high Breusch–Godfrey statistic, above 10.83, the critical value of $\chi^2(1)$ at the 0.1% level, and a low Durbin–Watson statistic, below 1.32, the critical value at the 1 per cent level. Equally it is no surprise to find that an AR(1) specification does not yield satisfactory results, the Durbin–Watson statistic now indicating negative autocorrelation.

By contrast the logarithmic specification appears entirely satisfactory, with a Breusch–Godfrey statistic of 0.85 and a Durbin–Watson statistic of 1.82 (d_U is 1.59 at the 5 per cent level). Comparing it with the ADL(1,1) specification, the F statistic for dropping the lagged variables is:

$$F(2, 46) = \frac{(1.084 - 1.020)/2}{1.020/46} = 1.44.$$

The critical value of $F(2, 40)$ at the 5 per cent level is 3.23. Hence we conclude that specification (3) is an acceptable simplification. Specifications (4) and (5) are inefficient, and this accounts for their larger standard errors.

12.15 Using the data on food in the Demand Functions data set, the following regressions were run, each with the logarithm of food as the dependent variable: (1) an OLS regression on a time trend T defined to be 1 in 1959, 2 in 1960, etc., (2) an AR(1) regression using the same specification, and (3) an OLS regression on T and the logarithm of food lagged one time period, with the results shown in the table (standard errors in parentheses).

	1: OLS	2: AR(1)	3: OLS
T	0.0181	0.0166	0.0024
	(0.0005)	(0.0021)	(0.0016)
$LGFOOD(-1)$	—	—	0.8551
			(0.0886)
constant	5.7768	5.8163	0.8571
	(0.0106)	(0.0586)	(0.5101)
$\widehat{\rho}$	—	0.8551	—
		(0.0886)	
R^2	0.9750	0.9931	0.9931
RSS	0.0327	0.0081	0.0081
d	0.2752	1.3328	1.3328
h	—	—	2.32

Discuss why each regression specification appears to be unsatisfactory. Explain why it was not possible to perform a common factor test.

Answer:

The Durbin–Watson statistic in regression (1) is very low, suggesting AR(1) autocorrelation. However, it remains below 1.40, d_{L} for a 5 per cent significance test with one explanatory variable and 35 observations, in the AR(1) specification in regression (2). The reason of course is that the model is very poorly specified, with two obvious major variables, income and price, excluded.

With regard to the impossibility of performing a common factor test, suppose that the original model is written:

$$LGFOOD_t = \beta_1 + \beta_2 T + u_t.$$

Lagging the model and multiplying through by ρ, we have:

$$\rho LGFOOD_{t-1} = \beta_1 \rho + \beta_2 \rho (T-1) + \rho u_{t-1}.$$

Subtracting and rearranging, we obtain the AR(1) specification:

$$\begin{aligned} LGFOOD_t &= \beta_1(1-\rho) + \rho LGFOOD_{t-1} + \beta_2 T - \beta_2 \rho (T-1) + u_t - \rho u_{t-1} \\ &= \beta_1(1-\rho) + \beta_2 \rho + \rho LGFOOD_{t-1} + \beta_2 (1-\rho) T + \varepsilon_t. \end{aligned}$$

However, this specification does not include any restrictions. The coefficient of $LGFOOD_{t-1}$ provides an estimate of ρ. The coefficient of T then provides an estimate of β_2. Finally, given these estimates, the intercept provides an estimate of β_1. The AR(1) and ADL(1,1) specifications are equivalent in this model, the reason being that the variable $(T-1)$ is merged into T and the intercept.

12.5 Answers to the additional exercises

A12.1 The Durbin–Watson statistic in the OLS regression is 0.49, causing us to reject the null hypothesis of no autocorrelation at the 1 per cent level. The Breusch–Godfrey statistic (not shown) is 25.12, also causing the null hypothesis of no autocorrelation to be rejected at a high significance level. Apart from a more satisfactory Durbin–Watson statistic, the results for the AR(1) specification are similar to those of the OLS one. The income and price elasticities are a little larger. The estimate of the population elasticity, negative in the OLS regression, is now effectively zero, suggesting that the direct effect of population on expenditure on food is offset by a negative income effect. The standard errors are larger than those for the OLS regression, but the latter are invalidated by the autocorrelation and therefore should not be taken at face value.

A12.2 All of the regressions exhibit strong evidence of positive autocorrelation. The Breusch–Godfrey test statistic for AR(1) autocorrelation is above the critical value of 10.82 (critical value of chi-squared with one degree of freedom at the 0.1% significance level) and the Durbin–Watson d statistic is below 1.20 (d_{L}, 1 per cent level, 45 observations, $k = 4$). The Durbin–Watson statistics for the AR(1) specification are generally much more healthy than those for the OLS one, being scattered around 2.

Breusch–Godfrey and Durbin–Watson statistics, logarithmic OLS regression including population					
	B–G	*d*		*B–G*	*d*
ADM	19.37	0.683	*GASO*	36.21	0.212
BOOK	25.85	0.484	*HOUS*	23.88	0.523
BUSI	24.31	0.507	*LEGL*	24.30	0.538
CLOT	18.47	0.706	*MAGS*	19.27	0.667
DENT	14.02	0.862	*MASS*	21.97	0.612
DOC	24.74	0.547	*OPHT*	31.64	0.328
FLOW	24.13	0.535	*RELG*	26.30	0.497
FOOD	24.95	0.489	*TELE*	30.08	0.371
FURN	22.92	0.563	*TOB*	27.84	0.421
GAS	23.41	0.569	*TOYS*	20.04	0.668

Since autocorrelation does not give rise to bias, one would not expect to see systematic changes in the point estimates of the coefficients. However, since multicollinearity is to some extent a problem for most categories, the coefficients do exhibit greater volatility than is usual when comparing OLS and AR(1) results. Fortunately, most of the major changes seem to be for the better. In particular, some implausibly high income elasticities are lower. Likewise, the population elasticities are a little less erratic, but most are still implausible, with large standard errors that reflect the continuing underlying problem of multicollinearity.

AR(1) logarithmic regression										
	LGDPI		*LGP*		*LGPOP*		$\hat{\rho}$		R^2	*d*
	coef.	s.e.	coef.	s.e.	coef.	s.e.	coef.	s.e.		
ADM	−0.34	0.34	0.00	0.20	3.73	0.95	0.76	0.08	0.992	2.03
BOOK	0.46	0.41	−1.06	0.29	2.73	1.25	0.82	0.10	0.990	1.51
BUSI	0.43	0.24	0.19	0.25	2.45	0.70	0.69	0.10	0.997	1.85
CLOT	1.07	0.16	−0.56	0.15	−0.49	0.71	0.84	0.08	0.999	2.19
DENT	1.14	0.18	−1.01	0.15	0.69	0.73	0.56	0.13	0.996	1.86
DOC	0.85	0.25	−0.30	0.26	1.26	0.77	0.83	0.10	0.997	1.61
FLOW	0.71	0.41	−1.04	0.44	0.74	1.33	0.78	0.09	0.994	1.97
FOOD	0.47	0.12	−0.36	0.12	0.07	0.38	0.88	0.09	0.997	1.56
FURN	1.73	0.36	−0.37	0.51	−1.62	1.55	0.92	0.06	0.994	2.00
GAS	−0.02	0.34	0.01	0.08	0.29	0.97	0.83	0.06	0.933	2.12
GASO	0.75	0.15	−0.14	0.03	−0.64	0.48	0.93	0.04	0.998	1.65
HOUS	0.27	0.08	−0.27	0.09	−0.03	0.54	0.98	0.00	0.997	1.66
LEGL	0.89	0.20	−0.19	0.22	−0.54	0.80	0.77	0.10	0.989	1.90
MAGS	0.98	0.30	−1.24	0.39	−0.23	0.92	0.73	0.12	0.983	1.73
MASS	0.06	0.28	−0.72	0.11	1.31	0.97	0.94	0.04	0.944	1.95
OPHT	1.99	0.60	−0.92	0.97	−1.45	1.85	0.90	0.08	0.991	1.67
RELG	0.86	0.18	−1.15	0.26	2.00	0.56	0.66	0.10	0.999	2.08
TELE	0.70	0.20	−0.56	0.13	2.44	0.71	0.87	0.10	0.999	1.51
TOB	0.38	0.22	−0.35	0.07	−0.99	0.66	0.79	0.10	0.960	2.37
TOYS	0.89	0.18	−0.58	0.13	1.61	0.66	0.75	0.12	0.999	1.77

A12.3 The table gives the residual sum of squares for the unrestricted ADL(1,1) specification and that for the restricted AR(1) one, the fourth column giving the chi-squared statistic for the common factor test.

Before performing the common factor test, one should check that the ADL(1,1) specification is itself free from autocorrelation using the Breusch–Godfrey test. The fifth column gives the *B–G* statistic for AR(1) autocorrelation. All but one of the statistics are below the critical value at the 5 per cent level, 3.84. The exception is that for *LEGL*. It should be remembered that the Breusch–Godfrey test is a large-sample tests and in this application, with only 44 observations, the sample is rather small.

Common factor test and tests of autocorrelation for ADL(1,1) model

	$RSS_{\text{ADL}(1,2)}$	$RSS_{\text{AR}(1)}$	Chi-squared	*B–G*
ADM	0.029792	0.039935	12.89	0.55
BOOK	0.070478	0.086240	8.88	1.25
BUSI	0.032074	0.032703	0.85	0.57
CLOT	0.009097	0.010900	7.96	1.06
DENT	0.019281	0.021841	5.49	1.22
DOC	0.025598	0.028091	4.09	0.33
FLOW	0.084733	0.084987	0.13	0.01
FOOD	0.005562	0.006645	7.83	3.12
FURN	0.050880	0.058853	6.41	0.29
GAS	0.035682	0.045433	10.63	0.66
GASO	0.006898	0.009378	13.51	2.91
HOUS	0.001350	0.002249	22.46	0.77
LEGL	0.026650	0.034823	11.77	8.04
MAGS	0.043545	0.051808	7.64	0.03
MASS	0.029125	0.033254	5.83	0.15
OPHT	0.139016	0.154629	4.68	0.08
RELG	0.013910	0.014462	1.71	0.32
TELE	0.014822	0.017987	8.52	0.97
TOB	0.021403	0.021497	0.19	3.45
TOYS	0.015313	0.015958	1.82	2.60

For the common factor test, the critical values of chi-squared are 7.81 and 11.34 at the 5 and 1 per cent levels, respectively, with 3 degrees of freedom. Summarising the results, we find:

- AR(1) specification not rejected: *BUSI*, *DENT*, *DOC*, *FLOW*, *FURN*, *MAGS*, *MASS*, *OPHT*, *RELG*, *TOB*, *TOYS*.
- AR(1) specification rejected at 5 per cent level: *BOOK*, *CLOT*, *FOOD*, *GAS*, *TELE*.
- AR(1) specification rejected at 1 per cent level: *ADM*, *GASO*, *HOUS*, *LEGL*.

A12.4 *Discuss whether specification (1) is an adequate representation of the data.*

The Breusch–Godfrey statistic is well in excess of the critical value at the 0.1 per cent significance level, 10.83. Likewise, the Durbin–Watson statistic is far below

1.15, d_L at the 1 per cent level with two explanatory variables and 36 observations. There is therefore strong evidence of either severe AR(1) autocorrelation or some serious misspecification.

Discuss whether specification (3) is an adequate representation of the data.

The only item that we can check is whether it is free from autocorrelation. The Breusch–Godfrey statistic is well under 3.84, the critical value at the 5 per cent significance level, and so there is no longer evidence of autocorrelation or misspecification.

Discuss whether specification (2) is an adequate representation of the data.

Let the original model be written:

$$\begin{aligned} LGLIFE &= \beta_1 + \beta_2 LGDPI + \beta_3 LGDPRLIFE + u \\ u_t &= \rho u_{t-1} + \varepsilon_t. \end{aligned}$$

The AR(1) specification is then:

$$\begin{aligned} LGLIFE &= \beta_1(1-\rho) + \rho LGLIFE(-1) + \beta_2 LGDPI - \beta_2\rho LGDPI(-1) \\ &\quad + \beta_3 LGDPRLIFE - \beta_3\rho LGPRLIFE(-1) + \varepsilon_t. \end{aligned}$$

This is a restricted version of the ADL(1,1) model because it incorporates nonlinear restrictions on the coefficients of $LGDPI(-1)$ and $LGPRLIFE(-1)$. In the ADL(1,1) specification, minus the product of the coefficients of $LGLIFE(-1)$ and $LGDPI$ is $-0.82 \times 0.42 = -0.34$. The coefficient of $LGDPI(-1)$ is smaller than this, but then its standard error is large. Minus the product of the coefficients of $LGLIFE(-1)$ and $LGPRLIFE$ is $-0.82 \times -0.59 = 0.48$. The coefficient of $LGPRLIFE(-1)$ is fairly close, bearing in mind that its standard error is also large. The coefficient of $LGLIFE(-1)$ is exactly equal to the estimate of ρ in the AR(1) specification.

The common factor test statistic is:

$$35 \log_{\mathrm{e}} \frac{0.799}{0.719} = 3.69.$$

The null hypothesis is that the two restrictions are valid. Under the null hypothesis, the test statistic has a chi-squared distribution with 2 degrees of freedom. Its critical value at the 5 per cent level is 5.99. Hence we do not reject the restrictions and the AR(1) specification therefore does appear to be acceptable.

Discuss whether specification (4) is an adequate representation of the data.

We note that $LGLDPI(-1)$ and $LGPRLIFE(-1)$ do not have significant t statistics, but since they are being dropped simultaneously, we should perform an F test of their joint explanatory power:

$$F(2,29) = \frac{(0.732 - 0.719)/2}{0.719/29} = 0.26.$$

Since this is less than 1, it is not significant at any significance level and so we do not reject the null hypothesis that the coefficients of $LGLDPI(-1)$ and

LGPRLIFE(−1) are both 0. Hence it does appear that we can drop these variables. We should also check for autocorrelation. The Breusch–Godfrey statistic indicates that there is no problem.

If you were presenting these results at a seminar, what would you say were your conclusions concerning the most appropriate of specifications (1) – (4)?

There is no need to mention (1). (3) is not a candidate because we have found acceptable simplifications that are likely to yield more efficient parameter estimates , and this is reflected in the larger standard errors compared with (2) and (4). We cannot discriminate between (2) and (4).

At the seminar a commentator points out that in specification (4) neither LGDPI nor LGPRLIFE have significant coefficients and so these variables should be dropped. As it happens, the researcher has considered this specification, and the results are shown as specification (5) in the table. What would be your answer to the commentator?

Comparing (3) and (5):

$$F(4,29) = \frac{(0.843 - 0.719)/4}{0.719/29} = 1.25.$$

The critical value of $F(4,29)$ at the 5 per cent level is 2.70, so it would appear that the joint explanatory power of the 4 income and price variables is not significant. However, it does not seem sensible to drop current income and current price from the model. The reason that they have so little explanatory power is that the short-run effects are small, life insurance being subject to long-term contracts and thus a good example of a category of expenditure with a large amount of inertia. The fact that income in the AR(1) specification has a highly significant coefficient is concrete evidence that it should not be dropped.

A12.5 *Looking at all five regressions together, evaluate the adequacy of:*

- *specification 1.*
- *specification 2.*
- *specification 3.*
- *specification 4.*

- Specification 1 has a very high Breusch–Godfrey statistic and a very low Durbin–Watson statistic. There is evidence of either severe autocorrelation or model misspecification.
- Specification 2 also has a very high Breusch–Godfrey statistic and a very low Durbin–Watson statistic. Further, there is evidence of multicollinearity: large standard errors (although comparisons are very dubious given low DW), and implausible coefficients.
- Specification 3 seems acceptable. In particular, there is no evidence of autocorrelation since the Breusch–Godfrey statistic is low.
- Specification 4: dropping $m(-1)$ may be expected to cause omitted variable bias since the t statistic for its coefficient was −3.0 in specification 3.

(Equivalently, the F statistic is:

$$F(1,46) = \frac{(0.0120 - 0.0100)/1}{0.0100/46} = 0.2 \times 46 = 9.2$$

the square of the t statistic and similarly significant.)

Explain why specification 5 is a restricted version of one of the other specifications, stating the restriction, and explaining the objective of the manipulations that lead to specification 5.

Write the original model and AR(1) process:

$$\begin{aligned} p_t &= \beta_1 + \beta_2 m_t + u_t \\ u_y &= \rho u_{t-1} + \varepsilon_t. \end{aligned}$$

Then fitting:

$$p_t = \beta_1(1-\rho) + \rho p_{t-1} + \beta_2 m_t - \beta_2 \rho m_{t-1} + \varepsilon_t$$

removes the autocorrelation. This is a restricted version of specification 3, with restriction that the coefficient of m_{t-1} is equal to minus the product of the coefficients of m_t and p_{t-1}.

Perform a test of the restriction embodied in specification 5.

Comparing specifications 3 and 5, the common factor test statistic is:

$$n \log_e \left(\frac{RSS_R}{RSS_U} \right) = 50 \log \left(\frac{0.0105}{0.0100} \right) = 50 \log 1.05 \cong 50 \times 0.05 = 2.5.$$

Under the null hypothesis that the restriction implicit in the specification is valid, the test statistic is distributed as chi-squared with one degree of freedom. The critical value at the 5 per cent significance level is 3.84, so we do not reject the restriction. Accordingly, specification 5 appears to be an adequate representation of the data.

Explain which would be your preferred specification.

Specifications (3) and (5) both appear to be adequate representations of the data. (5) should yield more efficient estimators of the parameters because, exploiting an apparently-valid restriction, it is less susceptible to multicollinearity, and this appears to be confirmed by the lower standard errors.

A12.6 The models are:

1. $p_t = \beta_1 + \beta_2 m_t + u_t$
2. $p_t = \beta_1 + \beta_2 m_t + \beta_3 m_{t-1} + \beta_4 m_{t-2} + \beta_5 m_{t-3} + u_t$
3. $p_t = \beta_1 + \beta_2 m_t + \beta_3 m_{t-1} + \beta_6 p_{t-1} + u_t$
4. $p_t = \beta_1 + \beta_2 m_t + \beta_6 p_{t-1} + u_t$
5. $p_t = \beta_1(1-\beta_6) + \beta_6 p_{t-1} + \beta_2 m_t - \beta_2 \beta_6 m_{t-1} + \varepsilon_t$ (writing $\rho = \beta_6$).

Hence we obtain the following estimates of $\partial p_t / \partial m_t$:

1. 0.95
2. 0.50
3. 0.40
4. 0.18
5. 0.90.

Putting p and m equal to equilibrium values, and ignoring the disturbance term, we have:

1. $\bar{p} = \beta_1 + \beta_2 \overline{m}$
2. $\bar{p} = \beta_1 + (\beta_2 + \beta_3 + \beta_4)\overline{m}$
3. $\bar{p} = \frac{1}{1-\beta_6}(\beta_1 + (\beta_2 + \beta_3)\overline{m})$
4. $\bar{p} = \frac{1}{1-\beta_6}(\beta_1 + \beta_2 \overline{m})$
5. $\bar{p} = \beta_1 + \beta_2 \overline{m}$.

Hence we obtain the following estimates of $\mathrm{d}\bar{p}/\mathrm{d}\overline{m}$:

1. 0.95
2. 0.95
3. 1.00
4. 0.90
5. 0.90.

A12.7 *Evaluate regression (1).*

Regression (1) has a very high Breusch–Godfrey statistic and a very low Durbin–Watson statistic. The null hypothesis of no autocorrelation is rejected at the 1 per cent level for both tests. Alternatively, the test statistics might indicate some misspecification problem.

Evaluate regression (2). Explain mathematically what assumptions were being made by the researcher when he used the AR(1) specification and why he hoped the results would be better than those obtained with regression (1).

Regression (2) has been run on the assumption that the disturbance term follows an AR(1) process:

$$u_t = \rho u_{t-1} + \varepsilon_t.$$

On the assumption that the regression model should be:

$$LGTAXI_t = \beta_1 + \beta_2 LGDPI_t + u_t,$$

the autocorrelation can be eliminated in the following way: lag the regression model by one time period and multiply through by ρ:

$$\rho LGTAXI_{t-1} = \beta_1 \rho + \beta_2 \rho LGDPI_{t-1} + \rho u_{t-1}.$$

Subtract this from the regression model:

$$LGTAXI_t - \rho LGTAXI_{t-1} = \beta_1(1-\rho) + \beta_2 LGDPI_t - \beta_2 \rho LGDPI_{t-1} + u_t - \rho u_{t-1}.$$

Hence one obtains a specification free from autocorrelation:

$$LGTAXI_t = \beta_1(1-\rho) + \rho LGTAXI_{t-1} + \beta_2 LGDPI_t - \beta_2\rho LGDPI_{t-1} + \varepsilon_t.$$

The Durbin–Watson statistic is still low, suggesting that fitting the AR(1) specification was an inappropriate response to the problem.

Evaluate regression (3).

In regression (3) the Breusch–Godfrey statistic suggests that, for this specification, there is not a problem of autocorrelation (the Durbin–Watson statistic is indecisive). This suggests that the apparent autocorrelation in the regression (1) is in fact attributable to the omission of the price variable.

This is corroborated by the diagrams, which show that large negative residuals occurred when the price rose and positive ones when it fell. The effect is especially obvious in the final years of the sample period.

Evaluate regression (4). In particular, discuss the possible reasons for the differences in the standard errors in regressions (3) and (4).

In regression (4), the Durbin–Watson statistic does not indicate a problem of autocorrelation. Overall, there is little to choose between regressions (3) and (4). It is possible that there was some autocorrelation in regression (3) and that it has been rectified by using AR(1) in regression (4). It is also possible that autocorrelation was not actually a problem in regression (3). Regressions (3) and (4) yield similar estimates of the income and price elasticities and in both cases the elasticities are significantly different from zero at a high significance level. If regression (4) is the correct specification, the lower standard errors in regression (3) should be disregarded because they are invalid. If regression (3) is the correct specification, AR(1) estimation will yield inefficient estimates; which could account for the higher standard errors in regression (4).

At a seminar one of the participants says that the researcher should consider adding lagged values of LGTAXI, LGDPI, and LGP to the specification. What would be your view?

Specifications (2) and (4) already contain the lagged values, with restrictions on the coefficients of $LGDPI(-1)$ and $LGP(-1)$.

A12.8 *Explain why the researcher was not satisfied with regression (1).*

The researcher was not satisfied with the results of regression (1) because the Breusch–Godfrey statistic was 4.42, above the critical value at the 5 per cent level, 3.84, and because the Durbin–Watson d statistic was only 0.99. The critical value of d_L with one explanatory variable and 30 observations is 1.35. Thus there is evidence that the specification may be subject to autocorrelation.

Evaluate regression (2). Explain why the coefficients of I(-1) and r(-1) are not reported, despite the fact that they are part of the regression specification.

Specification (2) is equally unsatisfactory. The fact that the Durbin–Watson statistic has remained low is an indication that the reason for the low d in (1) was not an AR(1) disturbance term. RSS is very high compared with those in specifications (4) – (6). The coefficient of $I(-1)$ is not reported as such because it is the estimate $\widehat{\rho}$. The coefficient of $r(-1)$ is not reported because it is constrained to be minus the product of $\widehat{\rho}$. and the coefficient of I.

Evaluate regression (3).

Specification (3) is the unrestricted ADL(1,1) model of which the previous AR(1) model was a restricted version and it suffers from the same problems. There is still evidence of positive autocorrelation, since the Breusch–Godfrey statistic, 4.24, is high and RSS is still much higher than in the three remaining specifications.

Evaluate regression (4).

Specification (4) seems fine. The null hypothesis of no autocorrelation is not rejected by either the Breusch–Godfrey statistic or the Durbin–Watson statistic. The coefficients are significant and have the expected signs.

Evaluate regression (5).

The AR(1) specification (5) does not add anything because there was no evidence of autocorrelation in (4). The estimate of ρ is not significantly different from zero.

Evaluate regression (6).

Specification (6) does not add anything either. t tests on the coefficients of the lagged variables indicate that they are individually not significantly different from zero. Likewise the joint hypothesis that their coefficients are all equal to zero is not rejected by an F test comparing RSS in (4) and (6):

$$F(3,23) = \frac{(27.4 - 23.5)/3}{23.5/23} = 1.27.$$

The critical value of $F(3,23)$ at the 5 per cent level is 3.03. [There is no point in comparing (5) and (6) using a common factor test, but for the record the test statistic is:

$$n \log_{\mathrm{e}} \frac{RSS_R}{RSS_U} = 29 \log_e \frac{26.8}{23.5} = 3.81.$$

The critical value of chi-squared with 2 degrees of freedom at the 5 per cent level is 5.99.]

Summarise your conclusions concerning the evaluation of the different regressions. Explain whether an examination of the figure supports your conclusions.

The overall conclusion is that the static model (4) is an acceptable representation of the data and the apparent autocorrelation in specifications (1) – (3) is attributable to the omission of g. Figure 12.3 shows very clearly that the residuals in specification (1) follow the same pattern as g, confirming that the apparent autocorrelation in the residuals is in fact attributable to the omission of g from the specification.

A12.9 *In Exercise A11.5 you performed a test of a restriction. The result of this test will have been invalidated if you found that the specification was subject to autocorrelation. How should the test be performed, assuming the correct specification is ADL(1,1)?*

If the ADL(1,1) model is written:

$$\begin{aligned}\log CAT &= \beta_1 + \beta_2 \log DPI + \beta_3 \log P + \beta_4 \log POP + \beta_5 \log CAT_{-1} \\ &\quad + \beta_6 logDPI_{-1} + \beta_7 \log P_{-1} + \beta_8 \log POP_{-1} + u\end{aligned}$$

the restricted version with expenditure per capita a function of income per capita is:

$$\begin{aligned}\log \frac{CAT}{POP} &= \beta_1 + \beta_2 \log \frac{DPI}{POP} + \beta_3 \log P + \beta_5 \log \frac{CAT_{-1}}{POP_{-1}} \\ &\quad + \beta_6 \log \frac{DPI_{-1}}{POP_{-1}} + \beta_7 \log P_{-1} + u.\end{aligned}$$

Comparing the two equations, we see that the restrictions are $\beta_4 = 1 - \beta_2$ and $\beta_8 = -\beta_5 - \beta_6$. The usual F statistic should be constructed and compared with the critical values of $F(2, 28)$.

A12.10 Let the AR(1) process be written:

$$u_t = \rho u_{t-1} + \varepsilon_t.$$

As the specification stands, OLS would yield inconsistent estimates because both the explanatory variable and the disturbance term depend on u_{t-1}. Applying the standard procedure, multiplying the lagged relationship by ρ and subtracting, one has:

$$Y_t - \rho Y_{t-1} = \beta_1(1-\rho) + \beta_2 Y_{t-1} - \beta_2 \rho Y_{t-1} + u_t - \rho u_{t-1}.$$

Hence:

$$Y_t = \beta_1(1-\rho) + (\beta_2 + \rho) Y_{t-1} - \beta_2 \rho Y_{t-2} + \varepsilon_t.$$

It follows that the model should be fitted as a second-order, rather than as a first-order, process. There are no restrictions on the coefficients. OLS estimators will be consistent, but subject to finite-sample bias.

A12.11 *Explain what is correct. incorrect, confused or incomplete in the following statements, giving a brief explaination if not correct.*

- *The disturbance term in a regression model is said to be autocorrelated if its values in a sample of observations are not distributed independently of each other.*

  Correct.

- *When the disturbance term is subject to autocorrelation, the ordinary least squares estimators are inefficient ...*

  Correct.

- *...and inconsistent...*

  Incorrect, unless there is a lagged dependent variable.

- *...but they are not biased...*

  Correct, unless there is a lagged dependent variable.

- *…and the t tests are invalid.*

  Correct.

- *It is a common problem in time series models because it always occurs when the dependent variable is correlated with its previous values.*

  Incorrect.

- *If this is the case, it could be eliminated by including the lagged value of the dependent variable as an explanatory variable.*

  In general, incorrect. However, a model requiring a lagged dependent variable could appear to exhibit autocorrelation if the lagged dependent variable were omitted, and including it could eliminate the apparent problem.

- *However, if the model is correctly specified and the disturbance term satisfies the regression model assumptions, adding the lagged value of the dependent variable as an explanatory variable will have the opposite effect and cause the disturbance term to be autocorrelated.*

  Nonsense.

- *A second way of dealing with the problem of autocorrelation is to use an instrumental variable.*

  More nonsense.

- *If the autocorrelation is of the AR(1) type, randomising the order of the observations will cause the Durbin–Watson statistic to be near 2...*

  Correct.

- *...thereby eliminating the problem.*

  Incorrect. The problem will have been disguised, not rectified.

Chapter 13
Introduction to nonstationary time series

13.1 Overview

This chapter begins by defining the concepts of stationarity and nonstationarity as applied to univariate time series and, in the case of nonstationary series, the concepts of difference-stationarity and trend-stationarity. It next describes the consequences of nonstationarity for models fitted using nonstationary time-series data and gives an account of the Granger–Newbold Monte Carlo experiment with random walks. Next the two main methods of detecting nonstationarity in time series are described, the graphical approach using correlograms and the more formal approach using Augmented Dickey–Fuller unit root tests. This leads to the topic of cointegration. The chapter concludes with a discussion of methods for fitting models using nonstationary time series: detrending, differencing, and error-correction models.

13.2 Learning outcomes

After working through the corresponding chapter in the text, studying the corresponding slideshows, and doing the starred exercises in the text and the additional exercises in this subject guide, you should be able to:

- explain what is meant by stationarity and nonstationarity.
- explain what is meant by a random walk and a random walk with drift
- derive the condition for the stationarity of an AR(1) process
- explain what is meant by an integrated process and its order of integration
- explain why Granger and Newbold obtained the results that they did
- explain what is depicted by a correlogram
- perform an Augmented Dickey–Fuller unit root test to test a time series for nonstationarity
- test whether a set of time series are cointegrated
- construct an error-correction model and describe its advantages over detrending and differencing.

13.3 Further material

Addition to Section 13.6 Cointegration

Section 13.6 contains the following paragraph on page 507:

In the case of a cointegrating relationship, least squares estimators can be shown to be superconsistent (Stock, 1987). An important consequence is that OLS may be used to fit a cointegrating relationship, even if it belongs to a system of simultaneous relationships, for any simultaneous equations bias tends to zero asymptotically.

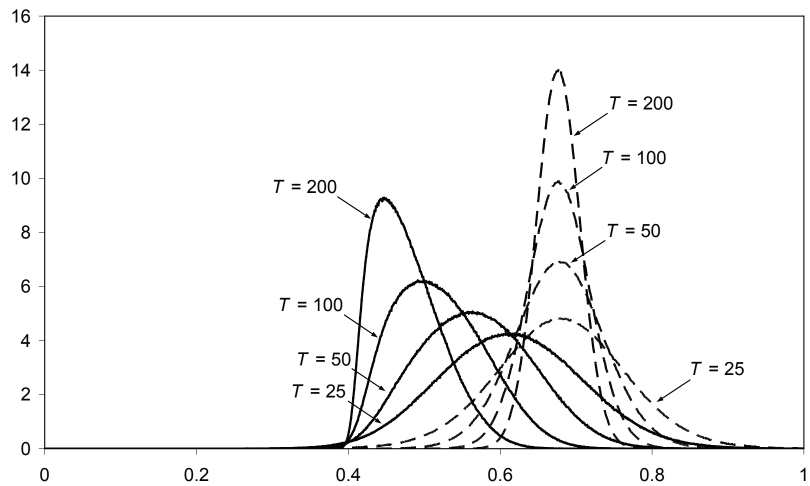
This cries out for an illustrative simulation, so here is one. Consider the model:

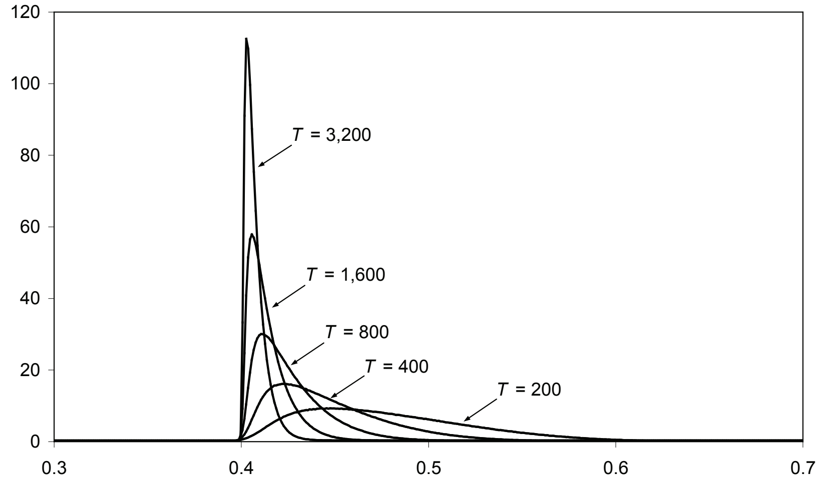
$$\begin{aligned} Y_t &= \beta_1 + \beta_2 X_t + \beta_3 Z_t + \varepsilon_{Yt} \\ X_t &= \alpha_1 + \alpha_2 Y_t + \varepsilon_{Xt} \\ Z_t &= \rho Z_{t-1} + \varepsilon_{Zt} \end{aligned}$$

where Y_t and X_t are endogenous variables, Z_t is exogenous, and ε_{Yt}, ε_{Xt}, and ε_{Zt} are iid $N(0,\,1)$ disturbance terms. We expect OLS estimators to be inconsistent if used to fit either of the first two equations. However, if $\rho = 1$, Z is nonstationary, and X and Y will also be nonstationary. So, if we fit the second equation, for example, the OLS estimator of α_2 will be superconsistent. This is illustrated by a simulation where the first two equations are:

$$\begin{aligned} Y_t &= 1.0 + 0.8X_t + 0.5Z_t + \varepsilon_{Yt} \\ X_t &= 2.0 + 0.4Y_t + \varepsilon_{Xt}. \end{aligned}$$

The distributions in the right of the figure below (dashed lines) are for the case $\rho = 0.5$. Z is stationary, and so are Y and X. You will have no difficulty in demonstrating that plim $\widehat{\alpha}_2^{OLS} = 0.68$. The distributions to the left of the figure (solid lines) are for $\rho = 1$, and you can see that in this case the estimator is consistent. But is it superconsistent? The variance seems to be decreasing relatively slowly, not fast, especially for small sample sizes. The explanation is that the superconsistency becomes apparent only for very large sample sizes, as shown in the second figure.





13.4 Additional exercises

A13.1 The Figure 13.1 plots the logarithm of the US population for the period 1959–2003. It is obviously nonstationary. Discuss whether it is more likely to be difference-stationary or trend-stationary.

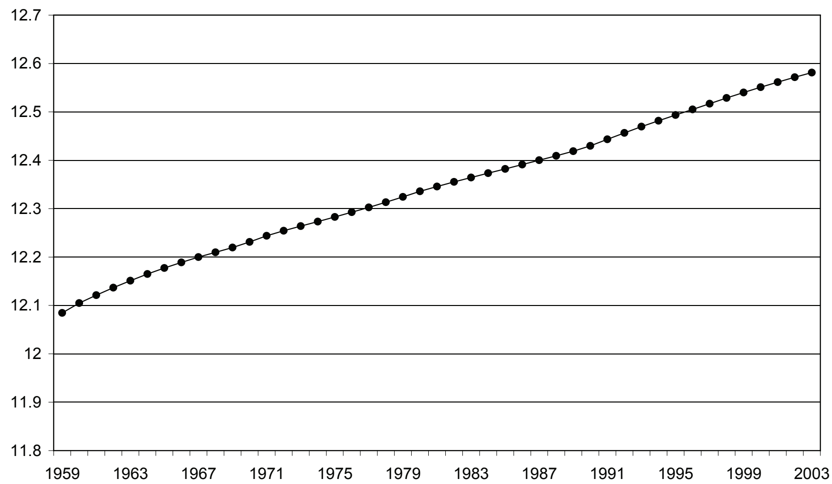


Figure 13.1: Logarithm of the US population.

A13.2 Figure 13.2 plots the first difference of the logarithm of the US population for the period 1959–2003. Explain why the vertical axis measures the proportional growth rate. Comment on whether the series appears to be stationary or nonstationary.

A13.3 The regression output below shows the results of ADF unit root tests on the logarithm of the US population, and its difference, for the period 1959–2003. Comment on the results and state whether they confirm or contradict your conclusions in Exercise A13.2.

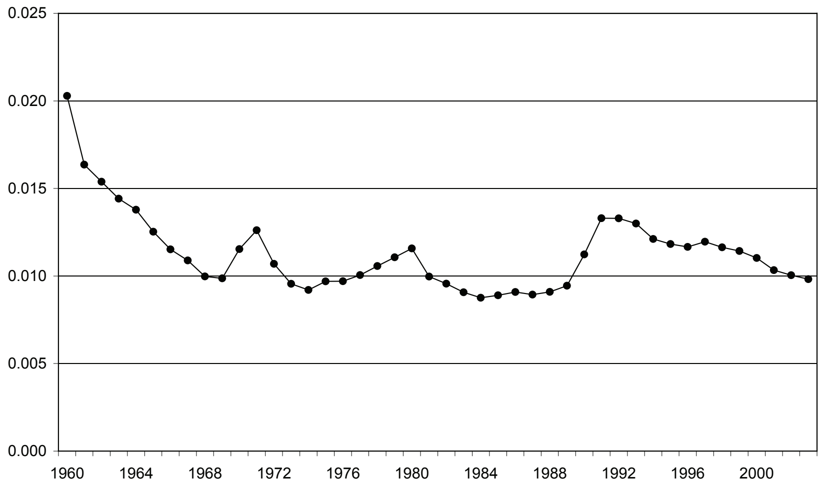


Figure 13.2: Logarithm of the US population, first difference.

```
      Augmented Dickey--Fuller Unit Root Test on LGPOP
============================================================
Null Hypothesis: LGPOP has a unit root
Exogenous: Constant, Linear Trend
Lag Length: 1 (Fixed)
============================================================
                                        t-Statistic  Prob.*
============================================================
Augmented Dickey--Fuller test statistic   -2.030967   0.5682
Test critical values1% level             -4.186481
                    5% level             -3.518090
                   10% level             -3.189732
============================================================
*MacKinnon (1996) one-sided p-values.

Augmented Dickey--Fuller Test Equation
Dependent Variable: D(LGPOP)
Method: Least Squares
Sample(adjusted): 1961 2003
Included observations: 43 after adjusting endpoints
============================================================
     Variable      Coefficient Std. Error t-Statistic  Prob.
============================================================
     LGPOP(-1)      -0.047182  0.023231  -2.030967   0.0491
   D(LGPOP(-1))      0.687772  0.058979   11.66139   0.0000
         C           0.574028  0.281358   2.040209   0.0481
   @TREND(1959)      0.000507  0.000246   2.060295   0.0461
============================================================
R-squared            0.839263    Mean dependent var 0.011080
Adjusted R-squared   0.826898    S.D. dependent var 0.001804
S.E. of regression   0.000750    Akaike info criter-11.46327
Sum squared resid    2.20E-05    Schwarz criterion -11.29944
Log likelihood       250.4603    F-statistic         67.87724
Durbin-Watson stat   1.164933    Prob(F-statistic)  0.000000
============================================================
```

```
      Augmented Dickey--Fuller Unit Root Test on DLGPOP
============================================================
Null Hypothesis: DLGPOP has a unit root
Exogenous: Constant, Linear Trend
Lag Length: 1 (Fixed)
============================================================
                                        t-Statistic  Prob.*
============================================================
Augmented Dickey--Fuller test statistic   -2.513668   0.3203
Test critical values1% level             -4.192337
                   5% level              -3.520787
                  10% level              -3.191277
============================================================
*MacKinnon (1996) one-sided p-values.

Augmented Dickey--Fuller Test Equation
Dependent Variable: D(DLGPOP)
Method: Least Squares
Sample(adjusted): 1962 2003
Included observations: 42 after adjusting endpoints
============================================================
     Variable      Coefficient Std. Error t-Statistic  Prob.
============================================================
    DLGPOP(-1)      -0.161563   0.064274  -2.513668   0.0163
   D(DLGPOP(-1))     0.294717   0.117766   2.502573   0.0167
         C           0.001714   0.000796   2.152327   0.0378
   @TREND(1959)     -1.32E-07   9.72E-06  -0.013543   0.9893
============================================================
R-squared             0.320511    Mean dependent var-0.000156
Adjusted R-squared    0.266867    S.D. dependent var 0.000827
S.E. of regression    0.000708    Akaike info criter-11.57806
Sum squared resid     1.90E-05    Schwarz criterion -11.41257
Log likelihood        247.1393    F-statistic        5.974780
Durbin-Watson stat    1.574084    Prob(F-statistic)  0.001932
============================================================
```

A13.4 A researcher believes that a time series is generated by the process:

$$X_t = \rho X_{t-1} + \varepsilon_t$$

where ε_t is a white noise series generated randomly from a normal distribution with mean zero, constant variance, and no autocorrelation. Explain why the null hypothesis for a test of nonstationarity is that the series is nonstationary, rather than stationary.

A13.5 A researcher correctly believes that a time series is generated by the process:

$$X_t = \rho X_{t-1} + \varepsilon_t$$

where ε_t is a white noise series generated randomly from a normal distribution with mean zero, constant variance, and no autocorrelation. Unknown to the researcher, the true value of ρ is 0.7. The researcher uses a unit root test to test the series for nonstationarity. The output is shown. Discuss the result of the test.

```
       Augmented Dickey--Fuller Unit Root Test on X
============================================================
ADF Test Statistic -2.528841     1%   Critical Value*-3.6289
                                 5%   Critical Value -2.9472
                                10%   Critical Value -2.6118
============================================================
*MacKinnon critical values for rejection of hypothesis of a unit root.

Augmented Dickey--Fuller Test Equation
Dependent Variable: D(X)
Method: Least Squares
Sample(adjusted): 2 36
Included observations: 35 after adjusting endpoints
============================================================
     Variable      Coefficient Std. Error t-Statistic  Prob.
============================================================
       X(-1)        -0.379661   0.150132  -2.528841   0.0164
         C           0.222066   0.203435   1.091580   0.2829
============================================================


R-squared             0.162331    Mean dependent var-0.052372
Adjusted R-squared    0.136947    S.D. dependent var 1.095782
S.E. of regression    1.017988    Akaike info criteri2.928979
Sum squared resid     34.19792    Schwarz criterion  3.017856
Log likelihood       -49.25714    F-statistic        6.395035
Durbin-Watson stat    1.965388    Prob(F-statistic)  0.016406
============================================================
```

A13.6 Test of cointegration. Perform a logarithmic regression of expenditure on your commodity on income, relative price, and population. Save the residuals and test them for stationarity. (Note: the critical values in the regression output do not apply to tests of cointegration. For the correct critical values, see the text.)

A13.7 A variable Y_t is generated by the autoregressive process:

$$Y_t = \beta_1 + \beta_2 Y_{t-1} + \varepsilon_t$$

where $\beta_2 = 1$ and ε_t satisfies the regression model assumptions. A second variable Z_t is generated as the lagged value of Y_t:

$$Z_t = Y_{t-1}.$$

Show that Y and Z are nonstationary processes. Show that nevertheless they are cointegrated.

A13.8 X_t and Z_t are independent I(1) (integrated of order 1) time series. W_t is a stationary time series. Y_t is generated as the sum of X_t, Z_t, and W_t. Not knowing this, a researcher regresses Y_t on X_t and Z_t. Explain whether he would find a cointegrating relationship.

A13.9 Two random walks RA_t and RB_t, and two stationary processes SA_t and SB_t are generated by the following processes:

$$\begin{aligned} RA_t &= RA_{t-1} + \varepsilon_{1t} \\ RB_t &= RB_{t-1} + \varepsilon_{2t} \\ SA_t &= \rho_A SA_{t-1} + \varepsilon_{3t}, \quad 0 < \rho_A < 1 \\ SB_t &= \rho_B SB_{t-1} + \varepsilon_{4t}, \quad 0 < \rho_B < 1 \end{aligned}$$

where ε_{1t}, ε_{2t}, ε_{3t}, and ε_{4t}, are iid $N(0, 1)$ (independently and identically distributed from a normal distribution with mean 0 and variance 1).

- Two series XA_t and XB_t are generated as:

$$\begin{aligned} XA_t &= RA_t + SA_t \\ XB_t &= RB_t + SB_t. \end{aligned}$$

Explain whether it is possible for XA_t and XB_t to be stationary.

Explain whether it is possible for them to be cointegrated.

- Two series YA_t and YB_t are generated as:

$$\begin{aligned} YA_t &= RA_t + SA_t \\ YB_t &= RA_t + SB_t. \end{aligned}$$

Explain whether it is possible for YA_t and YB_t to be cointegrated.

- Two series ZA_t and ZB_t are generated as:

$$\begin{aligned} ZA_t &= RA_t + RB_t + SA_t \\ ZB_t &= RA_t - RB_t + SB_t. \end{aligned}$$

Explain whether it is possible for ZA_t and ZB_t to be stationary.

Explain whether it is possible for them to be cointegrated.

13.5 Answers to the starred exercises in the textbook

13.1 Demonstrate that the MA(1) process:

$$X_t = \varepsilon_t + \alpha_2 \varepsilon_{t-1}$$

is stationary. Does the result generalise to higher-order MA processes?

Answer:

The expected value of X_t is zero and therefore independent of time:

$$E(X_t) = E(\varepsilon_t + \alpha_2 \varepsilon_{t-1}) = E(\varepsilon_t) + \alpha_2 E(\varepsilon_{t-1}) = 0 + 0 = 0.$$

Since ε_t and ε_{t-1} are uncorrelated:

$$\sigma_{X_t}^2 = \sigma_{\varepsilon_t}^2 + \alpha_2^2 \sigma_{\varepsilon_{t-1}}^2$$

and this is independent of time. Finally, because:

$$X_{t-1} = \varepsilon_{t-1} + \alpha_2 \varepsilon_{t-2},$$

the population covariance of X_t and X_{t-1} is given by:

$$\sigma_{X_t, X_{t-1}} = \alpha_2 \sigma_\varepsilon^2.$$

This is fixed and independent of time. The population covariance between X_t and X_{t-s} is zero for all $s > 1$ since then X_t and X_{t-1} have no elements in common. Thus the third condition for stationarity is also satisfied.

All MA processes are stationary, the general proof being a simple extension of that for the MA(1) case.

13.2 A stationary AR(1) process:

$$X_t = \beta_1 + \beta_2 X_{t-1} + \varepsilon_t$$

with $|\beta_2| < 1$, has initial value X_0, where X_0 is defined as:

$$X_0 = \frac{\beta_1}{1-\beta_2} + \sqrt{\frac{1}{1-\beta_2^2}}\varepsilon_0.$$

Demonstrate that X_0 is a random draw from the ensemble distribution for X.

Answer:

Lagging and substituting, it was shown, equation (13.12), that:

$$X_t = \beta_2^t X_0 + \beta_1 \frac{1-\beta_2^t}{1-\beta_2} + \beta_2^{t-1}\varepsilon_1 + \cdots + \beta_2^2 \varepsilon_{t-2} + \beta_2 \varepsilon_{t-1} + \varepsilon_t.$$

With the stochastic definition of X_0, we now have:

$$\begin{aligned} X_t &= \beta_2^t \left(\frac{\beta_1}{1-\beta_2} + \sqrt{\frac{1}{1-\beta_2^2}}\varepsilon_0 \right) + \beta_1 \frac{1-\beta_2^t}{1-\beta_2} + \beta_2^{t-1}\varepsilon_1 + \cdots + \beta_2^2 \varepsilon_{t-2} + \beta_2 \varepsilon_{t-1} + \varepsilon_t \\ &= \frac{\beta_1}{1-\beta_2} + \beta_2^t \sqrt{\frac{1}{1-\beta_2^2}}\varepsilon_0 + \beta_2^{t-1}\varepsilon_1 + \cdots + \beta_2^2 \varepsilon_{t-2} + \beta_2 \varepsilon_{t-1} + \varepsilon_t. \end{aligned}$$

Hence:

$$E(X_t) = \frac{\beta_1}{1-\beta_2}$$

and:

$$\begin{aligned} \mathrm{var}(X_t) &= \mathrm{var}\left(\beta_2^t \sqrt{\frac{1}{1-\beta_2^2}}\varepsilon_0 + \beta_2^{t-1}\varepsilon_1 + \cdots + \beta_2^2 \varepsilon_{t-2} + \beta_2 \varepsilon_{t-1} + \varepsilon_t \right) \\ &= \frac{\beta_2^{2t}}{1-\beta_2^2}\sigma_\varepsilon^2 + \left(\beta_2^{2t-2} + \cdots + \beta_2^4 + \beta_2^2 + 1 \right) \sigma_\varepsilon^2 \\ &= \frac{\beta_2^{2t}}{1-\beta_2^2}\sigma_\varepsilon^2 + \frac{1-\beta_2^{2t}}{1-\beta_2^2}\sigma_\varepsilon^2 = \frac{\sigma_\varepsilon^2}{1-\beta_2^2}. \end{aligned}$$

Given the generating process for X_0, one has:

$$E(X_0) = \frac{\beta_1}{1-\beta_2} \quad \text{and} \quad \operatorname{var}(X_0) = \frac{\sigma_\varepsilon^2}{1-\beta_2^2}.$$

Hence X_0 is a random draw from the ensemble distribution. Implicitly it has been assumed that the distributions of ε and X_0 are both normal. This should have been stated explicitly.

13.4 Suppose that Y_t is determined by the process:

$$Y_t = Y_{t-1} + \varepsilon_t + \lambda\varepsilon_{t-1}$$

where ε_t is iid. Show that the process for Y_t is nonstationary unless λ takes a certain value.

Answer:

Lagging and substituting back to time 0:

$$Y_t = Y_0 + \sum_{s=1}^{t} \varepsilon_t + \lambda \sum_{s=0}^{t-1} \varepsilon_t = Y_0 + (1+\lambda)\sum_{s=1}^{t-1} \varepsilon_t + \varepsilon_t + \lambda\varepsilon_0.$$

The expectation of Y_t, taken at time 0, is Y_0 and independent of time. The variance of Y_t is $\left((t-1)(1+\lambda)^2 + 1 + \lambda^2\right)\sigma_\varepsilon^2$. The process is nonstationary because the variance is dependent on time, unless $\lambda = -1$, in which case the process is stationary. It reduces to:

$$Y_t = Y_0 + \varepsilon_t - \varepsilon_0.$$

The covariance between Y_t and Y_{t-s} is zero for all s greater than 0 if ε_0 is taken as predetermined. It is equal to the variance of ε if ε_0 is treated as random. Either way, it is independent of time.

13.11 Suppose that a series is generated as:

$$X_t = \beta_2 X_{t-1} + \varepsilon_t$$

with β_2 equal to $1-\delta$, where δ is small. Demonstrate that, if δ is small enough that terms involving δ^2 may be neglected, the variance may be approximated as:

$$\begin{aligned} \sigma_{X_t}^2 &= \left((1-[2t-2]\delta) + \cdots + (1-2\delta) + 1\right)\sigma_\varepsilon^2 \\ &= \left(1-(t-1)\delta\right) t\,\sigma_\varepsilon^2 \end{aligned}$$

and draw your conclusions concerning the properties of the time series.

Answer:

$$X_t = \beta_2^t X_0 + \beta_2^{t-1}\varepsilon_1 + \cdots + \varepsilon_t.$$

Hence:

$$\begin{aligned} \sigma_{X_t}^2 &= \left(\beta_2^{2t-2} + \cdots + \beta_2^2 + 1\right)\sigma_\varepsilon^2 \\ &= \left((1-\delta)^{2t-2} + \cdots + (1-\delta)^2 + 1\right)\sigma_\varepsilon^2 \\ &= \left((1-(2t-2)\delta) + \cdots + (1-2\delta) + 1\right)\sigma_\varepsilon^2 \end{aligned}$$

assuming that δ is so small that terms involving δ^2 may be neglected. (Note that the expansion of $(1+x)^n$ is $\left(1+nx+\frac{n(n-1)}{2!}x^2+\cdots\right)$ and if x is so small that terms involving x^2 and higher powers of x may be neglected, the expansion reduces to $(1+nx)$.) Thus:

$$\begin{aligned}\sigma^2_{X_t} &= (t-2\delta(t-1+\cdots+1))\,\sigma^2_\varepsilon \\ &= (t-\delta t(t-1))\,\sigma^2_\varepsilon \\ &= (1-(t-1)\delta)\,t\,\sigma^2_\varepsilon.\end{aligned}$$

It follows that, for finite t, the variance is a function of t and hence that the series exhibits nonstationary behavior for finite t, even though it is stationary.

13.15 Demonstrate that, for Case (e), Y_t is determined by:

$$Y_t = t\,\beta_1 + \frac{t(t+1)}{2}\delta + Y_0 + \sum_{s=1}^{t}\varepsilon_s.$$

This implies that the process is a convex quadratic function of time, implausible empirically.

Answer:

The simplest proof is a proof by induction. Suppose that the expression is valid for time t. Then Y_{t+1} is given by:

$$\begin{aligned}Y_t &= \beta_1 + Y_t + \delta(t+1) + \varepsilon_{t+1} \\ &= \beta_1 + \left(t\,\beta_1 + \frac{t(t+1)}{2}\delta + Y_0 + \sum_{s=1}^{t}\varepsilon_s\right) + \delta(t+1) + \varepsilon_{t+1} \\ &= (t+1)\beta_1 + \frac{(t+1)(t+2)}{2}\delta + Y_0 + \sum_{s=1}^{t+1}\varepsilon_s\end{aligned}$$

and so it is valid for time $t+1$. But it is true for time 1. So it is valid for all $t \geq 1$.

13.17 Demonstrate that the OLS estimator of δ in the model:

$$Y_t = \beta_1 + \delta t + \varepsilon_t, \qquad t = 1, \ldots, T$$

is hyperconsistent. Show also that it is unbiased in finite samples, despite the fact that Y_t is nonstationary.

Answer:

Let $\widehat{\delta}$ be the OLS estimator of δ. Following the analysis in Chapter 2, $\widehat{\delta}$ may be decomposed as:

$$\widehat{\delta} = \delta + \sum_{t=1}^{T} a_t u_t$$

where:

$$a_t = \frac{t-0.5T}{\sum_{s=1}^{T}(s-0.5T)^2}.$$

Since a_t is deterministic:

$$E(\widehat{\delta}) = \delta + \sum_{t=1}^{T} a_t \, E(u_t) = \delta$$

and the estimator is unbiased. The variance of $\widehat{\delta}$, conditional on T, is:

$$\sigma^2_{\widehat{\delta}} = \frac{\sigma^2_\varepsilon}{\sum\limits_{t=1}^{T} (t - 0.5(T+1))^2}.$$

Now:

$$\begin{aligned}
\sum_{t=1}^{T} \left(t - \frac{1}{2}(T+1) \right)^2 &= \sum_{t=1}^{T} t^2 - (T+1) \sum_{t=1}^{T} t + \frac{1}{4} T(T+1)^2 \\
&= \frac{1}{6} T(T+1)(2T+1) - \frac{1}{2} T(T+1)^2 + \frac{1}{4} T(T+1)^2 \\
&= \frac{T+1}{12}(4T^2 + 2T - 6T^2 - 6T + 3T^2 + 3T) \\
&= \frac{T^3 - T}{12}.
\end{aligned}$$

Thus the variance is (asymptotically) inversely proportional to T^3 and the estimator is hyperconsistent.

13.6 Answers to the additional exercises

A13.1 The population series exhibits steady growth and is therefore obviously nonstationary. The growth is partly due to an excess of births over deaths and partly due to immigration. The question is whether variations in these factors are likely to be offsetting in the sense that a relatively large birth/ death excess one year is somehow automatically counterbalanced by a relatively small one in a subsequent year, or that a relatively large rate of immigration one year stimulates a reaction that leads to a relatively small one later. Such compensating mechanisms do not seem to exist, so trendstationarity may be ruled out. Population is a very good example of an integrated series with the effects of shocks being permanently incorporated in its level.

A13.2 It is difficult to come to any firm conclusion regarding this series. At first sight it looks like a random walk. On closer inspection, you will notice that after an initial decline in the first few years, the series appears to be stationary, with a high degree of correlation. The series is too short to allow one to discriminate between the two possibilities.

A13.3 As expected, given that the series is evidently nonstationary, the coefficient of $LGPOP(-1)$, -0.05, is close to zero and not significant. When we difference the series, the coefficient of $DLGPOP(-1)$ is -0.16 and not significant, even at the 5 per cent level. One possibility, which does not seem plausible, is that the population series is I(2). It is more likely that it is I(1), the first difference being stationary but highly autocorrelated.

A13.4 If the process is nonstationary, $\rho = 1$. If it is stationary, it could lie anywhere in the range $-1 < \rho < 1$. We must have a specific value for the null hypothesis. Hence we are forced to use nonstationarity as the null hypothesis, despite the inconvenience of having to compute alternative critical values of t.

A13.5 The model has been rewritten:

$$X_t - X_{t-1} = (\rho - 1)X_{t-1} + \varepsilon_t$$

so that the coefficient of X_{t-1} is zero under the null hypothesis of nonstationarity. We see that the null hypothesis is not rejected at any significance level, despite the fact that we know that the series is stationary. However, the estimate of the coefficient of X_{t-1}, -0.38, is not particularly close to zero. It implies an estimate of 0.67 for ρ, close to the actual value. This is a common outcome. Unit root tests generally have low power, making it generally difficult or impossible to discriminate between nonstationary processes and highly autocorrelated stationary processes.

A13.6 Where the hypothetical cointegrating relationship has a constant but no trend, as in the present case, the critical values of t are -3.34 and -3.90 at the 5 and 1 per cent levels, respectively (Davidson and MacKinnon, 1993). Hence the test indicates that we have a cointegrating relationship only for *DENT* and then only at the 5 per cent level. However, one knows in advance that the residuals are likely to be highly autocorrelated. Many of the coefficients are greater than 0.2 in absolute terms and perfectly compatible with a hypothesis of highly autocorrelated stationarity.

Test of cointegration

	$\widehat{\beta}_2$	s.e.	t		$\widehat{\beta}_2$	s.e.	t
ADM	-0.09	0.06	-1.69	*GASO*	-0.08	0.05	-1.62
BOOK	-0.17	0.08	-2.24	*HOUS*	-0.31	0.12	-2.52
BUSI	-0.23	0.09	-2.40	*LEGL*	-0.26	0.10	-2.59
CLOT	-0.41	0.13	-3.17	*MAGS*	-0.39	0.13	-3.03
DENT	-0.51	0.15	-3.51	*MASS*	-0.07	0.05	-1.48
DOC	-0.35	0.12	-2.99	*OPHT*	-0.14	0.08	-1.86
FLOW	-0.22	0.10	-2.14	*RELG*	-0.17	0.07	-2.35
FOOD	-0.29	0.11	-2.61	*TELE*	-0.22	0.09	-2.35
FURN	-0.32	0.10	-3.29	*TOB*	-0.16	0.10	-1.66
GAS	-0.24	0.09	-2.79	*TOYS*	-0.17	0.09	-1.96

A13.7 The expected value of Y_t is $\beta_1 t + Y_0$, and thus it is not independent of t, one of the conditions for stationarity. Similarly for Z_t. However:

$$Y_t - \beta_1 - \beta_2 Z_t = \varepsilon_t$$

and is therefore I(0).

A13.8

$$Y_t - X_t - Z_t = W_t.$$

Since W_t is stationary, the left side of the equation is a cointegrating relationship.

A13.9 *Two series* XA_t *and* XB_t *are generated as:*

$$\begin{aligned} XA_t &= RA_t + SA_t \\ XB_t &= RB_t + SB_t. \end{aligned}$$

Explain whether it is possible for XA_t *and* XB_t *to be stationary.*

Explain whether it is possible for them to be cointegrated.

A combination of a nonstationary process and a stationary one is nonstationary. Hence both X_A and X_B are nonstationary.

Since the nonstationary components of X_A and X_B are unrelated, there is no linear combination that is stationary, and so the series are not cointegrated.

Two series YA_t *and* YB_t *are generated as*

$$\begin{aligned} YA_t &= RA_t + SA_t \\ YB_t &= RA_t + SB_t. \end{aligned}$$

Explain whether it is possible for YA_t *and* YB_t *to be cointegrated.*

$$YA_t - YB_t = SA_t - SB_t.$$

This is a cointegrating relationship for YA_t and YB_t since $SA_t - SB_t$ is stationary.

Two series ZA_t *and* ZB_t *are generated as*

$$\begin{aligned} ZA_t &= RA_t + RB_t + SA_t \\ ZB_t &= RA_t - RB_t + SB_t. \end{aligned}$$

Explain whether it is possible for ZA_t *and* ZB_t *to be stationary.*

No linear combination of RA_t and RB_t can be stationary since they are independent random walks, and so ZA_t and ZB_t are both nonstationary.

Explain whether it is possible for them to be cointegrated.

No linear combination of ZA_t and ZB_t can eliminate both RA_t and RB_t, so there is no cointegrating relationship.

Chapter 14
Introduction to panel data

14.1 Overview

Increasingly, researchers are now using panel data where possible in preference to cross-sectional data. One major reason is that dynamics may be explored with panel data in a way that is seldom possible with crosssectional data. Another is that panel data offer the possibility of a solution to the pervasive problem of omitted variable bias. A further reason is that panel data sets often contain very large numbers of observations and the quality of the data is high. This chapter describes fixed effects regression and random effects regression, alternative techniques that exploit the structure of panel data.

14.2 Learning outcomes

After working through the corresponding chapter in the text, studying the corresponding slideshows, and doing the starred exercises in the text and the additional exercises in this subject guide, you should be able to:

- explain the differences between panel data, cross-sectional data, and time series data
- explain the benefits that can be obtained using panel data
- explain the differences between OLS pooled regressions, fixed effects regressions, and random effects regressions
- explain the potential advantages of the fixed effects model over pooled OLS
- explain the differences between the within-groups, first differences, and least squares dummy variables variants of the fixed effects model
- explain the assumptions required for the use of the random effects model
- explain the advantages of the random effects model over the fixed effects model when the assumptions are valid
- explain how to use a Durbin–Wu–Hausman test to determine whether the random effects model may be used instead of the fixed effects model.

14.3 Additional exercises

A14.1 The *NLSY2000* data set contains the following data for a sample of 2,427 males and 2,392 females for the years 1980–2000: years of work experience, EXP, years of schooling, S, and age, AGE. A researcher investigating the impact of schooling on willingness to work regresses EXP on S, including potential work experience, PWE, as a control. PWE was defined as:

$$PWE = AGE - S - 5.$$

The following regressions were performed for males and females separately:

(1) an ordinary least squares (OLS) regression pooling the observations

(2) a within-groups fixed effects regression

(3) a random effects regression.

The results of these regressions are shown in the table below. Standard errors are given in parentheses.

	Males			Females		
	OLS	FE	RE	OLS	FE	RE
S	0.78	0.65	0.72	0.89	0.71	0.85
	(0.01)	(0.01)	(0.01)	(0.01)	(0.02)	(0.01)
PWE	0.83	0.94	0.94	0.74	0.88	0.87
	(0.003)	(0.001)	(0.001)	(0.004)	(0.002)	(0.002)
constant	−10.16	dropped	−10.56	−11.11	dropped	−12.39
	(0.09)		(0.14)	(0.12)		(0.19)
R^2	0.79	—	—	0.71	—	—
n	24,057	24,057	24,057	18,758	18,758	18,758
DHW $\chi^2(2)$			10.76			1.43

- Explain why the researcher included PWE as a control.
- Evaluate the results of the Durbin–Wu–Hausman tests.
- For males and females separately, explain the differences in the coefficients of S in the OLS and FE regressions.
- For males and females separately, explain the differences in the coefficients of PWE in the OLS and FE regressions.

A14.2 Using the *NLSY2000* data set, a researcher fits OLS and fixed effects regressions of the logarithm of hourly wages on schooling, years of work experience, EXP, $ASVABC$ score, and dummies $MALE$, $ETHBLACK$, and $ETHHISP$ for being male, black, or hispanic. Schooling was split into years of high school, SH, and years of college, SC. The results are shown in the table below, with standard errors placed in parentheses.

	OLS	FE	RE
SH	0.026	0.005	0.016
	(0.002)	(0.007)	(0.004)
SC	0.063	0.073	0.067
	(0.001)	(0.004)	(0.002)
EXP	0.033	0.032	0.033
	(0.004)	(0.003)	(0.003)
ASVABC	0.012	—	0.011
	(0.003)		(0.001)
MALE	0.193		0.197
	(0.004)		(0.009)
ETHBLACK	−0.040	—	−0.030
	(0.007)		(0.015)
ETHHISP	0.047	—	0.033
	(0.008)		(0.018)
constant	5.639	—	5.751
	(0.028)		(0.051)
R^2	0.0367	—	—
DWH $\chi^2(3)$	—	—	9.31

If an individual reported being in high school or college, the observation for that individual for that year was deleted from the sample. As a consequence, the observations for most individuals in the sample begin when the formal education of that individual has been completed. However, a small minority of individuals, having apparently completed their formal education and having taken employment, subsequently resumed their formal education, either to complete high school with a general educational development (GED) degree equivalent to the high school diploma, or to complete one or more years of college.

- Discuss the differences in the estimates of the coefficient of *SH*.
- Discuss the differences in the estimates of the coefficient of *SC*.

A14.3 A researcher has data on G, the average annual rate of growth of GDP 2001–2005, and S, the average years of schooling of the workforce in 2005, for 28 European Union countries. She believes that G depends on S and on E, the level of entrepreneurship in the country, and a disturbance term u:

$$G = \beta_1 + \beta_2 S + \beta_3 E + u. \qquad (1)$$

u may be assumed to satisfy the usual regression model assumptions. Unfortunately the researcher does not have data on E.

- Explain intuitively and mathematically the consequences of performing a simple regression of G on S. For this purpose S and E may be treated as nonstochastic variables.

  The researcher does some more research and obtains data on G^*, the average annual rate of growth of GDP 1996–2000, and S^*, the average years of schooling of the workforce in 2000, for the same countries. She thinks that she can deal with the unobservable variable problem by regressing ΔG, the change in G, on ΔS, the change in S, where:

$$\begin{aligned}\Delta G &= G - G^* \\ \Delta S &= S - S^*\end{aligned}$$

assuming that E would be much the same for each country in the two periods. She fits the equation:

$$\Delta G = \delta_1 + \delta_2 \Delta S + w \qquad (2)$$

where w is a disturbance term that satisfies the usual regression model assumptions.

- Compare the properties of the estimators of the coefficient of S in (1) and of the coefficient of ΔS in (2).
- Explain why in principle you would expect the estimate of δ_1 in (2) not to be significant. Suppose that nevertheless the researcher finds that the coefficient is significant. Give two possible explanations.

Random effects regressions have potential advantages over fixed effect regressions.

- Could the researcher have used a random effects regression in the present case?

A14.4 A researcher has the following data for 3,763 respondents in the National Longitudinal Survey of Youth 1979– : hourly earnings in dollars in 1994 and 2000, years of schooling as recorded in 1994 and 2000, and years of work experience as recorded in 1994 and 2000. The respondents were aged 14–21 in 1979, so they were aged 29–36 in 1994 and 35–42 in 2000. 371 of the respondents had increased their formal schooling between 1994 and 2000, 210 by one year, 101 by two years, 47 by three years, and 13 by more than three years, mostly at college level in non-degree courses. The researcher performs the following regressions:

(1) the logarithm of hourly earnings in 1994 on schooling and work experience in 1994

(2) the logarithm of hourly earnings in 2000 on schooling and work experience in 2000

(3) the change in the logarithm of hourly earnings from 1994 to 2000 on the changes in schooling and work experience in that interval.

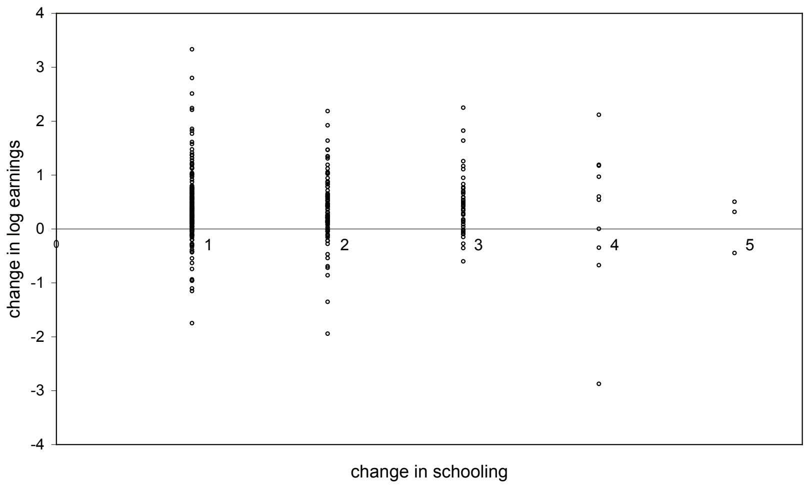
The results are shown in columns (1) – (3) in the table (t statistics in parentheses), and are presented at a seminar.

	(1)	(2)	(3)	(4)	(5)
Dependent variable	log earnings 1994	log earnings 2000	Change in log earnings 1994–2000	log earnings 2000	Change in log earnings 1994–2000
Schooling	0.114 (30.16)	0.116 (28.99)	—	0.108 (24.53)	—
Experience	0.052 (18.81)	0.038 (14.59)	—	0.037 (14.10)	— —
Cognitive ability score	—	—	—	0.004 (4.79)	—
Male	0.214 (12.03)	0.229 (11.77)	—	0.230 (11.88)	—
Black	−0.149 (−5.23)	−0.199 (−6.44)	—	−0.167 (−5.29)	—
Hispanic	0.039 (1.11)	0.053 (1.38)	—	0.071 (1.84)	—
Change in schooling	—	—	0.090 (5.00)	—	−0.006 (−0.16)
Change in experience	—	—	0.024 (2.75)	—	0.003 (0.15)
constant	4.899 (74.59)	5.023 (65.02)	0.102 (2.13)	4.966 (63.69)	0.389 (3.05)
R^2	0.265	0.243	0.007	0.248	0.0002
n	3,763	3,763	3,763	3,763	371

- The researcher is unable to explain why the coefficient of the change in schooling in regression (3) is so much lower than the schooling coefficients in (1) and (2). Someone says that it is because he has left out relevant variables such as cognitive ability, region of residence, etc, and the coefficients in (1) and (2) are therefore biased. Someone else says that cannot be the explanation because these variables are also omitted from regression (3). Explain what would be your view.
- He runs regressions (1) and (2) again, adding a measure of cognitive ability. The results for the 2000 regression are shown in column (4). The results for 1994 were very similar. Discuss possible reasons for the fact that the estimate of the schooling coefficient differs from those in (2) and (3).
- Someone says that the researcher should not have included a constant in regression (3). Explain why she made this remark and assess whether it is valid.
- Someone else at the seminar says that the reason for the relatively low coefficient of schooling in regression (3) is that it mostly represented non-degree schooling. Hence one would not expect to find the same relationship between schooling and earnings as for the regular pre-employment schooling of young people. Explain in general verbal terms what investigation the researcher should undertake in response to this suggestion.
- Another person suggests that the small minority of individuals who went back to school or college in their thirties might have characteristics different from

those of the individuals who did not, and that this could account for a different coefficient. Explain in general verbal terms what investigation the researcher should undertake in response to this suggestion.

- Finally, another person says that it might be a good idea to look at the relationship between earnings and schooling for the subsample who went back to school or college, restricting the analysis to these 371 individuals. The researcher responds by running the regression for that group alone. The result is shown in column (5) in the table. The researcher also plots a scatter diagram, reproduced below, showing the change in the logarithm of earnings and the change in schooling. For those with one extra year of schooling, the mean change in log earnings was 0.40. For those with two extra years, 0.37. For those with three extra years, 0.47. What conclusions might be drawn from the regression results?



A14.5 In the discussion of the DWH test, it was stated that the test compares the coefficients of those variables not dropped in the FE regression. Explain why the constant is not included in the comparison.

14.4 Answer to the starred exercise in the textbook

14.9 The *NLSY2000* data set contains the following data for a sample of 2,427 males and 2,392 females for the years 1980–2000: weight in pounds, years of schooling, age, marital status in the form of a dummy variable *MARRIED* defined to be 1 if the respondent was married, 0 if single, and height in inches. Hypothesizing that weight is influenced by schooling, age, marital status, and height, the following regressions were performed for males and females separately:

(1) an ordinary least squares (OLS) regression pooling the observations

(2) a within-groups fixed effects regression

(3) a random effects regression.

The results of these regressions are shown in the table. Standard errors are given in parentheses.

	Males			Females		
	OLS	FE	RE	OLS	FE	RE
Year of schooling	−0.98 (0.09)	−0.02 (0.23)	−0.45 (0.16)	−1.95 (0.12)	−0.60 (0.27)	−1.25 (0.18)
Age	1.61 (0.04)	1.64 (0.02)	1.65 (0.02)	2.03 (0.05)	1.66 (0.03)	1.72 (0.03)
Married	3.70 (0.48)	2.92 (0.33)	3.00 (0.32)	−8.27 (0.59)	3.08 (0.46)	1.98 (0.44)
Height	5.07 (0.08)	dropped	4.95 (0.18)	3.48 (0.10)	dropped	3.38 (0.21)
constant	−209.52 (5.39)	dropped	−209.81 (12.88)	−105.90 (6.62)	dropped	−107.61 (13.43)
R^2	0.27	—	—	0.17	—	—
n	17,299	17,299	17,299	13,160	13,160	13,160
DWH $\chi^2(3)$			7.22			92.94

Explain why height is excluded from the FE regression.

Evaluate, for males and females separately, whether the fixed effects or random effects model should be preferred.

For males and females separately, compare the estimates of the coefficients in the OLS and FE models and attempt to explain the differences.

Explain in principle how one might test whether individual-specific fixed effects jointly have significant explanatory power, if the number of individuals is small. Explain why the test is not practical in this case.

Answer:

Height is constant over observations. Hence, for each individual:

$$HEIGHT_{it} - \overline{HEIGHT}_i = 0$$

for all t, where $\overline{HEIGHT}_i$ is the mean height for individual i for the observations for that individual. Hence height has to be dropped from the regression model.

The critical value of chi-squared, with three degrees of freedom, is 7.82 at the 5 percent level and 16.27 at the 0.1 percent level. Hence there is a possibility that the random effects model may be appropriate for males, but it is definitely not appropriate for females.

Males

The OLS regression suggests that schooling has a small (one pound less per year of schooling) but highly significant negative effect on weight. The fixed effects regression eliminates the effect, indicating that an unobserved effect is responsible: males with unobserved qualities that have a positive effect on educational attainment, controlling for other measured variables, have lower weight as a consequence of the same unobserved qualities. We cannot compare estimates of the effect of height since it is dropped from the FE regression. The effect of age is the same in the two regressions. There is a small but highly significant positive effect of being married, the OLS estimate possibly being inflated by an unobserved effect.

Females

The main, and very striking, difference is in the marriage coefficient. The OLS regression suggests that marriage reduces weight by eight pounds, a remarkable amount. The FE regression suggests the opposite, that marriage leads to an *increase* in weight that is similar to that for males. The clear implication is that women who weigh less are relatively successful in the marriage market, but once they are married they put on weight.

For schooling the story is much the same as for males, except that the OLS coefficient is much larger and the coefficient remains significant at the 5 percent level in the FE regression. The effect of age appears to be exaggerated in the OLS regression, for reasons that are not obvious.

One might test whether individual-specific fixed effects jointly have significant explanatory power by performing a LSDV regression, eliminating the intercept in the model and adding a dummy variable for each individual. One would compare RSS for this regression with that for the regression without the dummy variables, using a standard F test. In the present case it is not a practical proposition because there are more than 17,000 males and 13,000 females.

14.5 Answers to the additional exercises

A14.1 *Explain why the researcher included PWE as a control.*

Clearly actual work experience is positively influenced by PWE. Omitting it would cause the coefficient of S to be biased downwards since PWE and S are negatively correlated.

Evaluate the results of the Durbin–Wu–Hausman tests

With two degrees of freedom, the critical value of chi-squared is 5.99 at the 5 percent level and 9.21 at the 1 percent level. Thus the random effects model is rejected for males but seemingly not for females.

For males and females separately, explain the differences in the coefficients of S in the OLS and FE regressions.

For both sexes the OLS estimate is greater than the FE estimate. One possible reason is that some unobserved characteristics, for example drive, are positively correlated with both acquiring schooling, and seeking and gaining employment.

For males and females separately, explain the differences in the coefficients of PWE in the OLS and FE regressions.

Since S and PWE are negatively correlated, these same unobserved characteristics would cause the OLS estimate of the coefficient of PWE to be biased downwards.

A14.2 First, note that the DWH statistic is significant at the 5 per cent level (critical value 7.82) but not at the 1 per cent level (critical value 11.35).

The coefficients of SH and SC in the OLS regression is an estimate of the impact of variations in years of high school and years of college among all the individuals in the sample. Most individuals in fact completed high school and so had $SH = 12$.

However, a small minority did not and this variation made possible the estimation of the *SH* coefficient. The majority of the remainder did not complete any years of college and therefore had $SC = 0$, but a substantial minority did have a partial or complete college education, some even pursuing postgraduate studies, and this variation made possible the estimation of the *SC* coefficient.

Most individuals completed their formal education before entering employment. For them, $SH_{it} = \overline{SH}_i$ for all t and hence $SH_{it} - \overline{SH}_i = 0$ for all t. As a consequence, the observations for such individuals provide no variation in the *SH* variable. Likewise they provide no variation in the *SC* variable. If all observations pertained to such individuals, schooling would be washed out in the FE regression along with other unchanging characteristics such as sex, ethnicity, and *ASVABC* score. The schooling coefficients in the FE regression therefore relate to those individuals who returned to formal education after a break in which they found employment.

The fact that these individuals account for a relatively small proportion of the observations in the data set has an adverse effect on the precision of the FE estimates of the coefficients of *SH* and *SC*. This is reflected in standard errors that are much larger than those obtained in the OLS pooled regression.

Discuss the differences in the estimates of the coefficient of SH.

Most of the variation in *SH* in the FE regressions come from individuals earning the GED degree. This degree provides an opportunity for high school drop-outs to make good their shortfall by taking courses and passing the examinations required for this diploma. These courses may be civilian or military adult education classes, but very often they are programmes offered to those in jail. In principle the GED should be equivalent to the high school diploma, but there is some evidence that standards are sometimes lower. The results in the table appear to corroborate this view. The OLS regression indicates that a year of high school raises earnings by 2.6 per cent, with the coefficient being highly significant, whereas the FE coefficient indicates that the effect is only 0.5 per cent and not significant.

Discuss the differences in the estimates of the coefficient of SC.

Some of the variation in *SC* in the FE regressions comes from individuals entering employment for a year or two after finishing high school and then going to college, resuming their formal education. However, most comes from individuals returning to college for a year or two after having been employment for a number of years. A typical example is a high school graduate who has settled down in an occupation and who has then decided to upgrade his or her professional skills by taking a two-year associate of arts degree. Similarly one encounters college graduates who upgrade to masters level after having worked for some time. One would expect such students to be especially well motivated – they are often undertaking studies that are relevant to an established career, and they are often bearing high opportunity costs from loss of earnings while studying – and accordingly one might expect the payoff in terms of increased earnings to be relatively high. This seems to be borne out in a comparison of the OLS and FE estimates of the coefficient of *SC*, though the difference is not dramatic.

On the surface, this exercise appeared to be about how one might use FE to eliminate the bias in OLS pooled regression caused by unobserved effects. Has the analysis been successful in this respect? Absolutely not. In particular, the apparent conclusion that high school education has virtually no effect on earnings should not be taken at face value. The reason is that the issue of biases attributable to unobserved effects has been overtaken by the much more important issue of the difference in the interpretation of the *SH* and *SC* coefficients discussed in the exercise. This illustrates a basic point in econometrics: understanding the context of the data is often just as important as being proficient at technical analysis.

A14.3 *Explain intuitively and mathematically the consequences of performing a simple regression of G on S. For this purpose S and E may be treated as nonstochastic variables.*

If one fits the regression:

$$\widehat{G} = \widehat{\beta}_1 + \widehat{\beta}_2 S$$

then

$$\begin{aligned}
\widehat{\beta}_2 &= \frac{\sum\left(S_i-\overline{S}\right)\left(G_i-\overline{G}\right)}{\sum\left(S_i-\overline{S}\right)^2} \\
&= \frac{\sum\left(S_i-\overline{S}\right)\left((\beta_1+\beta_2 S_i+\beta_3 E_i+u_i)-\left(\beta_1+\beta_2\overline{S}+\beta_2\overline{E}+\overline{u}\right)\right)}{\sum\left(S_i-\overline{S}\right)^2} \\
&= \beta_2+\beta_3\frac{\sum\left(S_i\overline{S}\right)\left(E_i-\overline{E}\right)}{\sum\left(S_i-\overline{S}\right)^2}+\frac{\sum\left(S_i-\overline{S}\right)\left(u_i-\overline{u}\right)}{\sum\left(S_i-\overline{S}\right)^2}.
\end{aligned}$$

Taking expectations, and making use of the invitation to treat S and E as nonstochastic:

$$\begin{aligned}
E(\widehat{\beta}_2) &= \beta_2+\beta_3\frac{\sum\left(S_i\overline{S}\right)\left(E_i-\overline{E}\right)}{\sum\left(S_i-\overline{S}\right)^2}+\frac{\sum\left(S_i-\overline{S}\right)E\left(u_i-\overline{u}\right)}{\sum\left(S_i-\overline{S}\right)^2} \\
&= \beta_2+\beta_3\frac{\sum\left(S_i-\overline{S}\right)\left(E_i-\overline{E}\right)}{\sum\left(S_i-\overline{S}\right)^2}.
\end{aligned}$$

Hence the estimator is biased unless S and E happen to be uncorrelated in the sample. As a consequence, the standard errors will be invalid.

Compare the properties of the estimators of the coefficient of S in (1) and of the coefficient of ΔS in (2).

Given (1), the differenced model should have been:

$$\Delta G = \delta_2 \Delta S + w$$

where $w = u - u^*$.

The estimator of the coefficient of ΔS in (2) should be unbiased, while that of S in (1) will be subject to omitted variable bias. However:

- it is possible that the bias in (1) may be small. This would be the case if E were a relatively unimportant determinant of G or if its correlation with S were low.
- it is possible that the variance in ΔS is smaller than that of S. This would be the case if S were changing slowly in each country, or if the rate of change of S were similar in each country.

Thus there may be a trade-off between bias and variance and it is possible that the estimator of β_2 using specification (1) could actually be superior according to some criterion such as the mean square error. It should be noted that the inclusion of δ_1 in (2) will make the estimation of δ_2 even less efficient.

Explain why in principle you would expect the estimate of δ_1 in (2) not to be significant. Suppose that nevertheless the researcher finds that the coefficient is significant. Give two possible explanations.

If specification (1) is correct, there should be no intercept in (2) and for this reason the estimate of the intercept should not be significantly different from zero. If it is significant, this could have occurred as a matter of Type I error. Alternatively, it might indicate a shift in the relationship between the two time periods. Suppose that (1) should have included a dummy variable set equal to 0 in the first time period and 1 in the second. $\widehat{\delta}_1$ would then be an estimate of its coefficient.

Could the researcher have used a random effects regression in the present case?

Random effects requires the sample to be drawn randomly from a population and for unobserved effects to be uncorrelated with the regressors. The first condition is not satisfied here, so random effects would be inappropriate.

A14.4 *The researcher is unable to explain why the coefficient of the change in schooling in regression (3) is so much lower than the schooling coefficients in (1) and (2). Someone says that it is because he has left out relevant variables such as cognitive ability, region of residence, etc, and the coefficients in (1) and (2) are therefore biased. Someone else says that cannot be the explanation because these variables are also omitted from regression (3). Explain what would be your view.*

Suppose that the true model is:

$$\begin{aligned} LGEARN &= \beta_1 + \beta_2 S + \beta_3 EXP + \beta_4 ASVABC + \beta_5 MALE \\ &\quad + \beta_6 ETHBLACK + \beta_7 ETHHISP + \beta_8 X_8 + u \end{aligned}$$

where X_8 is some further fixed characteristic of the respondent. $ASVABC$ and X_8 are absent from regressions (1) and (2) and so those regressions will be subject to omitted variable bias. In particular, since $ASVABC$ is likely to be positively correlated with S, and to have a positive coefficient, its omission will tend to bias the coefficient of S upwards.

However, if the specification is valid for both 1994 and 2000 and unchanged, one can eliminate the omitted variable bias by taking first differences as in regression (3):

$$\Delta LGEARN = \beta_2 \Delta S + \beta_3 \Delta EXP + \Delta u.$$

By fitting this specification one should obtain unbiased estimates of the coefficients of schooling and experience, and the former should therefore be smaller than in (1) and (2). Note that all the fixed characteristics have been washed out. The suggestion that $ASVABC$ should have been included in (3) is therefore incorrect.

Note that (3) should not have included an intercept. This is discussed later in the question.

He runs regressions (1) and (2) again, adding a measure of cognitive ability. The results for the 2000 regression are shown in column (4). The results for 1994 were very similar. Discuss possible reasons for the fact that the estimate of the schooling coefficient differs from those in (2) and (3).

The estimate of the coefficient of S differs from that in (2) because the omitted variable bias attributable to the omission of $ASVABC$ in that specification has now been corrected. However it is still biased if X_8 (representing other omitted characteristics) is a determinant of earnings and is correlated with S. This partial rectification of the omitted variable problem accounts for the fact that the coefficient of S in (4) lies between those in (2) and (3).

Someone says that the researcher should not have included a constant in regression (3). Explain why she made this remark and assess whether it is valid.

Given the specification in (1) and (2), there should have been no intercept in the first differences specification (3). One would therefore expect the estimate of the intercept to be somewhere near zero in the sense of not being significantly different from it. Nevertheless, it is significantly different at the 5 percent level. However, suppose that the relationship shifted between 1994 and 2000, and that the shift could be represented by a dummy variable D equal to zero in 1994 and 1 in 2000, with coefficient δ. Then (3) should have an intercept δ. Its estimate, 0.102, suggests that earnings grew by 10 percent from 1994 to 2000, holding other factors constant. This seems entirely reasonable, perhaps even a little low.

Alternatively, the apparently significant t statistic might have arisen as a matter of Type I error.

Someone else at the seminar says that the reason for the relatively low coefficient of schooling in regression (3) is that it mostly represented non-degree schooling. Hence one would not expect to find the same relationship between schooling and earnings as for the regular preemployment schooling of young people. Explain in general verbal terms what investigation the researcher should undertake in response to this suggestion.

Divide S into two variables, schooling as of 1994 and extra schooling as of 2000, with separate coefficients. Then use a standard F test (or t test) of a restriction to test whether the coefficients are significantly different.

Another person suggests that the small minority of individuals who went back to school or college in their thirties might have characteristics different from those of the individuals who did not, and that this could account for a different coefficient. Explain in general verbal terms what investigation the researcher should undertake in response to this suggestion.

The issue is sample selection bias and an appropriate procedure would be that proposed by Heckman. One would use probit analysis with an appropriate set of determinants to model the decision to return to school between 1994 and 2000, and a regression model to explain variations in the logarithm of earnings of those respondents who do return to school, linking the two models by allowing their disturbance terms to be correlated. One would test whether the estimate of this correlation is significantly different from zero.

Finally, another person says that it might be a good idea to look at the relationship between earnings and schooling for the subsample who went back to school or college, restricting the analysis to these 371 individuals. The researcher responds by running the regression for that group alone. The result is shown in column (5) in the table. The researcher also plots a scatter diagram, reproduced below, showing the change in the logarithm of earnings and the change in schooling. For those with one extra year of schooling, the mean change in log earnings was 0.40. For those with two extra years, 0.37. For those with three extra years, 0.47. What conclusions might be drawn from the regression results?

The schooling coefficient is effectively zero! [These are real data, incidentally.] The scatter diagram shows why. Irrespective of whether the respondent had one, two, or three years of extra schooling, the gain is about the same, on average. (These are the only categories with large numbers of observations, given the information at the beginning of the question, confirmed by the scatter diagram.) So the results indicate that the fact of going back to school, rather than the duration of the schooling, is the relevant determinant of the change in earnings. The intercept indicates that this subsample on average increased their earnings between 1994 and 2000 by 38.9 percent. (As a first approximation. The actual proportion would be better estimated as $e^{0.389} - 1 = 0.476$.) This figure is confirmed by the diagram, and it would appear to be much greater than the effect of regular schooling. One explanation could be sample selection bias, as already discussed. A more likely possibility is that the respondents were presented with opportunities to increase their earnings substantially if they undertook certain types of formal course, and they took advantage of these opportunities.

A14.5 In a random effects regression, the interpretation of an intercept is not affected by the estimation technique. In a fixed effects regression, the intercept is washed out. Hence there is no basis for a comparison. In general, the model is fitted without an intercept. The only case where an intercept should be included is in first-differences fixed effects estimation of a model containing a deterministic trend. For example, suppose one is fitting the model:

$$Y_{it} = \beta_1 + \beta_2 X_{it} + \delta t + u_{it}.$$

For individual i in the previous time period, one has:

$$Y_{i,t-1} = \beta_1 + \beta_2 X_{i,t-1} + \delta(t-1) + u_{i,t-1}.$$

Subtracting, one obtains:

$$Y_{it} - Y_{i,t-1} = \beta_2 (X_{it} - X_{i,t-1}) + \delta + u_{it} - u_{i,t-1}.$$

The model now does have an intercept, but its meaning is different from that in the original specification. It now provides an estimate of δ, not β_1.

Chapter 15

Regression analysis with linear algebra primer

15.1 Overview

This primer is intended to provide a mathematical bridge to a master's level course that uses linear algebra for students who have taken an undergraduate econometrics course that does not. Why should we make the mathematical shift? The most immediate reason is the huge double benefit of allowing us to generalise the core results to models with many explanatory variables while simultaneously permitting a great simplification of the mathematics. This alone justifies the investment in time – probably not more than ten hours – required to acquire the necessary understanding of basic linear algebra.

In fact, one could very well put the question the other way. Why do introductory econometrics courses not make this investment and use linear algebra from the start? Why do they (almost) invariably use ordinary algebra, leaving students to make the switch when they take a second course?

The answer to this is that the overriding objective of an introductory econometrics course must be to encourage the development of a solid intuitive understanding of the material and it is easier to do this with familiar, everyday algebra than with linear algebra, which for many students initially seems alien and abstract. An introductory course should ensure that at all times students understand the purpose and value of what they are doing. This is far more important than proofs and for this purpose it is usually sufficient to consider models with one, or at most two, explanatory variables. Even in the relatively advanced material, where we are forced to consider asymptotics because we cannot obtain finite-sample results, the lower-level mathematics holds its own. This is especially obvious when we come to consider finite-sample properties of estimators when only asymptotic results are available mathematically. We invariably use a simple model for a simulation, not one that requires a knowledge of linear algebra.

These comments apply even when it comes to proofs. It is usually helpful to see a proof in miniature where one can easily see exactly what is involved. It is then usually sufficient to know that in principle it generalises, without there being any great urgency to see a general proof. Of course, the linear algebra version of the proof will be general and often simpler, but it will be less intuitively accessible and so it is useful to have seen a miniature proof first. Proofs of the unbiasedness of the regression coefficients under appropriate assumptions are obvious examples.

At all costs, one wishes to avoid the study of econometrics becoming an extended exercise in abstract mathematics, most of which practitioners will never use again. They will use regression applications and as long as they understand what is happening in principle, the actual mechanics are of little interest.

This primer is not intended as an exposition of linear algebra as such. It assumes that a basic knowledge of linear algebra, for which there are many excellent introductory textbooks, has already been acquired. For the most part, it is sufficient that you should know the rules for multiplying two matrices together and for deriving the inverse of a square matrix, and that you should understand the consequences of a square matrix having a zero determinant.

15.2 Notation

Matrices and vectors will be written bold, upright, matrices upper case, for example $\mathbf{A}$, and vectors lower case, for example $\mathbf{b}$. The transpose of a matrix will be denoted by a prime, so that the transpose of $\mathbf{A}$ is $\mathbf{A}'$, and the inverse of a matrix will be denoted by a superscript -1, so that the inverse of $\mathbf{A}$ is $\mathbf{A^{-1}}$.

15.3 Test exercises

Answers to all of the exercises in this primer will be found at its end. If you are unable to answer the following exercises, you need to spend more time learning basic matrix algebra before reading this primer. The rules in Exercises 3–5 will be used frequently without further explanation.

1. Demonstrate that the inverse of the inverse of a matrix is the original matrix.
2. Demonstrate that if a (square) matrix possesses an inverse, the inverse is unique.
3. Demonstrate that, if $\mathbf{A} = \mathbf{BC}$, $\mathbf{A}' = \mathbf{C}'\mathbf{B}'$.
4. Demonstrate that, if $\mathbf{A} = \mathbf{BC}$, $\mathbf{A^{-1}} = \mathbf{C^{-1}B^{-1}}$, provided that $\mathbf{B^{-1}}$ and $\mathbf{C^{-1}}$ exist.
5. Demonstrate that $[\mathbf{A}']^{\mathbf{-1}} = [\mathbf{A^{-1}}]'$.

15.4 The multiple regression model

The most obvious benefit from switching to linear algebra is convenience. It permits an elegant simplification and generalisation of much of the mathematical analysis associated with regression analysis. We will consider the general multiple regression model:

$$Y_i = \beta_1 X_{i1} + \cdots + \beta_k X_{ik} + u_i \qquad (1)$$

where the second subscript identifies the variable and the first the observation. In the textbook, as far as the fourth edition, the subscripts were in the opposite order. The reason for the change of notation here, which will be adopted in the next edition of the textbook, is that it is more compatible with a linear algebra treatment.

Equation (1) is a row relating to observation i in a sample of n observations. The entire layout would be:

$$\begin{bmatrix} Y_1 \\ \vdots \\ Y_i \\ \vdots \\ Y_n \end{bmatrix} = \begin{bmatrix} \beta_1 X_{11} + \cdots + \beta_j X_{1j} + \cdots + \beta_k X_{1k} \\ \vdots \\ \beta_1 X_{i1} + \cdots + \beta_j X_{ij} + \cdots + \beta_k X_{ik} \\ \vdots \\ \beta_1 X_{n1} + \cdots + \beta_j X_{nj} + \cdots + \beta_k X_{nk} \end{bmatrix} + \begin{bmatrix} u_1 \\ \vdots \\ u_i \\ \vdots \\ u_n \end{bmatrix}.$$

This, of course, may be written in linear algebra form as:

$$\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \mathbf{u} \qquad (2)$$

where:

$$\mathbf{y} = \begin{bmatrix} Y_1 \\ \vdots \\ Y_i \\ \vdots \\ Y_n \end{bmatrix}, \quad \mathbf{X} = \begin{bmatrix} X_{11} \cdots X_{1j} \cdots X_{1k} \\ \vdots \\ X_{i1} \cdots X_{ij} \cdots X_{ik} \\ \vdots \\ X_{n1} \cdots X_{nj} \cdots X_{nk} \end{bmatrix}, \quad \boldsymbol{\beta} = \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_i \\ \vdots \\ \beta_n \end{bmatrix}, \quad \text{and } \mathbf{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_i \\ \vdots \\ u_n \end{bmatrix}$$

with the first subscript of X_{ij} relating to the row and the second to the column, as is conventional with matrix notation. This was the reason for the change in the order of the subscripts in equation (1).

Frequently, it is convenient to think of the matrix $\mathbf{X}$ as consisting of a set of column vectors:

$$\mathbf{X} = [\mathbf{x}_1 \cdots \mathbf{x}_j \cdots \mathbf{x}_k]$$

where:

$$\mathbf{x}_j = \begin{bmatrix} X_{1j} \\ \vdots \\ X_{ij} \\ \vdots \\ X_{nj} \end{bmatrix}.$$

$\mathbf{x}_j$ is the set of observations relating to explanatory variable j. It is written lower case, bold, not italic because it is a vector.

15.5 The intercept in a regression model

As described above, there is no special intercept term in the model. If, as is usually the case, one is needed, it is accommodated within the matrix framework by including an X variable, typically placed as the first, with value equal to 1 in all observations:

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ \vdots \\ 1 \\ \vdots \\ 1 \end{bmatrix}.$$

The coefficient of this unit vector is the intercept in the regression model. If it is included, and located as the first column, the $\mathbf{X}$ matrix becomes:

$$\mathbf{X} = \begin{bmatrix} 1 & X_{12} & \cdots & X_{1j} & \cdots & X_{1k} \\ & & & \vdots & & \\ 1 & X_{i2} & \cdots & X_{ij} & \cdots & X_{ik} \\ & & & \vdots & & \\ 1 & X_{n2} & \cdots & X_{nj} & \cdots & X_{nk} \end{bmatrix} = [\mathbf{1} \;\; \mathbf{x}_2 \;\; \cdots \;\; \mathbf{x}_j \;\; \cdots \;\; \mathbf{x}_k] \,.$$

15.6 The OLS regression coefficients

Using the matrix and vector notation, we may write the fitted equation:

$$\widehat{Y}_i = \widehat{\beta}_1 X_{i1} + \cdots + \widehat{\beta}_k X_{ik}$$

as:

$$\widehat{\mathbf{y}} = \mathbf{X}\widehat{\boldsymbol{\beta}}$$

with obvious definitions of $\widehat{\mathbf{y}}$ and $\widehat{\boldsymbol{\beta}}$. Then we may define the vector of residuals as:

$$\widehat{\mathbf{u}} = \mathbf{y} - \widehat{\mathbf{y}} = \mathbf{y} - \mathbf{X}\widehat{\boldsymbol{\beta}}$$

and the residual sum of squares as:

$$\begin{aligned} RSS = \widehat{\mathbf{u}}'\widehat{\mathbf{u}} &= (\mathbf{y} - \mathbf{X}\widehat{\boldsymbol{\beta}})'(\mathbf{y} - \mathbf{X}\widehat{\boldsymbol{\beta}}) \\ &= \mathbf{y}'\mathbf{y} - \mathbf{y}'\mathbf{X}\widehat{\boldsymbol{\beta}} - \widehat{\boldsymbol{\beta}}'\mathbf{X}'\mathbf{y} + \widehat{\boldsymbol{\beta}}'\mathbf{X}'\mathbf{X}\widehat{\boldsymbol{\beta}} \\ &= \mathbf{y}'\mathbf{y} - 2\mathbf{y}'\mathbf{X}\widehat{\boldsymbol{\beta}} + \widehat{\boldsymbol{\beta}}'\mathbf{X}'\mathbf{X}\widehat{\boldsymbol{\beta}} \end{aligned}$$

($\mathbf{y}'\mathbf{X}\widehat{\boldsymbol{\beta}} = \widehat{\boldsymbol{\beta}}'\mathbf{X}'\mathbf{y}$ since it is a scalar.) The next step is to obtain the normal equations:

$$\frac{\partial RSS}{\partial \widehat{\beta}_j} = 0$$

for $j = 1, \ldots, k$ and solve them (if we can) to obtain the least squares coefficients. Using linear algebra, the normal equations can be written:

$$\mathbf{X}'\mathbf{X}\widehat{\boldsymbol{\beta}} - \mathbf{X}'\mathbf{y} = \mathbf{0}.$$

The derivation is straightforward but tedious and has been consigned to Appendix A. $\mathbf{X}'\mathbf{X}$ is a square matrix with k rows and columns. If assumption A.2 is satisfied (that it is not possible to write one X variable as a linear combination of the others), $\mathbf{X}'\mathbf{X}$ has an inverse and we obtain the OLS estimator of the coefficients:

$$\widehat{\boldsymbol{\beta}} = [\mathbf{X}'\mathbf{X}]^{-\mathbf{1}}\mathbf{X}'\mathbf{y}. \qquad (3)$$

Exercises

6. If $Y = \beta_1 + \beta_2 X + u$, obtain the OLS estimators of β_1 and β_2 using (3).
7. If $Y = \beta_2 + u$, obtain the OLS estimator of β_2 using (3).
8. If $Y = \beta_1 + u$, obtain the OLS estimator of β_1 using (3).

15.7 Unbiasedness of the OLS regression coefficients

Substituting for $\mathbf{y}$ from (2) into (3), we have:

$$\begin{aligned}\widehat{\boldsymbol{\beta}} &= [\mathbf{X}'\mathbf{X}]^{-1}\mathbf{X}'(\mathbf{X}\boldsymbol{\beta}+\mathbf{u}) \\ &= [\mathbf{X}'\mathbf{X}]^{-1}\mathbf{X}'\mathbf{X}\boldsymbol{\beta}+[\mathbf{X}'\mathbf{X}]^{-1}\mathbf{X}'\mathbf{u} \\ &= \boldsymbol{\beta}+[\mathbf{X}'\mathbf{X}]^{-1}\mathbf{X}'\mathbf{u}.\end{aligned}$$

Hence each element of $\widehat{\boldsymbol{\beta}}$ is equal to the corresponding value of $\boldsymbol{\beta}$ plus a linear combination of the values of the disturbance term in the sample. Next:

$$E(\widehat{\boldsymbol{\beta}}\,|\,\mathbf{X}) = \boldsymbol{\beta}+E([\mathbf{X}'\mathbf{X}]^{-1}\mathbf{X}'\mathbf{u}\,|\,\mathbf{X}).$$

To proceed further, we need to be specific about the data generation process (DGP) for $\mathbf{X}$ and the assumptions concerning $\mathbf{u}$ and $\mathbf{X}$. In Model A, we have no DGP for $\mathbf{X}$: the data are simply taken as given. When we describe the properties of the regression estimators, we are either talking about the potential properties, before the sample has been drawn, or about the distributions that we would expect in repeated samples using those given data on $\mathbf{X}$. If we make the assumption $E(\mathbf{u}\,|\,\mathbf{X}) = \mathbf{0}$, then:

$$E(\widehat{\boldsymbol{\beta}}\,|\,\mathbf{X}) = \boldsymbol{\beta}+[\mathbf{X}'\mathbf{X}]^{-1}\mathbf{X}'E(\mathbf{u}\,|\,\mathbf{X}) = \boldsymbol{\beta}$$

and so $\widehat{\boldsymbol{\beta}}$ is an unbiased estimator of $\boldsymbol{\beta}$. It should be stressed that unbiasedness in Model A, along with all other properties of the regression estimators, are conditional on the actual given data for $\mathbf{X}$.

In Model B, we allow $\mathbf{X}$ to be drawn from a fixed joint distribution of the explanatory variables. The appropriate assumption for the disturbance term is that it is distributed independently of $\mathbf{X}$ and hence its conditional distribution is no different from its absolute distribution: $E(\mathbf{u}\,|\,\mathbf{X}) = E(\mathbf{u})$ for all $\mathbf{X}$. We also assume $E(\mathbf{u}) = \mathbf{0}$. The independence of the distributions of $\mathbf{X}$ and $\mathbf{u}$ allows us to write:

$$\begin{aligned}E(\widehat{\boldsymbol{\beta}}\,|\,\mathbf{X}) &= \boldsymbol{\beta}+E\left([\mathbf{X}'\mathbf{X}]^{-1}\mathbf{X}'\mathbf{u}|\mathbf{X}\right) \\ &= \boldsymbol{\beta}+E\left([\mathbf{X}'\mathbf{X}]^{-1}\mathbf{X}'\right)E(\mathbf{u}) \\ &= \boldsymbol{\beta}.\end{aligned}$$

15.8 The variance-covariance matrix of the OLS regression coefficients

We define the variance-covariance matrix of the disturbance term to be the matrix whose element in row i and column j is the population covariance of u_i and u_j. By assumption A.4, the covariance of u_i and u_j is constant and equal to σ_u^2 if $j = i$ and by assumption A.5 it is equal to zero if $j \neq i$. Thus the variance-covariance matrix is:

$$\begin{bmatrix} \sigma_u^2 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & \sigma_u^2 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & \sigma_u^2 & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & \sigma_u^2 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & \sigma_u^2 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & \sigma_u^2 \end{bmatrix}$$

that is, a matrix whose diagonal elements are all equal to σ_u^2 and whose off-diagonal elements are all zero. It may more conveniently be written $\mathbf{I}_n\sigma_u^2$ where $\mathbf{I}_n$ is the identity matrix of order n.

Similarly, we define the variance-covariance matrix of the regression coefficients to be the matrix whose element in row i and column j is the population covariance of $\widehat{\beta}_i$ and $\widehat{\beta}_j$:

$$\mathrm{cov}(\widehat{\beta}_i, \widehat{\beta}_j) = E\left[(\widehat{\beta}_i - E(\widehat{\beta}_i))(\widehat{\beta}_j - E(\widehat{\beta}_j))\right] = E\left[(\widehat{\beta}_i - \beta_i)(\widehat{\beta}_j - \beta_j)\right].$$

The diagonal elements are of course the variances of the individual regression coefficients. We denote this matrix $\mathrm{var}(\widehat{\boldsymbol{\beta}})$. If we are using the framework of Model A, everything will be conditional on the actual given data for $\mathbf{X}$, so we should refer to $\mathrm{var}(\widehat{\boldsymbol{\beta}} \,|\, \mathbf{X})$ rather than $\mathrm{var}(\widehat{\boldsymbol{\beta}})$. Then:

$$\begin{aligned} \mathrm{var}(\widehat{\boldsymbol{\beta}} \,|\, \mathbf{X}) &= E((\widehat{\boldsymbol{\beta}} - E(\widehat{\boldsymbol{\beta}}))(\widehat{\boldsymbol{\beta}} - E(\widehat{\boldsymbol{\beta}}))'|\mathbf{X}) \\ &= E((\widehat{\boldsymbol{\beta}} - \boldsymbol{\beta})(\widehat{\boldsymbol{\beta}} - \boldsymbol{\beta})' \,|\, \mathbf{X}) \\ &= E\left(([\mathbf{X}'\mathbf{X}]^{-\mathbf{1}}\mathbf{X}'\mathbf{u})([\mathbf{X}'\mathbf{X}]^{-\mathbf{1}}\mathbf{X}'\mathbf{u})' \,|\, \mathbf{X}\right) \\ &= E\left([\mathbf{X}'\mathbf{X}]^{-\mathbf{1}}\mathbf{X}'\mathbf{u}\mathbf{u}'\mathbf{X}[\mathbf{X}'\mathbf{X}]^{-\mathbf{1}} \,|\, \mathbf{X}\right) \\ &= [\mathbf{X}'\mathbf{X}]^{-\mathbf{1}}\mathbf{X}'E(\mathbf{u}\mathbf{u}'|\mathbf{X})\mathbf{X}[\mathbf{X}'\mathbf{X}]^{-\mathbf{1}} \\ &= [\mathbf{X}'\mathbf{X}]^{-\mathbf{1}}\mathbf{X}'\mathbf{I}_n\sigma_u^2\mathbf{X}[\mathbf{X}'\mathbf{X}]^{-\mathbf{1}} \\ &= [\mathbf{X}'\mathbf{X}]^{-\mathbf{1}}\sigma_u^2. \end{aligned}$$

If we are using Model B, we can obtain the unconditional variance of $\mathbf{b}$ using the standard decomposition of a variance in a joint distribution:

$$\mathrm{var}(\widehat{\boldsymbol{\beta}}) = E\left[\mathrm{var}(\widehat{\boldsymbol{\beta}} \,|\, \mathbf{X})\right] + \mathrm{var}\left[E(\widehat{\boldsymbol{\beta}} \,|\, \mathbf{X})\right].$$

Now $E(\widehat{\boldsymbol{\beta}} \,|\, \mathbf{X}) = \boldsymbol{\beta}$ for all $\mathbf{X}$, so $\mathrm{var}[E(\widehat{\boldsymbol{\beta}} \,|\, \mathbf{X})] = \mathrm{var}(\boldsymbol{\beta}) = \mathbf{0}$ since $\boldsymbol{\beta}$ is a constant vector, so:

$$\mathrm{var}(\widehat{\boldsymbol{\beta}}) = E\left([\mathbf{X}'\mathbf{X}]^{-\mathbf{1}}\sigma_u^2\right) = \sigma_u^2 E\left([\mathbf{X}'\mathbf{X}]^{-\mathbf{1}}\right)$$

the expectation being taken over the distribution of $\mathbf{X}$.

To estimate $\mathrm{var}(\widehat{\boldsymbol{\beta}})$, we need to estimate σ_u^2. An unbiased estimator is provided by $\widehat{\mathbf{u}}'\widehat{\mathbf{u}}/(n-k)$. For a proof, see Appendix B.

15.9 The Gauss–Markov theorem

We will demonstrate that the OLS estimators are the minimum variance unbiased estimators that are linear in y. For simplicity, we will do this within the framework of Model A, with the analysis conditional on the given data for $\mathbf{X}$. The analysis generalises straightforwardly to Model B, where the explanatory variables are stochastic but drawn from fixed distributions.

Consider the general estimator in this class:

$$\widehat{\boldsymbol{\beta}}^* = \mathbf{Ay}$$

where $\mathbf{A}$ is a k by n matrix. Let:

$$\mathbf{C} = \mathbf{A} - [\mathbf{X}'\mathbf{X}]^{-1}\mathbf{X}'.$$

Then:

$$\begin{aligned} \widehat{\boldsymbol{\beta}}^* &= \left([\mathbf{X}'\mathbf{X}]^{-1}\mathbf{X}' + \mathbf{C}\right)\mathbf{y} \\ &= \left([\mathbf{X}'\mathbf{X}]^{-1}\mathbf{X}' + \mathbf{C}\right)(\mathbf{X}\boldsymbol{\beta} + \mathbf{u}) \\ &= \boldsymbol{\beta} + \mathbf{CX}\boldsymbol{\beta} + [\mathbf{X}'\mathbf{X}]^{-1}\mathbf{X}'\mathbf{u} + \mathbf{Cu}. \end{aligned}$$

Unbiasedness requires:

$$\mathbf{CX} = \mathbf{0}_k$$

where $\mathbf{0}$ is a k by k matrix consisting entirely of zeros. Then, with $E(\widehat{\boldsymbol{\beta}}^*) = \boldsymbol{\beta}$, the variance-covariance matrix of $\widehat{\boldsymbol{\beta}}^*$ is given by:

$$\begin{aligned} E\left[(\widehat{\boldsymbol{\beta}}^* - \boldsymbol{\beta})(\widehat{\boldsymbol{\beta}}^* - \boldsymbol{\beta})'\right] &= E\left[\left([\mathbf{X}'\mathbf{X}]^{-1}\mathbf{X}' + \mathbf{C}\right)\mathbf{uu}'\left([\mathbf{X}'\mathbf{X}]^{-1}\mathbf{X}' + \mathbf{C}\right)'\right] \\ &= \left([\mathbf{X}'\mathbf{X}]^{-1}\mathbf{X}' + \mathbf{C}\right)\mathbf{I}_n\sigma_u^2\left([\mathbf{X}'\mathbf{X}]^{-1}\mathbf{X}' + \mathbf{C}\right)' \\ &= \left([\mathbf{X}'\mathbf{X}]^{-1}\mathbf{X}' + \mathbf{C}\right)\left([\mathbf{X}'\mathbf{X}]^{-1}\mathbf{X}' + \mathbf{C}\right)'\sigma_u^2 \\ &= \left([\mathbf{X}'\mathbf{X}]^{-1} + \mathbf{CC}'\right)'\sigma_u^2. \end{aligned}$$

Now diagonal element i of $\mathbf{CC}'$ is the inner product of row i of C and column i of $\mathbf{C}'$. These are the same, so it is given by:

$$\sum_{s=1}^{k} c_{ik}^2$$

which is positive unless $c_{is} = 0$ for all s. Hence minimising the variances of the estimators of all of the elements of $\boldsymbol{\beta}$ requires $\mathbf{C} = \mathbf{0}$. This implies that OLS provides the minimum variance unbiased estimator.

15.10 Consistency of the OLS regression coefficients

Since:

$$\widehat{\boldsymbol{\beta}} = \boldsymbol{\beta} + [\mathbf{X}'\mathbf{X}]^{-1}\mathbf{X}'\mathbf{u}$$

the probability limit of $\widehat{\boldsymbol{\beta}}$ is given by:

$$\begin{aligned}\text{plim } \widehat{\boldsymbol{\beta}} &= \boldsymbol{\beta} + \text{ plim } [\mathbf{X}'\mathbf{X}]^{-1}\mathbf{X}'\mathbf{u} \\ &= \boldsymbol{\beta} + \text{ plim } \left(\left[\frac{1}{n}\mathbf{X}'\mathbf{X} \right]^{-1} \frac{1}{n}\mathbf{X}'\mathbf{u} \right).\end{aligned}$$

Now, if we are working with cross-sectional data with the explanatory variables drawn from fixed (joint) distributions, it can be shown that:

$$\text{plim } \left[\frac{1}{n}\mathbf{X}'\mathbf{X} \right]^{-1}$$

has a limiting matrix and that:

$$\text{plim } \frac{1}{n}\mathbf{X}'\mathbf{u} = 0.$$

Hence we can decompose:

$$\text{plim } \left(\left[\frac{1}{n}\mathbf{X}'\mathbf{X} \right]^{-1} \frac{1}{n}\mathbf{X}'\mathbf{u} \right) = \text{ plim } \left[\frac{1}{n}\mathbf{X}'\mathbf{X} \right]^{-1} \text{ plim } \frac{1}{n}\mathbf{X}'\mathbf{u} = 0$$

and so plim $\widehat{\boldsymbol{\beta}} = \boldsymbol{\beta}$. Note that this is only an outline of the proof. For a proper proof and a generalisation to less restrictive assumptions, see Greene pp.64–65.

15.11 Frisch–Waugh–Lovell theorem

We will precede the discussion of the Frisch–Waugh–Lovell (FWL) theorem by introducing the residual-maker matrix. We have seen that, when we fit:

$$\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \mathbf{u}$$

using OLS, the residuals are given by:

$$\widehat{\mathbf{u}} = \mathbf{y} - \widehat{\mathbf{y}} = \mathbf{y} - \mathbf{X}\widehat{\boldsymbol{\beta}}.$$

Substituting for $\widehat{\boldsymbol{\beta}}$, we have:

$$\begin{aligned}\widehat{\mathbf{u}} &= \mathbf{y} - \mathbf{X}[\mathbf{X}'\mathbf{X}]^{-1}\mathbf{X}'\mathbf{y} \\ &= \left[\mathbf{I} - \mathbf{X}[\mathbf{X}'\mathbf{X}]^{-1}\mathbf{X}'\right]\mathbf{y} \\ &= \mathbf{M}\mathbf{y}\end{aligned}$$

where:

$$\mathbf{M} = \mathbf{I} - \mathbf{X}[\mathbf{X}'\mathbf{X}]^{-1}\mathbf{X}'.$$

$\mathbf{M}$ is known as the 'residual-maker' matrix because it converts the values of $\mathbf{y}$ into the residuals of $\mathbf{y}$ when regressed on $\mathbf{X}$. Note that $\mathbf{M}$ is symmetric, because $\mathbf{M}' = \mathbf{M}$, and idempotent, meaning that $\mathbf{M}\mathbf{M} = \mathbf{M}$.

Now suppose that we divide the k variables comprising $\mathbf{X}$ into two subsets, the first s and the last $k - s$. (For the present purposes, it makes no difference whether there is or is not an intercept in the model, and if there is one, whether the vector of ones responsible for it is in the first or second subset.) We will partition $\mathbf{X}$ as:

$$\mathbf{X} = [\mathbf{X_1} \quad \mathbf{X_2}]$$

where $\mathbf{X} + \mathbf{1}$ comprises the first s columns and $\mathbf{X_2}$ comprises the last $k - s$, and we will partition $\boldsymbol{\beta}$ similarly, so that the theoretical model may be written:

$$\mathbf{y} = [\mathbf{X_1} \quad \mathbf{X_2}] \begin{bmatrix} \boldsymbol{\beta_1} \\ \boldsymbol{\beta_2} \end{bmatrix} + \mathbf{u}.$$

The FWL theorem states that the OLS estimates of the coefficients in $\boldsymbol{\beta_1}$ are the same as those that would be obtained by the following procedure: regress $\mathbf{y}$ on the variables in $\mathbf{X_2}$ and save the residuals as $\widehat{\mathbf{u}}_\mathbf{y}$. Regress each of the variables in $\mathbf{X_1}$ on $\mathbf{X_2}$ and save the matrix of residuals as $\widehat{\mathbf{u}}_{\mathbf{X_1}}$. If we regress $\widehat{\mathbf{u}}_\mathbf{y}$ on $\widehat{\mathbf{u}}_{\mathbf{X_1}}$, we will obtain the same estimates of the coefficients of $\boldsymbol{\beta_1}$ as we did in the straightforward multiple regression. (Why we might want to do this is another matter. We will come to this later.) Applying the preceding discussion relating to the residual-maker, we have:

$$\widehat{\mathbf{u}}_\mathbf{y} = \mathbf{M_2 y}$$

where:

$$\mathbf{M_2} = \mathbf{I} - \mathbf{X_2}[\mathbf{X_2}'\mathbf{X_2}]^{-1}\mathbf{X_2}'$$

and:

$$\widehat{\mathbf{u}}_{\mathbf{X_1}} = \mathbf{M_2 X_1}.$$

Let the vector of coefficients obtained when we regress $\widehat{\mathbf{u}}_\mathbf{y}$ on $\widehat{\mathbf{u}}_{\mathbf{X_1}}$ be denoted $\widehat{\boldsymbol{\beta}}_\mathbf{1}^*$. Then:

$$\begin{aligned}
\widehat{\boldsymbol{\beta}}_\mathbf{1}^* &= [\widehat{\mathbf{u}}'_{\mathbf{X_1}}\widehat{\mathbf{u}}_{\mathbf{X_1}}]^{-1}\widehat{\mathbf{u}}'_{\mathbf{X_1}}\widehat{\mathbf{u}}_\mathbf{y} \\
&= [\mathbf{X_1}'\mathbf{M_2}'\mathbf{M_2}\mathbf{X_1}]^{-1}\mathbf{X_1}'\mathbf{M_2}'\mathbf{M_2}\mathbf{y} \\
&= [\mathbf{X_1}'\mathbf{M_2}\mathbf{X_1}]^{-1}\mathbf{X_1}\mathbf{M_2}\mathbf{y}.
\end{aligned}$$

(Remember that $\mathbf{M_2}$ is symmetric and idempotent.) Now we will derive an expression for $\widehat{\boldsymbol{\beta}}_\mathbf{1}$ from the orthodox multiple regression of $\mathbf{y}$ on $\mathbf{X}$. For this purpose, it is easiest to start with the normal equations:

$$\mathbf{X}'\mathbf{X}\widehat{\boldsymbol{\beta}} - \mathbf{X}'\mathbf{y} = \mathbf{0}.$$

We partition $\widehat{\boldsymbol{\beta}}$ as $\begin{bmatrix} \widehat{\boldsymbol{\beta}}_\mathbf{1} \\ \widehat{\boldsymbol{\beta}}_\mathbf{2} \end{bmatrix}$. $\mathbf{X}'$ is $\begin{bmatrix} \mathbf{X_1}' \\ \mathbf{X_2}' \end{bmatrix}$, and we have the following:

$$\begin{aligned}
\mathbf{X}'\mathbf{X} &= \begin{bmatrix} \mathbf{X_1}'\mathbf{X_1} & \mathbf{X_1}'\mathbf{X_2} \\ \mathbf{X_2}'\mathbf{X_1} & \mathbf{X_2}'\mathbf{X_2} \end{bmatrix} \\
\mathbf{X}'\mathbf{X}\widehat{\boldsymbol{\beta}} &= \begin{bmatrix} \mathbf{X_1}'\mathbf{X_1} & \mathbf{X_1}'\mathbf{X_2} \\ \mathbf{X_2}'\mathbf{X_1} & \mathbf{X_2}'\mathbf{X_2} \end{bmatrix} \begin{bmatrix} \widehat{\boldsymbol{\beta}}_\mathbf{1} \\ \widehat{\boldsymbol{\beta}}_\mathbf{2} \end{bmatrix} = \begin{bmatrix} \mathbf{X_1}'\mathbf{X_1}\widehat{\boldsymbol{\beta}}_\mathbf{1} + \mathbf{X_1}'\mathbf{X_2}\widehat{\boldsymbol{\beta}}_\mathbf{2} \\ \mathbf{X_2}'\mathbf{X_1}\widehat{\boldsymbol{\beta}}_\mathbf{1} + \mathbf{X_2}'\mathbf{X_2}\widehat{\boldsymbol{\beta}}_\mathbf{2} \end{bmatrix} \\
\mathbf{X}'\mathbf{y} &= \begin{bmatrix} \mathbf{X_1}'\mathbf{y} \\ \mathbf{X_2}'\mathbf{y} \end{bmatrix}.
\end{aligned}$$

Hence, splitting the normal equations into their upper and lower components, we have:

$$\mathbf{X_1}'\mathbf{X_1}\widehat{\boldsymbol{\beta}}_\mathbf{1} + \mathbf{X_1}'\mathbf{X_2}\widehat{\boldsymbol{\beta}}_\mathbf{2} - \mathbf{X_1}'\mathbf{y} = \mathbf{0}$$

and:

$$\mathbf{X_2}'\mathbf{X_1}\widehat{\boldsymbol{\beta}}_\mathbf{1} + \mathbf{X_2}'\mathbf{X_2}\widehat{\boldsymbol{\beta}}_\mathbf{2} - \mathbf{X_2}'\mathbf{y} = \mathbf{0}.$$

From the second we obtain:

$$\mathbf{X_2}'\mathbf{X_2}\widehat{\boldsymbol{\beta}}_\mathbf{2} = \mathbf{X_2}'\mathbf{y} - \mathbf{X_2}'\mathbf{X_1}\widehat{\boldsymbol{\beta}}_\mathbf{1}$$

and so:

$$\widehat{\boldsymbol{\beta}}_\mathbf{2} = [\mathbf{X_2}'\mathbf{X_2}]^{-1}[\mathbf{X_2}'\mathbf{y} - \mathbf{X_2}'\mathbf{X_1}\widehat{\boldsymbol{\beta}}_\mathbf{1}].$$

Substituting for $\widehat{\boldsymbol{\beta}}_\mathbf{2}$ in the first normal equation:

$$\mathbf{X_1}'\mathbf{X_1}\widehat{\boldsymbol{\beta}}_\mathbf{1} + \mathbf{X_1}'\mathbf{X_2}[\mathbf{X_2}'\mathbf{X_2}]^{-1}[\mathbf{X_2}'\mathbf{y} - \mathbf{X_2}'\mathbf{X_1}\widehat{\boldsymbol{\beta}}_\mathbf{1}] - \mathbf{X_1}'\mathbf{y} = \mathbf{0}.$$

Hence:

$$\mathbf{X_1}'\mathbf{X_1}\widehat{\boldsymbol{\beta}}_\mathbf{1} - \mathbf{X_1}'\mathbf{X_2}[\mathbf{X_2}'\mathbf{X_2}]^{-1}\mathbf{X_2}'\mathbf{X_1}\widehat{\boldsymbol{\beta}}_\mathbf{1} = \mathbf{X_1}'\mathbf{y} - \mathbf{X_1}'\mathbf{X_2}[\mathbf{X_2}'\mathbf{X_2}]^{-1}\mathbf{X_2}'\mathbf{y}$$

and so:

$$\mathbf{X_1}'\left[\mathbf{I} - \mathbf{X_2}[\mathbf{X_2}'\mathbf{X_2}]^{-1}\mathbf{X}_\mathbf{2}'\right]\mathbf{X_1}\widehat{\boldsymbol{\beta}}_\mathbf{1} = \mathbf{X_1}'\left[\mathbf{I} - \mathbf{X_2}[\mathbf{X_2}'\mathbf{X_2}]^{-1}\mathbf{X_2}'\right]\mathbf{y}.$$

Hence:

$$\mathbf{X_1}'\mathbf{M_2}\mathbf{X_1}\widehat{\boldsymbol{\beta}}_\mathbf{1} = \mathbf{X_1}'\mathbf{M_2}\mathbf{y}$$

and:

$$\widehat{\boldsymbol{\beta}}_\mathbf{1} = [\mathbf{X_1}'\mathbf{M_2}\mathbf{X_1}]^{-1}\mathbf{X_1}'\mathbf{M_2}\mathbf{y} = \widehat{\boldsymbol{\beta}}_\mathbf{1}^*.$$

Why should we be interested in this result? The original purpose remains instructive. In early days, econometricians working with time series data, especially macroeconomic data, were concerned to avoid the problem of spurious regressions. If two variables both possessed a time trend, it was very likely that 'significant' results would be obtained when one was regressed on the other, even if there were no genuine relationship between them. To avoid this, it became the custom to detrend the variables before using them by regressing each on a time trend and then working with the residuals from these regressions. Frisch and Waugh (1933) pointed out that this was an unnecessarily laborious procedure. The same results would be obtained using the original data, if a time trend was added as an explanatory variable.

Generalising, and this was the contribution of Lovell, we can infer that, in a multiple regression model, the estimator of the coefficient of any one variable is not influenced by any of the other variables, irrespective of whether they are or are not correlated with the variable in question. The result is so general and basic that it should be understood by all students of econometrics. Of course, it fits neatly with the fact that the multiple regression coefficients are unbiased, irrespective of any correlations among the variables.

A second reason for being interested in the result is that it allows one to depict graphically the relationship between the observations on the dependent variable and those on any single explanatory variable, controlling for the influence of all the other explanatory variables. This is described in the textbook in Section 3.2.

Exercise

9. Using the FKL theorem, demonstrate that, if a multiple regression model contains an intercept, the same slope coefficients could be obtained by subtracting the means of all of the variables from the data for them and then regressing the model omitting an intercept.

15.12 Exact multicollinearity

We will assume, as is to be expected, that k, the number of explanatory variables (including the unit vector, if there is one), is less than n, the number of observations. If the explanatory variables are independent, the $\mathbf{X}$ matrix will have rank k and likewise $\mathbf{X}'\mathbf{X}$ will have rank k and will possess an inverse. However, if one or more linear relationships exist among the explanatory variables, the model will be subject to exact multicollinearity. The rank of $\mathbf{X}$, and hence of $\mathbf{X}'\mathbf{X}$, will then be less than k and $\mathbf{X}'\mathbf{X}$ will not possess an inverse.

Suppose we write $\mathbf{X}$ as a set of column vectors $\mathbf{x_j}$, each corresponding to the observations on one of the explanatory variables:

$$\mathbf{X} = [\mathbf{x}_1 \quad \cdots \quad \mathbf{x}_j \quad \cdots \quad \mathbf{x}_k]$$

where:

$$\mathbf{x}_j = \begin{bmatrix} x_{1j} \\ \vdots \\ x_{ij} \\ \vdots \\ x_{nj} \end{bmatrix}.$$

Then:

$$\mathbf{X}' = \begin{bmatrix} \mathbf{x}_1' \\ \vdots \\ \mathbf{x}_j' \\ \vdots \\ \mathbf{x}_k' \end{bmatrix}$$

and the normal equations:

$$\mathbf{X}'\mathbf{X}\widehat{\boldsymbol{\beta}} - \mathbf{X}'\mathbf{y} = \mathbf{0}$$

may be written:

$$\begin{bmatrix} \mathbf{x}_1'\mathbf{X}\widehat{\boldsymbol{\beta}} \\ \vdots \\ \mathbf{x}_j'\mathbf{X}\widehat{\boldsymbol{\beta}} \\ \vdots \\ \mathbf{x}_k'\mathbf{X}\widehat{\boldsymbol{\beta}} \end{bmatrix} - \begin{bmatrix} \mathbf{x}_1'\mathbf{y} \\ \vdots \\ \mathbf{x}_j'\mathbf{y} \\ \vdots \\ \mathbf{x}_k'\mathbf{y} \end{bmatrix} = \mathbf{0}.$$

Now suppose that one of the explanatory variables, say the last, can be written as a linear combination of the others:

$$\mathbf{x}_k = \sum_{i=1}^{k-1} \lambda_i \mathbf{x}_i.$$

Then the last of the normal equations is that linear combination of the other $k-1$. Hence it is redundant, and we are left with a set of $k-1$ equations for determining the k unknown regression coefficients. The problem is not that there is no solution. It is the opposite: there are too many possible solutions, in fact an infinite number. One coefficient could be chosen arbitrarily, and then the normal equations would provide a solution for the other $k-1$. Some regression applications deal with this situation by dropping one of the variables from the regression specification, effectively assigning a value of zero to its coefficient.

Exact multicollinearity is unusual because it mostly occurs as a consequence of a logical error in the specification of the regression model. The classic example is the dummy variable trap. This occurs when a set of dummy variables D_j, $j = 1, \ldots, s$ are defined for a qualitative characteristic that has s categories. If all s dummy variables are included in the specification, in observation i we will have:

$$\sum_{j=1}^{s} D_{ij} = 1$$

since one of the dummy variables must be equal to 1 and the rest are all zero. But this is the (unchanging) value of the unit vector. Hence the sum of the dummy variables is equal to the unit vector. As a consequence, if the unit vector and all of the dummy variables are simultaneously included in the specification, there will be exact multicollinearity. The solution is to drop one of the dummy variables, making it the reference category, or, alternatively, to drop the intercept (and hence unit vector), effectively making the dummy variable coefficient for each category the intercept for that category. As explained in the textbook, it is illogical to wish to include a complete set of dummy variables as well as the intercept, for then no interpretation can be given to the dummy variable coefficients.

15.13 Estimation of a linear combination of regression coefficients

Suppose that one wishes to estimate a linear combination of the regression parameters:

$$\sum_{j=1}^{k} \lambda_j \beta_j.$$

In matrix notation, we may write this as $\boldsymbol{\lambda}'\boldsymbol{\beta}$ where:

$$\lambda = \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_j \\ \vdots \\ \lambda_k \end{bmatrix}.$$

The corresponding linear combination of the regression coefficients, $\boldsymbol{\lambda}'\widehat{\boldsymbol{\beta}}$, provides an unbiased estimator of $\boldsymbol{\lambda}'\boldsymbol{\beta}$. However, we will often be interested also in its standard error, and this is not quite so straightforward. We obtain it via the variance:

$$\mathrm{var}(\boldsymbol{\lambda}'\widehat{\boldsymbol{\beta}}) = E\left[(\boldsymbol{\lambda}'\widehat{\boldsymbol{\beta}} - E(\boldsymbol{\lambda}'\widehat{\boldsymbol{\beta}}))^2\right] = E\left[(\boldsymbol{\lambda}'\widehat{\boldsymbol{\beta}} - \boldsymbol{\lambda}'\boldsymbol{\beta})^2\right].$$

Since $(\boldsymbol{\lambda}'\widehat{\boldsymbol{\beta}} - \boldsymbol{\lambda}'\boldsymbol{\beta})$ is a scalar, it is equal to its own transpose, and so $(\boldsymbol{\lambda}'\widehat{\boldsymbol{\beta}} - \boldsymbol{\lambda}'\boldsymbol{\beta})^2$ may be written:

$$\begin{aligned} \mathrm{var}(\boldsymbol{\lambda}'\widehat{\boldsymbol{\beta}}) &= E\left[\boldsymbol{\lambda}'(\widehat{\boldsymbol{\beta}} - \boldsymbol{\beta})(\widehat{\boldsymbol{\beta}} - \boldsymbol{\beta})'\boldsymbol{\lambda}\right] \\ &= \boldsymbol{\lambda}'E\left[(\widehat{\boldsymbol{\beta}} - \boldsymbol{\beta})(\widehat{\boldsymbol{\beta}} - \boldsymbol{\beta})'\right]\boldsymbol{\lambda} \\ &= \boldsymbol{\lambda}'[\mathbf{X}'\mathbf{X}]^{-\mathbf{1}}\boldsymbol{\lambda}\sigma_u^2. \end{aligned}$$

The square root of this expression provides the standard error of $\boldsymbol{\lambda}'\widehat{\boldsymbol{\beta}}$ after we have replaced σ_u^2 by its estimator $\widehat{\mathbf{u}}'\widehat{\mathbf{u}}/(n-k)$ in the usual way.

15.14 Testing linear restrictions

An obvious application of the foregoing is its use in testing a linear restriction. Suppose that one has a hypothetical restriction:

$$\sum_{j=1}^{k} \lambda_j \beta_j = \lambda_0.$$

We can perform a t test of the restriction using the t statistic:

$$t = \frac{\boldsymbol{\lambda}'\widehat{\boldsymbol{\beta}} - \lambda_0}{\mathrm{s.e.}(\boldsymbol{\lambda}'\widehat{\boldsymbol{\beta}})}$$

where the standard error is obtained via the variance-covariance matrix as just described. Alternatively, we could reparameterise the regression specification so that one of the coefficients is $\boldsymbol{\lambda}'\boldsymbol{\beta}$. In practice, this is often more convenient since it avoids having to work with the variancecovariance matrix. If there are multiple restrictions that should be tested simultaneously, the appropriate procedure is an F test comparing RSS for the unrestricted and fully restricted models.

15.15 Weighted least squares and heteroskedasticity

Suppose that the regression model:

$$\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \mathbf{u}$$

satisfies the usual regression model assumptions and suppose that we premultiply the elements of the model by the n by n matrix $\mathbf{A}$ whose diagonal elements are A_{ii},

$i = 1, \ldots, n$, and whose off-diagonal elements are all zero:

$$\mathbf{A} = \begin{bmatrix} A_{11} & \cdots & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & A_{ii} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & \cdots & A_{nn} \end{bmatrix}.$$

The model becomes:

$$\mathbf{Ay} = \mathbf{AX}\boldsymbol{\beta} + \mathbf{Au}.$$

If we fit it using least squares, the point estimates of the coefficients are given by:

$$\widehat{\boldsymbol{\beta}}^{\text{WLS}} = [\mathbf{X}'\mathbf{A}'\mathbf{AX}]^{-1}\mathbf{X}'\mathbf{A}'\mathbf{Ay}$$

(WLS standing for weighted least squares). This is unbiased but heteroskedastic because the disturbance term in observation i is $A_{ii}u_i$ and has variance $A_{ii}^2\sigma_u^2$.

Now suppose that the disturbance term in the original model was heteroskedastic, with variance $\sigma_{u_i}^2$ in observation i. If we define the matrix $\mathbf{A}$ so that the diagonal elements are determined by:

$$A_{ii} = \frac{1}{\sqrt{\sigma_{u_i}^2}}$$

the corresponding variance in the weighted regression will be 1 for all observations and the WLS model will be homoskedastic. The WLS estimator is then:

$$\widehat{\boldsymbol{\beta}}^{\text{WLS}} = [\mathbf{X}'\mathbf{CX}]^{-1}\mathbf{X}'\mathbf{Cy}$$

where:

$$\mathbf{C} = \mathbf{A}'\mathbf{A} = \begin{bmatrix} \frac{1}{\sigma_{u_1}^2} & \cdots & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & \frac{1}{\sigma_{u_i}^2} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & \cdots & \frac{1}{\sigma_{u_n}^2} \end{bmatrix}.$$

The variance-covariance matrix of the WLS coefficients, conditional on the data for $\mathbf{X}$, is:

$$\begin{aligned}
\text{var}(\widehat{\boldsymbol{\beta}}^{\text{WLS}}) &= E\left[(\widehat{\boldsymbol{\beta}}^{\text{WLS}} - E(\widehat{\boldsymbol{\beta}}^{\text{WLS}}))(\widehat{\boldsymbol{\beta}}^{\text{WLS}} - E(\widehat{\boldsymbol{\beta}}^{\text{WLS}}))'\right] \\
&= E\left[(\widehat{\boldsymbol{\beta}}^{\text{WLS}} - \boldsymbol{\beta}))(\widehat{\boldsymbol{\beta}}^{\text{WLS}} - \boldsymbol{\beta}))'\right] \\
&= E\left[([\mathbf{X}'\mathbf{A}'\mathbf{AX}]^{-1}\mathbf{X}'\mathbf{A}'\mathbf{Au})([\mathbf{X}'\mathbf{A}'\mathbf{AX}]^{-1}\mathbf{X}'\mathbf{A}'\mathbf{Au})'\right] \\
&= E\left[[\mathbf{X}'\mathbf{A}'\mathbf{AX}]^{-1}\mathbf{X}'\mathbf{A}'\mathbf{Auu}'\mathbf{A}'\mathbf{AX}[\mathbf{X}'\mathbf{A}'\mathbf{AX}]^{-1}\right] \\
&= \left[[\mathbf{X}'\mathbf{A}'\mathbf{AX}]^{-1}\mathbf{X}'\mathbf{A}'\mathbf{A}E(\mathbf{uu}')\mathbf{A}'\mathbf{AX}[\mathbf{X}'\mathbf{A}'\mathbf{AX}]^{-1}\right] \\
&= [\mathbf{X}'\mathbf{A}'\mathbf{AX}]^{-1}\mathbf{X}'\mathbf{A}'\mathbf{AX}[\mathbf{X}'\mathbf{A}'\mathbf{AX}]^{-1} \\
&= [\mathbf{X}'\mathbf{CX}]^{-1}\mathbf{X}'\mathbf{CX}[\mathbf{X}'\mathbf{CX}]^{-1}\sigma_u^2 \\
&= [\mathbf{X}'\mathbf{CX}]^{-1}\sigma_u^2
\end{aligned}$$

since $\mathbf{A}$ has been defined so that:

$$\mathbf{A}E(\mathbf{uu}')\mathbf{A}' = \mathbf{I}.$$

Of course, in practice we seldom know $\sigma_{u_i}^2$, but if it is appropriate to hypothesise that the standard deviation is proportional to some measurable variable Z_i, then the WLS regression will be homoskedastic if we define $\mathbf{A}$ to have diagonal element i equal to the reciprocal of Z_i.

15.16 IV estimators and TSLS

Suppose that we wish to fit the model:

$$\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \mathbf{u}$$

where one or more of the explanatory variables is not distributed independently of the disturbance term. For convenience, we will describe such variables as 'endogenous', irrespective of the reason for the violation of the independence requirement. Given a sufficient number of suitable instruments, we may consider using the IV estimator:

$$\widehat{\boldsymbol{\beta}}^{\mathrm{IV}} = [\mathbf{W}'\mathbf{X}]^{-\mathbf{1}}\mathbf{W}'\mathbf{y} \qquad (4)$$

where $\mathbf{W}$ is the matrix of instruments. In general $\mathbf{W}$ will be a mixture of (1) those original explanatory variables that are distributed independently of the disturbance term (these are then described as acting as instruments for themselves), and (2) new variables that are correlated with the endogenous variables but distributed independently of the disturbance term. If we substitute for $\mathbf{y}$:

$$\widehat{\boldsymbol{\beta}}^{\mathrm{IV}} = [\mathbf{W}'\mathbf{X}]^{-\mathbf{1}}\mathbf{W}'(\mathbf{X}\boldsymbol{\beta} + \mathbf{u}) = \boldsymbol{\beta} + [\mathbf{W}'\mathbf{X}]^{-\mathbf{1}}\mathbf{W}'\mathbf{u}.$$

We cannot obtain a closed-form expression for the expectation of the error term, so instead we take plims:

$$\operatorname{plim} \widehat{\boldsymbol{\beta}}^{\mathrm{IV}} = \boldsymbol{\beta} + \operatorname{plim} \left(\left[\frac{1}{n}\mathbf{W}'\mathbf{X} \right]^{-1} \frac{1}{n}\mathbf{W}'\mathbf{u} \right).$$

Now if we are using cross-sectional data, it is usually reasonable to suppose that:

$$\operatorname{plim} \left(\left[\frac{1}{n}\mathbf{W}'\mathbf{X} \right]^{-1} \right) \quad \text{and} \quad \operatorname{plim} \left(\frac{1}{n}\mathbf{W}'\mathbf{u} \right)$$

both exist, in which case we can decompose the plim of the error term:

$$\operatorname{plim} \widehat{\boldsymbol{\beta}}^{\mathrm{IV}} = \boldsymbol{\beta} + \operatorname{plim} \left(\left[\frac{1}{n}\mathbf{W}'\mathbf{X} \right]^{-1} \right) \operatorname{plim} \left(\frac{1}{n}\mathbf{W}'\mathbf{u} \right).$$

Further, if the matrix of instruments has been correctly chosen, it can be shown that:

$$\operatorname{plim} \left(\frac{1}{n}\mathbf{W}'\mathbf{u} \right) = 0$$

and hence the IV estimator is consistent.

It is not possible to derive a closed-form expression for the variance of the IV estimator in finite samples. The best we can do is to invoke a central limit theorem that gives the limiting distribution asymptotically and work backwards from that, as an approximation, for finite samples. A central limit theorem can be used to establish that:

$$\sqrt{n}(\widehat{\boldsymbol{\beta}}^{\mathrm{IV}} - \boldsymbol{\beta}) \xrightarrow{d} N\left(\mathbf{0}, \left\{\sigma_u^2 \operatorname{plim} \left[\frac{1}{n}\mathbf{W}'\mathbf{X}\right]^{-1} \operatorname{plim} \left[\frac{1}{n}\mathbf{W}'\mathbf{W}\right] \operatorname{plim} \left[\frac{1}{n}\mathbf{X}'\mathbf{W}\right]^{-1}\right\}\right).$$

From this, we may infer, that as an approximation, for sufficiently large samples:

$$\widehat{\boldsymbol{\beta}}^{\mathrm{IV}} \sim N\left(\boldsymbol{\beta}, \left\{\frac{\sigma_u^2}{n} \operatorname{plim} \left[\frac{1}{n}\mathbf{W}'\mathbf{X}\right]^{-1} \operatorname{plim} \left[\frac{1}{n}\mathbf{W}'\mathbf{W}\right] \operatorname{plim} \left[\frac{1}{n}\mathbf{X}'\mathbf{W}\right]^{-1}\right\}\right). \quad (5)$$

We have implicitly assumed so far that $\mathbf{W}$ has the same dimensions as $\mathbf{X}$ and hence that $\mathbf{W}'\mathbf{X}$ is a square k by k matrix. However, the model may be overidentified, with the number of columns of $\mathbf{W}$ exceeding k. In that case, the appropriate procedure is two-stage least squares. One regresses each of the variables in $\mathbf{X}$ on $\mathbf{W}$ and saves the fitted values. The matrix of fitted values is then used as the instrument matrix in place of $\mathbf{W}$.

Exercises

10. Using (4) and (5), demonstrate that, for the simple regression model:

$$Y_i = \beta_1 = \beta_2 X_i + u_i$$

with Z acting as an instrument for X (and the unit vector acting as an instrument for itself):

$$\begin{aligned}\widehat{\beta}_1^{\mathrm{IV}} &= \overline{Y} - \widehat{\beta}_2^{\mathrm{IV}}\overline{X} \\ \widehat{\beta}_2^{\mathrm{IV}} &= \frac{\sum\left(Z_i - \overline{Z}\right)\left(Y_i - \overline{Y}\right)}{\sum\left(Z_i - \overline{Z}\right)\left(X_i - \overline{X}\right)}\end{aligned}$$

and, as an approximation:

$$\operatorname{var}(\widehat{\beta}_2^{\mathrm{IV}}) = \frac{\sigma_u^2}{\sum\left(X_i - \overline{X}\right)^2} \times \frac{1}{r_{XZ}^2}$$

where Z is the instrument for X and r_{XZ} is the correlation between X and Z.

11. Demonstrate that any variable acting as an instrument for itself is unaffected by the first stage of two-stage least squares.

12. Demonstrate that TSLS is equivalent to IV if the equation is exactly identified.

15.17 Generalised least squares

The final topic in this introductory primer is generalised least squares and its application to autocorrelation (autocorrelated disturbance terms). One of the basic regression model assumptions is that the disturbance terms in the observations in a sample are distributed identically and independently of each other. If this is the case, the variance-covariance matrix of the disturbance terms is the identity matrix of order n, multiplied by σ_u^2. We have encountered one type of violation, heteroskedasticity, where the values of the disturbance term are independent but not identical. The consequence was that the off-diagonal elements of the variance-covariance matrix remained zero, but the diagonal elements differed. Mathematically, autocorrelation is complementary. It occurs when the values of the disturbance term are not independent and as a consequence some, or all, of the off-diagonal elements are non-zero. It is usual in initial treatments to retain the assumption of identical distributions, so that the diagonal elements of the variance-covariance matrix are the same. Of course, in principle one could have both types of violation at the same time.

In abstract, it is conventional to denote the variance-covariance matrix of the disturbance term $\mathbf{\Omega}\sigma_u^2$, where Ω is the Greek upper case omega, writing the model:

$$\mathbf{y} = \mathbf{X}\boldsymbol{\beta} \quad \text{with} \quad E(\mathbf{uu}') = \mathbf{\Omega}\sigma_u^2. \tag{6}$$

If the values of the disturbance term are iid, $\mathbf{\Omega} = \mathbf{I}$. If they are not iid, OLS is in general inefficient and the standard errors are estimated incorrectly. Then, it is desirable to transform the model so that the transformed disturbance terms are iid. One possible way of doing this is to multiply through by some suitably chosen matrix $\mathbf{P}$, fitting:

$$\mathbf{Py} = \mathbf{PX}\boldsymbol{\beta} + \mathbf{Pu}$$

choosing $\mathbf{P}$ so that $E(\mathbf{Puu}'\mathbf{P}') = \mathbf{I}\alpha$ where α is some scalar. The solution for heteroskedasticity was a simple example of this type. We had:

$$\mathbf{\Omega} = \begin{bmatrix} \sigma_{u_1}^2 & \cdots & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & \sigma_{u_i}^2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & \cdots & \sigma_{u_n}^2 \end{bmatrix}$$

and the appropriate choice of $\mathbf{P}$ was:

$$\mathbf{P} = \begin{bmatrix} \sqrt{\frac{1}{\sigma_{u_1}^2}} & \cdots & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & \sqrt{\frac{1}{\sigma_{u_i}^2}} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & \cdots & \sqrt{\frac{1}{\sigma_{u_n}^2}} \end{bmatrix}.$$

In the case of heteroskedasticity, the choice of $\mathbf{P}$ is obvious, provided, of course, that one knows the values of the diagonal elements of $\mathbf{\Omega}$. The more general theory requires an understanding of eigenvalues and eigenvectors that will be assumed. $\mathbf{\Omega}$ is a symmetric matrix since $\text{cov}(u_i, u_j)$ is the same as $\text{cov}(u_j, u_i)$. Hence all its eigenvalues are real. Let $\mathbf{\Lambda}$ be the diagonal matrix with the eigenvalues as the diagonal elements. Then there exists a matrix of eigenvectors, C, such that:

$$\mathbf{C}'\mathbf{\Omega}\mathbf{C} = \mathbf{\Lambda}. \tag{7}$$

$\mathbf{C}$ has the properties that $\mathbf{CC}' = \mathbf{I}$ and $\mathbf{C}' = \mathbf{C}^{-1}$. Since Λ is a diagonal matrix, if its eigenvalues are all positive (which means that it is what is known as a 'positive definite' matrix), it can be factored as $\mathbf{\Lambda} = \mathbf{\Lambda}^{1/2}\mathbf{\Lambda}^{1/2}$ where $\mathbf{\Lambda}^{1/2}$ is a diagonal matrix whose diagonal elements are the square roots of the eigenvalues. It follows that the inverse of $\mathbf{\Lambda}$ can be factored as $\mathbf{\Lambda}^{-1} = \mathbf{\Lambda}^{-1/2}\mathbf{\Lambda}^{-1/2}$. Then, in view of (7):

$$\mathbf{\Lambda}^{-1/2}[\mathbf{C}'\mathbf{\Omega}\mathbf{C}]\mathbf{\Lambda}^{-1/2} = \mathbf{\Lambda}^{-1/2}\mathbf{\Lambda}\mathbf{\Lambda}^{-1/2} = \mathbf{\Lambda}^{-1/2}\mathbf{\Lambda}^{1/2}\mathbf{\Lambda}^{1/2}\mathbf{\Lambda}^{-1/2} = \mathbf{I}. \tag{8}$$

This, if we define $\mathbf{P} = \mathbf{\Lambda}^{-1/2}\mathbf{C}'$, (8) becomes:

$$\mathbf{P}\mathbf{\Omega}\mathbf{P}' = \mathbf{I}.$$

As a consequence, if we premultiply (6) through by $\mathbf{P}$, we have:

$$\mathbf{Py} = \mathbf{PX}\boldsymbol{\beta} + \mathbf{Pu}$$

or:

$$\mathbf{y}^* = \mathbf{X}^*\boldsymbol{\beta} + \mathbf{u}^*$$

where $\mathbf{y}^* = \mathbf{Py}$, $\mathbf{X}^* = \mathbf{PX}$, and $\mathbf{u}^* = \mathbf{Pu}$, and $E(\mathbf{u}^*\mathbf{u}^{*\prime}) = \mathbf{I}\boldsymbol{\sigma}_u^2$. An OLS regression of $\mathbf{y}^*$ on $\mathbf{X}^*$ will therefore satisfy the usual regression model assumptions and the estimator of $\boldsymbol{\beta}$ will have the usual properties. Of course, the approach usually requires the estimation of $\mathbf{\Omega}$, $\mathbf{\Omega}$ being positive definite, and there being no problems in extracting the eigenvalues and determining the eigenvectors.

Exercise

13. Suppose that the disturbance term in a simple regression model (with an intercept) is subject to AR(1) autocorrelation with $|\rho| < 1$, and suppose that the sample consists of just two observations. Determine the variance-covariance matrix of the disturbance term, find its eigenvalues, and determine its eigenvectors. Hence determine $\mathbf{P}$ and state the transformed model. Verify that the disturbance term in the transformed model is iid.

15.18 Appendix A: Derivation of the normal equations

We have seen that RSS is given by:

$$RSS = \mathbf{y}'\mathbf{y} - 2\mathbf{y}'\mathbf{X}\widehat{\boldsymbol{\beta}} + \widehat{\boldsymbol{\beta}}'\mathbf{X}'\mathbf{X}\widehat{\boldsymbol{\beta}}. \tag{A.1}$$

The normal equations are:

$$\frac{\partial RSS}{\partial \widehat{\beta}_j} = 0 \tag{A.2}$$

for $j = 1, \ldots, k$. We will show that they can be written:

$$\mathbf{X}'\mathbf{X}\widehat{\boldsymbol{\beta}} - \mathbf{X}'\mathbf{y} = \mathbf{0}.$$

The proof is mathematically unchallenging but tedious because one has to keep careful track of the dimensions of all of the elements in the equations. As far as I know, it is of no intrinsic interest and once one has seen it there should never be any reason to look at it again.

First note that the term $\mathbf{y}'\mathbf{y}$ in (A.1) is not a function of any of the b_j and disappears in (A.2). Accordingly we will restrict our attention to the other two terms on the right side of (A.1). Suppose that we write the $\mathbf{X}$ matrix as a set of column vectors:

$$\mathbf{X} = [\mathbf{x}_1 \quad \cdots \quad \mathbf{x}_j \quad \cdots \quad \mathbf{x}_k] \tag{A.3}$$

where:

$$\mathbf{x}_j = \begin{bmatrix} X_{1j} \\ \vdots \\ X_{ij} \\ \vdots \\ X_{nj} \end{bmatrix}.$$

Then:

$$\mathbf{y}'\mathbf{X}\widehat{\boldsymbol{\beta}} = [\mathbf{y}'\mathbf{x}_1 \quad \cdots \quad \mathbf{y}'\mathbf{x}_j \quad \cdots \quad \mathbf{y}'\mathbf{x}_k] \begin{bmatrix} \widehat{\beta}_1 \\ \vdots \\ \widehat{\beta}_j \\ \vdots \\ \widehat{\beta}_k \end{bmatrix} = [\mathbf{y}'\mathbf{x}_1\widehat{\beta}_1 + \cdots + \mathbf{y}'\mathbf{x}_j\widehat{\beta}_j + \cdots + \mathbf{y}'\mathbf{x}_k\widehat{\beta}_k].$$

Hence:

$$\frac{\partial \mathbf{y}'\mathbf{X}\widehat{\boldsymbol{\beta}}}{\partial \widehat{\beta}_j} = \mathbf{y}'\mathbf{x}_j.$$

We now consider the $\widehat{\boldsymbol{\beta}}'\mathbf{X}'\mathbf{X}\widehat{\boldsymbol{\beta}}$ term. Using (A.3):

$$\begin{aligned} \widehat{\boldsymbol{\beta}}'\mathbf{X}'\mathbf{X}\widehat{\boldsymbol{\beta}} &= [\mathbf{x}_1\widehat{\beta}_1 + \cdots + \mathbf{x}_j\widehat{\beta}_j + \cdots + \mathbf{x}_k\widehat{\beta}_k]'[\mathbf{x}_1\widehat{\beta}_1 + \cdots + \mathbf{x}_j\widehat{\beta}_j + \cdots + \mathbf{x}_k\widehat{\beta}_k] \\ &= \sum_{p=1}^{k}\sum_{q=1}^{k} \widehat{\beta}_p\widehat{\beta}_q\mathbf{x}_p'\mathbf{x}_q. \end{aligned}$$

The subset of terms including $\widehat{\beta}_j$ is:

$$\sum_{q=1}^{k} \widehat{\beta}_j\widehat{\beta}_q\mathbf{x}_j'\mathbf{x}_q + \sum_{p=1}^{k} \widehat{\beta}_p\widehat{\beta}_j\mathbf{x}_p'\mathbf{x}_j.$$

Hence:

$$\frac{\partial \widehat{\boldsymbol{\beta}}'\mathbf{X}'\mathbf{X}\widehat{\boldsymbol{\beta}}}{\partial \widehat{\beta}_j} = \sum_{q=1}^{k} \widehat{\beta}_q\mathbf{x}_j'\mathbf{x}_q + \sum_{p=1}^{k} \widehat{\beta}_p\mathbf{x}_p'\mathbf{x}_j = 2\sum_{p=1}^{k} \widehat{\beta}_p\mathbf{x}_p'\mathbf{x}_j.$$

Putting these results together:

$$\frac{\partial RSS}{\partial \widehat{\beta}_j} = \frac{\partial[\mathbf{y}'\mathbf{y} - 2\mathbf{y}'\mathbf{X}\widehat{\boldsymbol{\beta}} + \widehat{\boldsymbol{\beta}}'\mathbf{X}'\mathbf{X}\widehat{\boldsymbol{\beta}}]}{\partial \widehat{\beta}_j} = -2\mathbf{y}'\mathbf{x}_j + 2\sum_{p=1}^{k} \widehat{\beta}_p \mathbf{x}_p'\mathbf{x}_j.$$

Hence the normal equation $\partial RSS/\partial\widehat{\beta}_j = 0$ is:

$$\sum_{p=1}^{k} \widehat{\beta}_p \mathbf{x}_j'\mathbf{x}_p = \mathbf{x}_j'\mathbf{y}.$$

(Note that $\mathbf{x}_p'\mathbf{x}_j = \mathbf{x}_j'\mathbf{x}_p$ and $\mathbf{y}'\mathbf{x}_j = \mathbf{x}_j'\mathbf{y}$) since they are scalars.) Hence:

$$\mathbf{x}_j' \left[\sum_{p=1}^{k} \widehat{\beta}_p \mathbf{x}_p \right] = \mathbf{x}_j'\mathbf{y}.$$

Hence:

$$\mathbf{x}_j'\mathbf{X}\widehat{\boldsymbol{\beta}} = \mathbf{x}_j'\mathbf{y}$$

since:

$$\mathbf{X}\widehat{\boldsymbol{\beta}} = [\mathbf{x}_1 \quad \cdots \quad \mathbf{x}_p \quad \cdots \quad \mathbf{x}_k] \begin{bmatrix} \widehat{\beta}_1 \\ \vdots \\ \widehat{\beta}_p \\ \vdots \\ \widehat{\beta}_k \end{bmatrix} = \sum_{p=1}^{k} \mathbf{x}_p \widehat{\beta}_p.$$

Hence, stacking the k normal equations:

$$\begin{bmatrix} \mathbf{x}_1{}'\mathbf{X}\widehat{\boldsymbol{\beta}} \\ \vdots \\ \mathbf{x}_j{}'\mathbf{X}\widehat{\boldsymbol{\beta}} \\ \vdots \\ \mathbf{x}_k{}'\mathbf{X}\widehat{\boldsymbol{\beta}} \end{bmatrix} = \begin{bmatrix} \mathbf{x}_1{}'\mathbf{y} \\ \vdots \\ \mathbf{x}_j{}'\mathbf{y} \\ \vdots \\ \mathbf{x}_k{}'\mathbf{y} \end{bmatrix}.$$

Hence:

$$\begin{bmatrix} \mathbf{x}_1' \\ \vdots \\ \mathbf{x}_j' \\ \vdots \\ \mathbf{x}_k' \end{bmatrix} \mathbf{X}\widehat{\boldsymbol{\beta}} = \begin{bmatrix} \mathbf{x}_1' \\ \vdots \\ \mathbf{x}_j' \\ \vdots \\ \mathbf{x}_k' \end{bmatrix} \mathbf{y}.$$

Hence:

$$\mathbf{X}'\mathbf{X}\widehat{\boldsymbol{\beta}} = \mathbf{X}'\mathbf{y}.$$

15.19 Appendix B: Demonstration that $\widehat{u}'\widehat{u}/(n-k)$ is an unbiased estimator of σ_u^2

This classic proof is both elegant, in that it is much shorter than any proof not using matrix algebra, and curious, in that it uses the trace of a matrix, a feature that I have never seen used for any other purpose. The trace of a matrix, defined for square matrices only, is the sum of its diagonal elements. We will first need to demonstrate that, for any two conformable matrices whose product is square:

$$tr(\mathbf{AB}) = tr(\mathbf{BA}).$$

Let $\mathbf{A}$ have n rows and m columns, and let $\mathbf{B}$ have m rows and n columns. Diagonal element i of $\mathbf{AB}$ is:

$$\sum_{p=1}^{m} a_{ip} b_{pi}.$$

Hence:

$$tr(\mathbf{AB}) = \sum_{i=1}^{n} \left(\sum_{p=1}^{m} a_{ip} b_{pi} \right).$$

Similarly, diagonal element i of $\mathbf{BA}$ is:

$$\sum_{p=1}^{n} b_{ip} a_{pi}.$$

Hence:

$$tr(\mathbf{BA}) = \sum_{i=1}^{m} \left(\sum_{p=1}^{n} b_{ip} a_{pi} \right).$$

What we call the symbols used to index the summations makes no difference. Re-writing p as i and i as p, and noting that the order of the summation makes no difference, we have $tr(\mathbf{BA}) = tr(\mathbf{AB})$.

We also need to note that:

$$tr(\mathbf{A} + \mathbf{B}) = tr(\mathbf{A}) + tr(\mathbf{B})$$

where $\mathbf{A}$ and $\mathbf{B}$ are square matrices of the same dimension. This follows immediately from the way that we sum conformable matrices.

By definition:

$$\hat{\mathbf{u}} = \mathbf{y} - \hat{\mathbf{y}} = \mathbf{y} - \mathbf{X}\hat{\boldsymbol{\beta}}.$$

Using:

$$\hat{\boldsymbol{\beta}} = [\mathbf{X}'\mathbf{X}]^{-1}\mathbf{X}'\mathbf{y}$$

we have:

$$\begin{aligned}
\hat{\mathbf{u}} &= \mathbf{y} - \mathbf{X}[\mathbf{X}'\mathbf{X}]^{-1}\mathbf{X}'\mathbf{y} \\
&= \mathbf{X}\boldsymbol{\beta} + \mathbf{u} - \mathbf{X}[\mathbf{X}'\mathbf{X}]^{-1}\mathbf{X}'(\mathbf{X}\boldsymbol{\beta} + \mathbf{u}) \\
&= \mathbf{I}_n\mathbf{u} - \mathbf{X}[\mathbf{X}'\mathbf{X}]^{-1}\mathbf{X}'\mathbf{u} \\
&= \mathbf{Mu}
\end{aligned}$$

where $\mathbf{I}_n$ is an identity matrix of dimension n and:

$$\mathbf{M} = \mathbf{I}_n - \mathbf{X}[\mathbf{X}'\mathbf{X}]^{-1}\mathbf{X}'.$$

Hence:

$$\widehat{\mathbf{u}}'\widehat{\mathbf{u}} = \mathbf{u}'\mathbf{M}'\mathbf{M}\mathbf{u}.$$

Now $\mathbf{M}$ is symmetric and idempotent: $\mathbf{M}' = \mathbf{M}$ and $\mathbf{MM} = \mathbf{M}$. Hence:

$$\widehat{\mathbf{u}}'\widehat{\mathbf{u}} = \mathbf{u}'\mathbf{M}\mathbf{u}$$

$\widehat{\mathbf{u}}'\widehat{\mathbf{u}}$ is a scalar, and so the expectation of $\widehat{\mathbf{u}}'\widehat{\mathbf{u}}$ and the expectation of the trace of $\widehat{\mathbf{u}}'\widehat{\mathbf{u}}$ are the same. So:

$$E(\widehat{\mathbf{u}}'\widehat{\mathbf{u}}) = E(tr(\widehat{\mathbf{u}}'\widehat{\mathbf{u}})) = E(tr(\mathbf{u}'\mathbf{M}\mathbf{u})) = E(tr(\mathbf{M}\mathbf{u}\mathbf{u}')) = tr(E(\mathbf{M}\mathbf{u}\mathbf{u}')).$$

The penultimate line uses $tr(\mathbf{AB}) = tr(\mathbf{BA})$. The last line uses the fact that the expectation of the sum of the diagonal elements of a matrix is equal to the sum of their individual expectations. Assuming that $\mathbf{X}$, and hence $\mathbf{M}$, is nonstochastic:

$$\begin{aligned}
E(\widehat{\mathbf{u}}'\widehat{\mathbf{u}}) &= tr(\mathbf{M}E(\mathbf{u}\mathbf{u}')) \\
&= tr(\mathbf{M}\mathbf{I}_n\sigma_u^2) \\
&= \sigma_u^2 tr(\mathbf{M}) \\
&= \sigma_u^2 tr(\mathbf{I}_n - \mathbf{X}[\mathbf{X}'\mathbf{X}]^{-\mathbf{1}}\mathbf{X}') \\
&= \sigma_u^2 (tr(\mathbf{I}_n) - tr(\mathbf{X}[\mathbf{X}'\mathbf{X}]^{-\mathbf{1}}\mathbf{X}')).
\end{aligned}$$

The last step uses $tr((A) + \mathbf{B}) = tr(\mathbf{A}) + tr(\mathbf{B})$. The trace of an identity matrix is equal to its dimension. Hence:

$$E(\widehat{\mathbf{u}}'\widehat{\mathbf{u}}) = \sigma_u^2(n - tr(\mathbf{X}[\mathbf{X}'\mathbf{X}]^{-\mathbf{1}}\mathbf{X}')) = \sigma_u^2(n - tr(\mathbf{X}'\mathbf{X}[\mathbf{X}'\mathbf{X}]^{-\mathbf{1}})) = \sigma_u^2(n - tr(\mathbf{I}_k)) = \sigma_u^2(n-k).$$

Hence $\widehat{\mathbf{u}}'\widehat{\mathbf{u}}/(n-k)$ is an unbiased estimator of $\boldsymbol{\sigma_u^2}$.

15.20 Appendix C: Answers to the exercises

1. Given any square matrix $\mathbf{C}$, another matrix $\mathbf{D}$ is said to be its inverse if and only if $\mathbf{CD} = \mathbf{DC} = \mathbf{I}$. Thus, if $\mathbf{B}$ is the inverse of $\mathbf{A}$, $\mathbf{AB} = \mathbf{BA} = \mathbf{I}$. Now focus on the matrix $\mathbf{B}$. Since $\mathbf{BA} = \mathbf{AB} = \mathbf{I}$, $\mathbf{A}$ is its inverse. Hence the inverse of an inverse is the original matrix.

2. Suppose that two different matrices $\mathbf{B}$ and $\mathbf{C}$ both satisfied the conditions for being the inverse of $\mathbf{A}$. Then $\mathbf{BA} = \mathbf{I}$ and $\mathbf{AC} = \mathbf{I}$. Consider the matrix $\mathbf{BAC}$. Using $\mathbf{BA} = \mathbf{I}$, $\mathbf{BAC} = \mathbf{C}$. However, using $\mathbf{AC} = \mathbf{I}$, $\mathbf{BAC} = \mathbf{B}$. Hence $\mathbf{B} = \mathbf{C}$ and it is not possible for $\mathbf{A}$ to have two separate inverses.

3. A_{ij}, and hence A'_{ji}, is the inner product of row i of $\mathbf{B}$ and column j of $\mathbf{C}$. If one writes $\mathbf{D} = \mathbf{C}'\mathbf{B}'$, D_{ji} is the inner product of row j of $\mathbf{C}'$ and column i of $\mathbf{B}'$, that is, column j of $\mathbf{C}$ and row i of $\mathbf{B}$. Hence $D_{ji} = A_{ij}$, so $\mathbf{D} = \mathbf{A}'$ and $\mathbf{C}'\mathbf{B}' = (\mathbf{BC})'$.

4. Let $\mathbf{D}$ be the inverse of $\mathbf{A}$. Then $\mathbf{D}$ must satisfy $\mathbf{AD} = \mathbf{DA} = \mathbf{I}$. Now $\mathbf{A} = \mathbf{BC}$, so $\mathbf{D}$ must satisfy $\mathbf{BCD} = \mathbf{DBC} = \mathbf{I}$. $\mathbf{C}^{-\mathbf{1}}\mathbf{B}^{-\mathbf{1}}$ satisfies both of these conditions, since $\mathbf{BCC}^{-\mathbf{1}}\mathbf{B}^{-\mathbf{1}} = \mathbf{BIB}^{-\mathbf{1}} = \mathbf{I}$ and $\mathbf{C}^{-\mathbf{1}}\mathbf{B}^{-\mathbf{1}}\mathbf{BC} = \mathbf{C}^{-\mathbf{1}}\mathbf{IC} = \mathbf{I}$. Hence $\mathbf{C}^{-\mathbf{1}}\mathbf{B}^{-\mathbf{1}}$ is the inverse of $\mathbf{BC}$ (assuming that B^{-1} and C^{-1} exist).

5. Let $\mathbf{B} = \mathbf{A^{-1}}$. Then $\mathbf{BA} = \mathbf{AB} = \mathbf{I}$. Hence, using the result from Exercise 3, $\mathbf{A}'\mathbf{B}' = \mathbf{B}'\mathbf{A}' = \mathbf{I}' = \mathbf{I}$. Hence $\mathbf{B}'$ is the inverse of $\mathbf{A}'$. In other words, $[\mathbf{A^{-1}}]' = [\mathbf{A}']^{\mathbf{-1}}$.

6. The relationship $Y = \beta_1 + \beta_2 X + u$ may be written in linear algebra form as $\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \mathbf{u}$ where $\mathbf{X} = [\mathbf{1}\ \ \mathbf{x}]$ and $\mathbf{1}$ is the unit vector and:

$$\mathbf{x} = \begin{bmatrix} X_1 \\ \vdots \\ X_i \\ \vdots \\ X_n \end{bmatrix}.$$

Then:

$$\mathbf{X}'\mathbf{X} = \begin{bmatrix} \mathbf{1}' \\ \mathbf{x}' \end{bmatrix} [\mathbf{1}\ \ \mathbf{x}] = \begin{bmatrix} \mathbf{1}'\mathbf{1} & \mathbf{1}'\mathbf{x} \\ \mathbf{x}'\mathbf{1} & \mathbf{x}'\mathbf{x} \end{bmatrix} = \begin{bmatrix} n & \sum X_i \\ \sum X_i & \sum X_i^2 \end{bmatrix}.$$

The determinant of $\mathbf{X}'\mathbf{X}$ is:

$$n \sum X_i^2 - \left(\sum X_i \right)^2 = n \sum X_i^2 - n^2 \overline{X}^2.$$

Hence:

$$[\mathbf{X}'\mathbf{X}]^{\mathbf{-1}} = \frac{1}{n \sum X_i^2 - n^2 \overline{X}^2} \begin{bmatrix} \sum X_i^2 & -n\overline{X} \\ -n\overline{X} & n \end{bmatrix}.$$

We also have:

$$\mathbf{X}'\mathbf{y} = \begin{bmatrix} \mathbf{1}'\mathbf{y} \\ \mathbf{x}'\mathbf{y} \end{bmatrix} = \begin{bmatrix} \sum Y_i \\ \sum X_i Y_i \end{bmatrix}.$$

So:

$$\begin{aligned}
\widehat{\boldsymbol{\beta}} &= [\mathbf{X}'\mathbf{X}]\mathbf{X}'\mathbf{y} \\
&= \frac{1}{n \sum X_i^2 - n^2 \overline{X}^2} \begin{bmatrix} \sum X_i^2 & -n\overline{X} \\ -n\overline{X} & n \end{bmatrix} \begin{bmatrix} n\overline{Y} \\ \sum X_i Y_i \end{bmatrix} \\
&= \frac{1}{n \sum X_i^2 - n^2 \overline{X}^2} \begin{bmatrix} n\overline{Y} \sum X_i^2 - n\overline{X} \sum X_i Y_i \\ -n^2 \overline{XY} + n \sum X_i Y_i \end{bmatrix} \\
&= \frac{1}{\sum \left(X_i - \overline{X} \right)^2} \begin{bmatrix} \overline{Y} \sum X_i^2 - \overline{X} \sum X_i Y_i \\ \sum \left(X_i - \overline{X} \right) \left(Y_i \overline{Y} \right) \end{bmatrix}.
\end{aligned}$$

Thus:

$$\widehat{\beta}_2 = \frac{\sum \left(X_i - \overline{X} \right) \left(Y_i - \overline{Y} \right)}{\sum \left(X_i - \overline{X} \right)^2}$$

and:

$$\widehat{\beta}_1 = \frac{\overline{Y} \sum X_i^2 - \overline{X} \sum X_i Y_i}{\sum \left(X_i - \overline{X} \right)^2}.$$

$\widehat{\beta}_1$ may be written in its more usual form as follows:

$$\begin{aligned}
b_1 &= \frac{\overline{Y}\left(\sum X_i^2 - n\overline{X}^2\right) + \overline{Y}n\overline{X}^2 - \overline{X}\sum X_iY_i}{\sum\left(X_i - \overline{X}\right)^2} \\
&= \frac{\overline{Y}\left(\sum\left(X_i - \overline{X}\right)^2\right) - \overline{X}\left(\sum X_iY_i - n\overline{XY}\right)}{\sum\left(X_i - \overline{X}\right)^2} \\
&= \overline{Y} - \frac{\overline{X}\left(\sum\left(X_i - \overline{X}\right)\left(Y_i - \overline{Y}\right)\right)}{\sum\left(X_i - \overline{X}\right)^2} \\
&= \overline{Y} - \widehat{\beta}_2\overline{X}.
\end{aligned}$$

7. If $Y = \beta_2 X + u$, $\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \mathbf{u}$ where:

$$\mathbf{X} = \mathbf{x} = \begin{bmatrix} X_1 \\ \vdots \\ X_i \\ \vdots \\ X_n \end{bmatrix}.$$

Then:

$$\mathbf{X}'\mathbf{X} = \mathbf{x}'\mathbf{x} = \sum X_i^2.$$

The inverse of $\mathbf{X}'\mathbf{X}$ is $1/\sum X_i^2$. In this model, $\mathbf{X}'\mathbf{y} = \mathbf{x}'\mathbf{y} = \sum X_iY_i$. So:

$$\widehat{\boldsymbol{\beta}} = [\mathbf{X}'\mathbf{X}]^{-1}\mathbf{X}'\mathbf{y} = \frac{\sum X_iY_i}{\sum X_i^2}.$$

8. If $Y = \beta_1 + u$, $\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \mathbf{u}$ where $\mathbf{X} = \mathbf{1}$, the unit vector. Then $\mathbf{X}'\mathbf{X} = \mathbf{1}'\mathbf{1} = n$ and its inverse is $1/n$.

$$\mathbf{X}'\mathbf{y} = \mathbf{1}'\mathbf{y} = \sum Y_i = n\overline{Y}.$$

So:

$$\widehat{\boldsymbol{\beta}} = [\mathbf{X}'\mathbf{X}]^{-1}\mathbf{X}'\mathbf{y} = \frac{1}{n}n\overline{Y} = \overline{Y}.$$

9. We will start with Y. If we regress it on the intercept, we are regressing it on $\mathbf{1}$, the unit vector, and, as we saw in Exercise 8, the coefficient is $\overline{Y}$. Hence the residual in observation i is $Y_i - \overline{Y}$. The same is true for each of the X variables when regressed on the intercept. So when we come to regress the residuals of Y on the residuals of the X variables, we are in fact using the demeaned data for Y and the demeaned data for the X variables.

10. The general form of the IV estimator is $\widehat{\boldsymbol{\beta}}^{\mathrm{IV}} = [\mathbf{W}'\mathbf{X}]^{-1}\mathbf{W}'\mathbf{y}$. In the case of the simple regression model, with Z acting as an instrument for X and the unit vector acting as an instrument for itself, $\mathbf{W} = [\mathbf{1}\ \ \mathbf{z}]$ and $\mathbf{X} = [\mathbf{1}\ \ \mathbf{x}]$. Thus:

$$\mathbf{W}'\mathbf{X} = \begin{bmatrix} \mathbf{1}' \\ \mathbf{z}' \end{bmatrix}[\mathbf{1}\ \ \mathbf{x}] = \begin{bmatrix} \mathbf{1}'\mathbf{1} & \mathbf{1}'\mathbf{x} \\ \mathbf{z}'\mathbf{1} & \mathbf{z}'\mathbf{x} \end{bmatrix} = \begin{bmatrix} n & \sum X_i \\ \sum Z_i & \sum Z_iX_i \end{bmatrix}.$$

The determinant of $\mathbf{W}'\mathbf{X}$ is:

$$n\sum Z_iX_i - \left(\sum Z_i\right)\left(\sum X_i\right) = n\sum Z_iX_i - n^2\overline{ZX}.$$

Hence:

$$[\mathbf{W}'\mathbf{X}]^{-1} = \frac{1}{n\sum Z_iX_i - n^2\overline{\overline{ZX}}} = \begin{bmatrix} \sum Z_iX_i & -n\overline{X} \\ -n\overline{Z} & n \end{bmatrix}.$$

We also have:

$$\mathbf{W}'\mathbf{y} = \begin{bmatrix} \mathbf{1}'\mathbf{y} \\ \mathbf{z}'\mathbf{y} \end{bmatrix} = \begin{bmatrix} \sum Y_i \\ \sum Z_iY_i \end{bmatrix}.$$

So:

$$\begin{aligned}
\widehat{\boldsymbol{\beta}}^{\mathrm{IV}} &= [\mathbf{W}'\mathbf{X}]^{-\mathbf{1}}\mathbf{W}'\mathbf{y} \\
&= \frac{1}{n\sum Z_iX_i - n^2\overline{\overline{ZX}}}\begin{bmatrix} \sum Z - iX_i & -n\overline{X} \\ -n\overline{Z} & n \end{bmatrix}\begin{bmatrix} n\overline{Y} \\ \sum Z_iY_i \end{bmatrix} \\
&= \frac{1}{n\sum Z_iX_i - n^2\overline{\overline{ZX}}}\begin{bmatrix} n\overline{Y}\sum Z_iX_i - n\overline{X}\sum Z_iY_i \\ -n^2\overline{ZX} + n\sum Z_iY_i \end{bmatrix} \\
&= \frac{1}{\sum\left(Z_i - \overline{Z}\right)\left(X_i - \overline{X}\right)}\begin{bmatrix} \overline{Y}\sum Z_iX_i - \overline{X}\sum Z_iY_i \\ \sum\left(Z_i - \overline{Z}\right)\left(Y_i - \overline{Y}\right) \end{bmatrix}.
\end{aligned}$$

Thus:

$$\widehat{\beta}_2^{\mathrm{IV}} = \frac{\sum\left(Z_i - \overline{Z}\right)\left(Y_i - \overline{Y}\right)}{\sum\left(Z_i - \overline{Z}\right)\left(X_i - \overline{X}\right)}$$

and:

$$\widehat{\beta}_1^{\mathrm{IV}} = \frac{\bar{Y}\sum Z_iX_i - \overline{X}\sum Z_iY_i}{\sum\left(Z_i - \overline{Z}\right)\left(X_i - \overline{X}\right)}.$$

$\widehat{\beta}_1^{\mathrm{IV}}$ may be written in its more usual form as follows:

$$\begin{aligned}
\widehat{\beta}_1^{\mathrm{IV}} &= \frac{\overline{Y}\left(\sum Z_iX_i - n\overline{ZX}\right) + \overline{Y}n\overline{ZX} - \overline{X}\sum Z_iY_i}{\sum\left(Z_i - \overline{Z}\right)\left(X_i - \overline{X}\right)} \\
&= \frac{\overline{Y}\left(\sum\left(Z_i - \overline{Z}\right)\left(X_i - \overline{X}\right)\right) - \overline{X}\left(\sum Z_iY_i - n\overline{ZY}\right)}{\sum\left(Z_i - \overline{Z}\right)\left(X_i - \overline{X}\right)} \\
&= \overline{Y} - \frac{\overline{X}\left(\sum\left(Z_i - \overline{Z}\right)\left(Y_i - \overline{Y}\right)\right)}{\sum\left(Z_i - \overline{Z}\right)\left(X_i - \overline{X}\right)} \\
&= \overline{Y} - \widehat{\beta}_2^{\mathrm{IV}}\overline{X}.
\end{aligned}$$

11. By definition, if one of the variables in $\mathbf{X}$ is acting as an instrument for itself, it is included in the $\mathbf{W}$ matrix. If it is regressed on $\mathbf{W}$, a perfect fit is obtained by

assigning its column in $\mathbf{W}$ a coefficient of 1 and assigning zero values to all the other coefficients. Hence its fitted values are the same as its original values and it is not affected by the first stage of Two-Stage Least Squares.

12. If the variables in $\mathbf{X}$ are regressed on $\mathbf{W}$ and the matrix of fitted values of $\mathbf{X}$ saved:

$$\widehat{\mathbf{X}} = \mathbf{W}[\mathbf{W}'\mathbf{W}]^{-1}\mathbf{W}'\mathbf{X}.$$

If $\widehat{\mathbf{X}}$ is used as the matrix of instruments:

$$\begin{aligned}\widehat{\boldsymbol{\beta}}^{\mathrm{TSLS}} &= [\widehat{\mathbf{X}}'\mathbf{X}]^{-1}\widehat{\mathbf{X}}'\mathbf{y}\\ &= [\mathbf{X}'\mathbf{W}[\mathbf{W}'\mathbf{W}]^{-1}\mathbf{W}'\mathbf{X}]^{-1}\mathbf{X}'\mathbf{W}[\mathbf{W}'\mathbf{W}]^{-1}\mathbf{W}'\mathbf{y}\\ &= [\mathbf{W}'\mathbf{X}]^{-1}\mathbf{W}'\mathbf{W}[\mathbf{X}'\mathbf{W}]^{-1}\mathbf{X}'\mathbf{W}[\mathbf{W}'\mathbf{W}]^{-1}\mathbf{W}'\mathbf{y}\\ &= [\mathbf{W}'\mathbf{X}]^{-1}\mathbf{W}'\mathbf{y}\\ &= \widehat{\boldsymbol{\beta}}^{\mathrm{IV}}.\end{aligned}$$

Note that, in going from the second line to the third, we have used $[\mathbf{ABC}]^{-1} = \mathbf{C}^{-1}\mathbf{B}^{-1}\mathbf{A}^{-1}$, and we have exploited the fact that $\mathbf{W}'\mathbf{X}$ is square and possesses an inverse.

13. The variance-covariance matrix of u is:

$$\begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}$$

and hence the characteristic equation for the eigenvalues is:

$$(1-\lambda)^2 - \rho^2 = 0.$$

The eigenvalues are therefore $1-\rho$ and $1+\rho$. Since we are told $|\rho| < 1$, the matrix is positive definite.

Let:

$$\mathbf{c} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}.$$

If $\lambda = 1-\rho$, the matrix $\mathbf{A} - \lambda\mathbf{I}$ is given by:

$$\mathbf{A} - \lambda\mathbf{I} = \begin{bmatrix} \rho & \rho \\ \rho & \rho \end{bmatrix}$$

and hence the equation:

$$[\mathbf{A} - \lambda\mathbf{I}]\mathbf{c} = \mathbf{0}$$

yields:

$$\rho c_1 + \rho c_2 = 0.$$

Hence, also imposing the normalisation:

$$\mathbf{c}'\mathbf{c} = c_1^2 + c_2^2 = 1$$

we have $c_1 = 1/\sqrt{2}$ and $c_2 = -1/\sqrt{2}$, or vice versa. If $\lambda = 1 + \rho$:

$$\mathbf{A} - \lambda\mathbf{I} = \begin{bmatrix} -\rho & \rho \\ \rho & -\rho \end{bmatrix}$$

and hence $[\mathbf{A} - \lambda\mathbf{I}]\mathbf{c} = \mathbf{0}$ yields:

$$-\rho c_1 + \rho c_2 = 0.$$

Hence, also imposing the normalisation:

$$\mathbf{c}'\mathbf{c} = c_1^2 + c_2^2 = 1$$

we have $c_1 = c_2 = 1/\sqrt{2}$. Thus:

$$\mathbf{C} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

and:

$$\mathbf{P} = \mathbf{\Lambda}^{-1/2}\mathbf{C}' = \begin{bmatrix} \frac{1}{\sqrt{1-\rho}} & 0 \\ 0 & \frac{1}{\sqrt{1+\rho}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} \frac{1}{\sqrt{1-\rho}} & -\frac{1}{\sqrt{1-\rho}} \\ \frac{1}{\sqrt{1+\rho}} & \frac{1}{\sqrt{1+\rho}} \end{bmatrix}.$$

It may then be verified that $\mathbf{P}\mathbf{\Omega}\mathbf{P}' = \mathbf{I}$:

$$\begin{aligned} & \frac{1}{\sqrt{2}} \begin{bmatrix} \frac{1}{\sqrt{1-\rho}} & -\frac{1}{\sqrt{1-\rho}} \\ \frac{1}{\sqrt{1+\rho}} & \frac{1}{\sqrt{1+\rho}} \end{bmatrix} \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} \frac{1}{\sqrt{1-\rho}} & \frac{1}{\sqrt{1+\rho}} \\ -\frac{1}{\sqrt{1-\rho}} & \frac{1}{\sqrt{1+\rho}} \end{bmatrix} \\ = {} & \frac{1}{2} \begin{bmatrix} \frac{1}{\sqrt{1-\rho}} & -\frac{1}{\sqrt{1-\rho}} \\ \frac{1}{\sqrt{1+\rho}} & \frac{1}{\sqrt{1+\rho}} \end{bmatrix} \begin{bmatrix} \frac{1-\rho}{\sqrt{1-\rho}} & \frac{1+\rho}{\sqrt{1+\rho}} \\ \frac{\rho-1}{\sqrt{1-\rho}} & \frac{1+\rho}{\sqrt{1+\rho}} \end{bmatrix} \\ = {} & \frac{1}{2} \begin{bmatrix} \frac{1}{\sqrt{1-\rho}} & -\frac{1}{\sqrt{1-\rho}} \\ \frac{1}{\sqrt{1+\rho}} & \frac{1}{\sqrt{1+\rho}} \end{bmatrix} \begin{bmatrix} \sqrt{1-\rho} & \sqrt{1+\rho} \\ -\sqrt{1-\rho} & \sqrt{1+\rho} \end{bmatrix} \\ = {} & \frac{1}{2} \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \end{aligned}$$

The transformed model has:

$$\mathbf{y}^* = \frac{1}{\sqrt{2}} \begin{bmatrix} \frac{1}{\sqrt{1-\rho}}(y_1 - y_2) \\ \frac{1}{\sqrt{1+\rho}}(y_1 + y_2) \end{bmatrix}$$

and parallel transformations for the $\mathbf{X}$ variables and $\mathbf{u}$. Given that:

$$\mathbf{u}^* = \frac{1}{\sqrt{2}} \begin{bmatrix} \frac{1}{\sqrt{1-\rho}}(u_1 - u_2) \\ \frac{1}{\sqrt{1+\rho}}(u_1 + u_2) \end{bmatrix}$$

none of its elements is the white noise ε in the AR(1) process, but nevertheless its elements are iid.

$$\begin{aligned}
\operatorname{var}(u_1^*) &= \frac{1}{2}\frac{1}{1-\rho}\left(\operatorname{var}(u_1)+\operatorname{var}(u_2)-2\operatorname{cov}(u_1,u_2)\right)\\
&= \frac{1}{2}\frac{1}{1-\rho}\left(\sigma_u^2+\sigma_u^2-2\rho\sigma_u^2\right)=\sigma_u^2\\
\operatorname{var}(u_2^*) &= \frac{1}{2}\frac{1}{1+\rho}\left(\operatorname{var}(u_1)+\operatorname{var}(u_2)+2\operatorname{cov}(u_1,u_2)\right)\\
&= \frac{1}{2}\frac{1}{1+\rho}\left(\sigma_u^2+\sigma_u^2+2\rho\sigma_u^2\right)=\sigma_u^2\\
\operatorname{cov}(u_1^*,u_2^*) &= \frac{1}{2}\frac{1}{\sqrt{1-\rho^2}}\operatorname{cov}\left((u_1-u_2),(u_1+u_2)\right)\\
&= \frac{1}{2}\frac{1}{\sqrt{1-\rho^2}}\left(\operatorname{var}(u_1)+\operatorname{cov}(u_1,u_2)-\operatorname{cov}(u_2,u_1)-\operatorname{var}(u_2)\right)\\
&= 0.
\end{aligned}$$

Hence $E(\mathbf{u}^*\mathbf{u}^{*\prime})=\mathbf{I}\sigma_u^2$. Of course, this was the objective of the $\mathbf{P}$ transformation.

Appendix A
Syllabus for the EC2020 Elements of econometrics examination

This syllabus is intended to provide an explicit list of all the mathematical formulae and proofs that you are expected to know for the **EC2020 Elements of Econometrics** examination. You are warned that the examination is intended to be an opportunity for you to display your understanding of the material, rather than of your ability to reproduce standard items.

A.1 Review: Random variables and sampling theory

Probability distribution of a random variable. Expected value of a random variable. Expected value of a function of a random variable. Population variance of a discrete random variable and alternative expression for it. Expected value rules. Independence of two random variables. Population covariance, covariance and variance rules, and correlation. Sampling and estimators. Unbiasedness. Efficiency. Loss functions and mean square error. Estimators of variance, covariance and correlation. The normal distribution. Hypothesis testing. Type II error and the power of a test. t tests. Confidence intervals. One-sided tests. Convergence in probability and plim rules. Consistency. Convergence in distribution (asymptotic limiting distributions) and the role of central limit theorems.

Formulae and proofs: This chapter is concerned with statistics, not econometrics, and is not examinable. However, you are expected to know the results in this chapter and to be able to use them.

A.2 Chapter 1 Simple regression analysis

Simple regression model. Derivation of linear regression coefficients. Interpretation of a regression equation. Goodness of fit.

Formulae and proofs: You are expected to know, and be able to derive, the expressions for the regression coefficients in a simple regression model, including variations where either the intercept or the slope coefficient may be assumed to be zero. You are expected to know the definition of R^2 and how it is related to the residual sum of squares. You are expected to know the relationship between R^2 and the correlation between the actual and fitted values of the dependent variable, but not to be able to prove it.

A.3 Chapter 2 Properties of the regression coefficients

Types of data and regression model. Assumptions for Model A. Regression coefficients as random variables. Unbiasedness of the regression coefficients. Precision of the regression coefficients. Gauss–Markov theorem. t test of a hypothesis relating to a regression coefficient. Type I error and Type II error. Confidence intervals. One-sided tests. F test of goodness of fit.

Formulae and proofs: You are expected to know the regression model assumptions for Model A. You are expected to know, though not be able to prove, that, in the case of a simple regression model, an F test on the goodness of fit is equivalent to a two-sided t test on the slope coefficient. You are expected to know how to make a theoretical decomposition of an estimator and hence how to investigate whether or not it is biased. In particular, you are expected to be able to show that the OLS estimator of the slope coefficient in a simple regression model can be decomposed into the true value plus a weighted linear combination of the values of the disturbance term in the sample. You are expected to be able to derive the expression for the variance of the slope coefficient in a simple regression model. You are expected to know how to estimate the variance of the disturbance term, given the residuals, but you are not expected to be able to derive the expression. You are expected to understand the Gauss–Markov theorem, but you are not expected to be able to prove it.

A.4 Chapter 3 Multiple regression analysis

Multiple regression with two explanatory variables. Graphical representation of a relationship in a multiple regression model. Properties of the multiple regression coefficients. Population variance of the regression coefficients. Decomposition of their standard errors. Multicollinearity. F tests in a multiple regression model. Hedonic pricing models. Prediction.

Formulae and proofs: You are expected to know how, in principle, the multiple regression coefficients are derived, but you do not have to remember the expressions, nor do you have to be able to derive them mathematically. You are expected to know, but not to be able to derive, the expressions for the population variance of a slope coefficient and its standard error in a model with two explanatory variables. You are expected to be able to perform F tests on the goodness of fit of the model as a whole and for the improvement in fit when a group of explanatory variables is added to the model. You are expected to be able to demonstrate the properties of predictions within the context of the classical linear regression model. In particular, you are expected to be able to demonstrate that the expected value of the prediction error is 0, if the model is correctly specified and the regression model assumptions are satisfied. You are not expected to know the population variance of the prediction error.

A.5 Chapter 4 Transformation of variables

Linearity and nonlinearity. Elasticities and double-logarithmic models. Semilogarithmic models. The disturbance term in nonlinear models. Box–Cox transformation. Models with quadratic and interactive variables. Nonlinear regression.

Formulae and proofs: You are expected to know how to perform a Box–Cox transformation for comparing the goodness of fit of alternative versions of a model with Y and $\log Y$ as the dependent variable.

A.6 Chapter 5 Dummy variables

Dummy variables. Dummy classification with more than two categories. The effects of changing the reference category. Multiple sets of dummy variables. Slope dummy variables. Chow test. Relationship between Chow test and dummy group test.

Formulae and proofs: You are expected to be able to perform a Chow test and a test of the explanatory power of a group of dummy variables, and to understand the relationship between them.

A.7 Chapter 6 Specification of regression variables

Omitted variable bias. Consequences of the inclusion of an irrelevant variable. Proxy variables. F test of a linear restriction. Reparameterisation of a regression model (see the Further material handout). t test of a restriction. Tests of multiple restrictions. Tests of zero restrictions.

Formulae and proofs: You are expected to be able to derive the expression for omitted variable bias when the true model has two explanatory variables and the fitted model omits one of them. You are expected to know how to perform an F test on the validity of a linear restriction, given appropriate data on the residual sum of squares. You are expected to understand the logic behind the t test of a linear restriction and to be able to reparameterise a regression specification to perform such a test in a simple context. You are expected to be able to perform F tests of multiple linear restrictions.

A.8 Chapter 7 Heteroskedasticity

Meaning of heteroskedasticity. Consequences of heteroskedasticity. Goldfeld–Quandt and White tests for heteroskedasticity. Elimination of heteroskedasticity using weighted or logarithmic regressions. Use of heteroskedasticity-consistent standard errors.

Formulae and proofs: You are expected to know how to perform the Goldfeld–Quandt and White tests for heteroskedasticity.

A.9 Chapter 8 Stochastic regressors and measurement errors

Stochastic regressors. Assumptions for models with stochastic regressors. Finite sample and asymptotic properties of the regression coefficients in models with stochastic regressors. Measurement error and its consequences. Friedman's Permanent Income Hypothesis. Instrumental variables (IV). Asymptotic properties of IV estimators, including the asymptotic limiting distribution of $\sqrt{n}(\widehat{\beta}_2^{\mathrm{IV}} - \beta_2)$. $\widehat{\beta}_2^{\mathrm{IV}}$ is the IV estimator of β_2 in a simple regression model. Use of simulation to investigate the finite-sample properties of estimators when only asymptotic properties can be determined analytically. Application of the Durbin–Wu–Hausman test.

Formulae and proofs: You are expected to be able to demonstrate that, in a simple regression model, the OLS estimator of the slope coefficient is inconsistent when there is measurement error in the explanatory variable. You should know the expression for the bias and be able to derive it. You should be able to explain the consequences of measurement error in the dependent variable. You should know the expression for an instrumental variable estimator of the slope coefficient in a simple regression model and be able to demonstrate that it yields consistent estimates, provided that certain assumptions are satisfied. You should also know the expression for the asymptotic population variance of an instrumental variable estimator in a simple regression model and to understand why it provides only an approximation for finite samples. You are not expected to know the formula for the Durbin–Wu–Hausman test.

A.10 Chapter 9 Simultaneous equations estimation

Definitions of endogenous variables, exogenous variables, structural equations and reduced form. Inconsistency of OLS. Use of instrumental variables. Exact identification, underidentification, and overidentification. Two-stage least squares (TSLS). Order condition for identification. Application of the Durbin–Wu–Hausman test.

Formulae and proofs: You are expected to be able to derive an expression for simultaneous equations bias in a simple regression equation and to be able to demonstrate the consistency of an IV estimator in a simple regression equation. You are expected to be able to explain in general terms why TSLS is used in overidentified models.

A.11 Chapter 10 Binary choice models and maximum likelihood estimation

Linear probability model. Logit model. Probit model. Maximum likelihood estimation of the population mean and variance of a random variable. Maximum likelihood estimation of regression coefficients. Likelihood ratio tests.

Formulae and proofs: You are expected to know the expression for the probability of an event occurring in the logit model, and to know the expressions for the marginal functions in the logit and probit models. You would not be expected to calculate marginal effects in an examination, but you should be able to explain how they are calculated and to comment on calculations of them. You are expected to be able to derive a maximum likelihood estimator in a simple example. In more complex examples, you would only be expected to explain how the estimates are obtained, in principle. You are expected to be able to perform, from first principles, likelihood ratio tests in a simple context.

A.12 Chapter 11 Models using time series data

Static demand functions fitted using aggregate time series data. Lagged variables and naive attempts to model dynamics. Autoregressive distributed lag (ADL) models with applications in the form of the partial adjustment and adaptive expectations models. Error correction models. Asymptotic properties of OLS estimators of ADL models, including asymptotic limiting distributions. Use of simulation to investigate the finite sample properties of parameter estimators for the ADL(1,0) model. Use of predetermined variables as instruments in simultaneous equations models using time series data. (Section 11.7 of the textbook, *Alternative dynamic representations*..., is not in the syllabus.)

Formulae and proofs: You are expected to be able to analyse the short-run and long-run dynamics inherent in ADL(1,0) models in general and the adaptive expectations and partial adjustment models in particular. You are expected to be able to explain why the OLS estimators of the parameters of ADL(1,0) models are subject to finite-sample bias and, within the context of the model $Y_t = \beta_1 + \beta_2 Y_{t-1} + u_t$ to be able to demonstrate that they are consistent.

A.13 Chapter 12 Autocorrelation

Assumptions for regressions with time series data. Assumption of the independence of the disturbance term and the regressors. Definition of autocorrelation. Consequences of autocorrelation. Breusch–Godfrey, Lagrange multiplier and Durbin–Watson d tests for autocorrelation. AR(1) nonlinear regression. Potential advantages and disadvantages of such estimation, in comparison with OLS. Autocorrelation with a lagged dependent variable. Common factor test and implications for model selection. Apparent autocorrelation caused by variable or functional misspecification. General-to-specific versus specific-to-general model specification.

Formulae and proofs: You are expected to know how to perform the tests for autocorrelation mentioned above and to know how to perform a common factor test. You are expected to be able to explain why the properties of estimators obtained by fitting the AR(1) nonlinear regression specification are not necessarily superior to those obtained using OLS.

A.14 Chapter 13 Introduction to nonstationary processes

Stationary and nonstationary processes. Granger–Newbold experiments with random walks. Unit root tests. Akaike Information Criterion and Schwarz's Bayes Information Criterion. Cointegration. Error correction models.

Formulae and proofs: You are expected to be able to determine whether a simple random process is stationary or nonstationary. You would not be expected to perform a unit root test in an examination, but you are expected to understand the test and to be able to comment on the results of such a test.

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