Digital Signal Processing for Communications

Solutions' Manual

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Chapter 2

Solution 2.1: Review of complex numbers. Recall that

$$\sum_{i=0}^{k} z^k = \begin{cases} \frac{1-z^{k+1}}{1-z} & \text{for } z \neq 1 \\ N+1 & \text{for } z = 1\,. \end{cases}$$

Proof for $z \neq 0$ (for $z = 1$ is trivial)

$$\begin{aligned} s &= 1 + z + z^2 + \ldots + z^N\,, \\ -zs &= -z - z^2 - \ldots - z^N - z^{N+1}\,. \end{aligned}$$

Summing the above two equations gives

$$(1-z)s = 1 - z^{N+1} \Rightarrow s = \frac{1-z^{N+1}}{1-z}\,.$$

Similarly

$$\sum_{k=N_1}^{N_2} z^k = z^{N_1} \sum_{k=0}^{N_2-N_1} z^k = \frac{z^{N_1} - z^{N_2} + 1}{1-z}\,.$$

(a) We have

$$\begin{aligned} \sum_{n=1}^{N} s[n] &= \sum_{n=1}^{N} 2^{-n} + j \sum_{n=1}^{N} 3^{-n} \\ &= \frac{1}{2} \cdot \frac{1-2^{-N}}{1-2^{-1}} + j\frac{1}{3} \cdot \frac{1-3^{-N}}{1-3^{-1}} = (1-2^{-N}) + j\frac{1}{2}(1-3^{-N}). \end{aligned}$$

Now,

$$\lim_{N\to\infty} 2^{-N} = \lim_{N\to\infty} 3^{-N} = 0\,.$$

Therefore,

$$\sum_{n=1}^{\infty} s[n] = 1 + \frac{1}{2} j \, .$$

(b) We can write

$$\sum_{k=1}^{N} s[k] = \frac{j}{3} \cdot \frac{1-(j/3)^N}{1-j/3} \, .$$

Since $\left|\frac{j}{3}\right| = \frac{1}{3} < 1$, we have $\lim_{N\to\infty}(j/3)^N = 0$. Therefore,

$$\sum_{k=1}^{\infty} s[k] = \frac{1}{3j-1} = -\frac{1}{10} + j \cdot \frac{3}{10} \, .$$

(c) From $z^* = z^{-1}$ with $z \in \mathbb{C}$, we have

$$z z^* = 1, \quad \forall\, z \neq 0.$$

Therefore, $|z|^2 = 1$ and, consequently, $|z| = 1$. It follows that all the z such that $z^* = z^{-1}$ describe the unit circle.

(d) Remark that $e^{2k\pi} = 1$, for all $k \in \mathbb{Z}$. Therefore, $z_k = e^{\frac{2k\pi}{3}}$ is such that $z_k^3 = 1$. Now z_k is periodic of period 3, i.e. $z_k = z_{k+3l}$, for all $l \in \mathbb{Z}$. Therefore the (only) three different complex numbers are

$$z_0 = 1, \quad z_1 = e^{\frac{2\pi}{3}} \quad \text{and} \quad z_2 = e^{\frac{4k\pi}{3}} \, .$$

(e) We have

$$\prod_{n=1}^{N} e^{j\frac{\pi}{2^n}} = e^{j\pi \sum_{n=1}^{N} 2^{-n}} = e^{j\pi \frac{1}{2} \cdot \frac{1-2^{-N}}{1-1/2}} \, .$$

Since $\lim_{N\to\infty} 2^{-N} = 0$,

$$\prod_{n=1}^{\infty} e^{j\frac{\pi}{2^n}} = e^{j\pi} = -1 \, .$$

Solution 2.2: Periodic signals.

(a) Not periodic.

(b) Not periodic.

(c) 14-periodic.

(d) 100-periodic.

Chapter 3

Solution 3.1: Elementary operators.

(a) $\mathscr{D}\{\alpha x[n]\} = \alpha x[n-1] = \alpha\mathscr{D}\{x[n]\}$
$\mathscr{D}\{x[n]+y[n]\} = x[n-1]+y[n-1] = \mathscr{D}\{x[n]\}+\mathscr{D}\{y[n]\}$.

(b) $\mathscr{V}$ is a linear combination of the original signal with the linear operator $\mathscr{D}$, therefore it is also linear.

(c) $\mathscr{S}\{\alpha x[n]\} = \alpha^2 x^2[n-1] = \alpha^2\mathscr{S}\{x[n]\} \neq \alpha\mathscr{S}\{x[n]\}$.

(d)
$$\mathbf{V} = \mathbf{I} - \mathbf{D} = \begin{bmatrix} 1 & 0 & 0 & -1 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}.$$

(e) The matrix realizes an integration operation over a vector in $\mathbb{C}^4$.

Solution 3.2: Bases. Suppose by contradiction that the vector $\mathbf{z} \in S$ admits two distinct representations in the basis $\{\mathbf{x}^{(k)}\}_{k=0,\dots,N-1}$. In other words, there exist scalars $\alpha_0,\dots,\alpha_{N-1}$ and $\beta_0,\dots,\beta_{N-1}$ with $(\alpha_0,\dots,\alpha_{N-1}) \neq (\beta_0,\dots,\beta_{N-1})$ such that $\mathbf{z} = \Sigma_{k=0}^{N-1}\alpha_k\mathbf{x}^{(k)}$ and also $\mathbf{z} = \Sigma_{k=0}^{N-1}\beta_k\mathbf{x}^{(k)}$. Consequently

$$\Sigma_{k=0}^{N-1}\alpha_k\mathbf{x}^{(k)} = \Sigma_{k=0}^{N-1}\beta_k\mathbf{x}^{(k)}$$

or, equivalently,

$$\Sigma_{k=0}^{N-1}(\alpha_k - \beta_k)\mathbf{x}^{(k)} = 0.$$

Since $\{\mathbf{x}^{(k)}\}_{k=0,\ldots,N-1}$ is a basis, it is a set of independent vectors and by definition the above equation admits only the trivial solution $\alpha_k - \beta_k = 0$, $\forall k = 0,\ldots,N-1$. Thus, $\alpha_k = \beta_k$, $\forall k = 0,\ldots,N-1$.

Solution 3.3: Vector spaces and signals.

(a) It is straightforward to verify that the set of all ordered n-tuples $[a_1, a_2, \ldots, a_n]$ with the natural definition for the sum: $[a_1, a_2, \ldots, a_n]+[b_1, b_2, \ldots, b_n] = [a_1+b_1, a_2+b_2, \ldots, a_n+b_n]$ and the multiplication by a scalar: $\alpha[a_1, a_2, \ldots, a_n] = [\alpha a_1, \alpha a_2, \ldots, \alpha a_n]$ satisfies all the properties of a vector space:

- Addition is commutative.
- Addition is associative.
- Scalar multiplication is distributive.
- There exists a null vector: $[0,0,\ldots,0]$.
- Additive inverse: $[-a_1, -a_2, \ldots, -a_n]$.
- Identity element for scalar multiplication: 1.

The dimension of this vector space is n and a basis is:

$$[1,0,\ldots,0],[0,1,\ldots,0],...,[0,0,\ldots,1].$$

(b) The set of signals of the form $y(x) = a\cos(x)+b\sin(x)$ (for arbitrary a, b) with the usual addition and multiplication by a scalar form a vector space:

- Addition is commutative.
- Addition is associative.
- Scalar multiplication is distributive.
- There exists a null vector: $a, b = 0$.
- Additive inverse: $-y(x) = -a\cos(x) - b\sin(x)$.
- Identity element for scalar multiplication: 1.

The dimension of this vector space is 2 and one possible basis is:

$$y_1(x) = \cos(x), y_2(x) = \sin(x).$$

(c) The eight vertices of the cube can be represented by the following four vectors:

$$v_1 = [0,0,0],\ v_2 = [1,0,0],\ v_3 = [0,1,0],\ v_4 = [1,1,0],$$
$$v_5 = [0,0,1],\ v_6 = [1,0,1],\ v_7 = [0,1,1],\ v_8 = [1,1,1].$$

and the four associated diagonals:

$$d_1 = \{v_1, v_8\} = v_8 - v_1 = [1,1,1].$$
$$d_2 = \{v_2, v_7\} = v_7 - v_2 = [-1,1,1].$$
$$d_3 = \{v_3, v_6\} = v_6 - v_3 = [1,-1,1].$$
$$d_4 = \{v_4, v_5\} = v_5 - v_4 = [-1,-1,1].$$

Two vectors are orthogonal if their inner product is zero. In this case,

$$< d_i, d_j > \neq 0 \text{ for all } i, j.$$

Therefore, the four diagonals of a cube are not orthogonal. Alternatively, just remark that it is not possible to have 4 orthogonal vectors in a space of dimension 3.

(d) $\delta[n] = u[n] - u[n-1]$ and $u[n] = \Sigma_{k=0}^{\infty} \delta[n-k]$.

(e) $f_o(t) = \frac{f(t) - f(-t)}{2}$ and $f_e(t) = \frac{f(t) + f(-t)}{2}$.

Solution 3.4: The Haar basis.

(a) For a square matrix to have full rank, it is enough to prove that the determinant is nonzero.

(b) Using Matlab, we define

```
h=[1 -1 0 0 0 0 0 0; 0 0 1 -1 0 0 0 0;...
0 0 0 0 1 -1 0 0; 0 0 0 0 0 0 1 -1;...
1 1 -1 -1 0 0 0 0; 0 0 0 0 1 1 -1 -1;...
1 1 1 1 -1 -1 -1 -1; 1 1 1 1 1 1 1 1];
```

and we have that `det(H)` returns -128.

(c) Using Matlab, `H*H'` returns a diagonal matrix.

(d) We type `H*ones(8,1)` and we get the result $[0\ 0\ 0\ 0\ 0\ 0\ 0\ 8]^T$. In the Haar basis, constant signals have only nonzero coefficients in the high end of the vector.

(e) Similarly, `H*(-1).^(0:7)'` returns $[2\ 2\ 2\ 2\ 0\ 0\ 0\ 0]^T$. In the Haar basis, alternating signals have nonzero coefficients in the low end of the vector.

Chapter 4

Solution 4.1: DFT of elementary functions. We have:

$$\begin{aligned} x[n] &= \frac{e^{j\phi}}{2} e^{j(2\pi/N)Ln} + \frac{e^{-j\phi}}{2} e^{-j(2\pi/N)Ln} \\ &= \frac{e^{j\phi}}{2} e^{j(2\pi/N)Ln} + \frac{e^{-j\phi}}{2} e^{-j(2\pi/N)Ln} e^{j(2\pi/N)Nn} \\ &= \frac{e^{j\phi}}{2} e^{j(2\pi/N)Ln} + \frac{e^{-j\phi}}{2} e^{j(2\pi/N)(N-L)n}. \end{aligned}$$

Therefore we can write in vector notation:

$$\mathbf{x} = \frac{e^{j\phi}}{2}\mathbf{w}^{(L)} + \frac{e^{-j\phi}}{2}\mathbf{w}^{(N-L)},$$

and the result follows from the linearity of the expansion formula

$$\begin{aligned} X[k] &= \langle \mathbf{w}^{(k)}, \mathbf{x} \rangle \\ &= \left\langle \mathbf{w}^{(k)}, \frac{e^{j\phi}}{2}\mathbf{w}^{(L)} + \frac{e^{-j\phi}}{2}\mathbf{w}^{(N-L)} \right\rangle = \frac{e^{j\phi}}{2}\langle \mathbf{w}^{(k)}, \mathbf{w}^{(L)} \rangle + \frac{e^{-j\phi}}{2}\langle \mathbf{w}^{(k)}, \mathbf{w}^{(N-L)} \rangle \\ &= \begin{cases} \frac{N}{2} e^{j\phi} & \text{if } k = L \\ \frac{N}{2} e^{-j\phi} & \text{if } k = N-L \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Solution 4.2: Real DFT. A sufficient condition for the DFT to be real is that the signal is real symmetric. From the definition of symmetry for finite-length signals (see Figure 4.14) it is enough to have $b = d$ for the DFT to be real.

Solution 4.3: Limits. The limit can be "read" as the DTFT of $\cos(\omega_0 n)$ computed in $\omega = 0$ so that, in a signal processing sense,

$$\lim_{N\to\infty} \sum_{n=-N}^{N} \cos(\omega_0 n) = \tilde{\delta}(\omega - \omega_0)|_{\omega=0} = 0.$$

Solution 4.4: Estimating the DFT graphically. By simple visual inspection we can see that $x[n] = a[n] + b[n] + c[n]$ with

$$\begin{aligned} a[n] &= 2 \\ b[n] &= 3\cos(3(2\pi/64)n) \\ c[n] &= \sin(7(2\pi/64)n) = -\cos(7(2\pi/64)n + \pi/2). \end{aligned}$$

The DFT coefficients are $X[k] = A[k] + B[k] + C[k]$, with

$$\begin{aligned} A[k] &= 2N\delta[k] \\ B[k] &= (3N/2)\delta[k-3] + (3N/2)\delta[k-61] \\ C[k] &= -(jN/2)\delta[k-7] + (jN/2)\delta[k-57] \end{aligned}$$

and $N = 64$, so that in the end we have

$$\begin{aligned} X[0] &= 128 \\ X[3] &= 96 \\ X[7] &= -32j \\ X[57] &= 32j \\ X[61] &= 96 \end{aligned}$$

and $X[k] = 0$ for all the other values of k.

Solution 4.5: The structure of DFT formulas. Let $f[n] = \text{DFT}\{x[n]\}$. We have:

$$\begin{aligned} y[n] &= \sum_{k=0}^{N-1} f[k] e^{-j\frac{2\pi}{N}nk} \\ &= \sum_{k=0}^{N-1} \left\{ \sum_{i=0}^{N-1} x[i] e^{-j\frac{2\pi}{N}ik} \right\} e^{-j\frac{2\pi}{N}nk} \\ &= \sum_{i=0}^{N-1} x[i] \sum_{k=0}^{N-1} e^{-j\frac{2\pi}{N}(i+n)k} \end{aligned}$$

Now,

$$\sum_{k=0}^{N-1} e^{-j\frac{2\pi}{N}(i+n)k} = \begin{cases} N & \text{for } (i+n)=0,N,2N,3N,\ldots \\ 0 & \text{otherwise} \end{cases} = N\delta[(i+n) \bmod N]$$

so that

$$y[n] = \begin{cases} Nx[0] & \text{for } n=0 \\ Nx[N-n] & \text{otherwise} \end{cases}$$

In other words, if $x=\{1,2,3,4,5\}$ then DFT{DFT{x}} $=\{1,5,4,3,2\}$

Solution 4.6: Two DFTs for the price of one. Both $X[k]$ and $Y[k]$ have Hermitian symmetry. If we separate real and imaginary parts so that $X[k]=X_R[k]+jX_I[k]$, we have

$$\begin{aligned} X_R[k] &= X_R[N-k] \\ X_I[k] &= -X_I[N-k] \end{aligned}$$

and the same holds for $Y[k]$.
Since the DFT is a linear operator, we have

$$\begin{aligned} C[k] &= X[k]+jY[k] \\ &= X_R[k]+jX_I[k]+jY_R[k]-Y_I[k]; \end{aligned}$$

and by using the symmetry properties for $X[k]$ and $Y[k]$:

$$\begin{aligned} C[N-k] &= X_R[N-k]+jX_I[N-k]+jY_R[N-k]-Y_I[N-k] \\ &= X_R[k]-jX_I[k]+jY_R[k]+Y_I[k] \end{aligned}$$

and therefore

$$\begin{aligned} C^*[N-k] &= X_R[k]+jX_I[k]-jY_R[k]+Y_I[k] \\ &= X[k]-jY[k]. \end{aligned}$$

From this,

$$C[k]+C^*[N-k]=2X[k]$$

and

$$C[k]-C^*[N-k]=2jY[k]$$

Solution 4.7: The Plancherel-Parseval equality.

(a) The inner product in $l_2(\mathbb{Z})$ is defined as

$$\langle x[n], y[n]\rangle = \Sigma_n x^*[n] y[n],$$

and in $L_2([-\pi,\pi])$ as

$$\langle X(e^{j\omega}), Y(e^{j\omega})\rangle = \int_{-\pi}^{\pi} X^*(e^{j\omega})Y(e^{j\omega})d\omega.$$

Thus,

$$\begin{aligned}
\frac{1}{2\pi}\int_{-\pi}^{\pi} X^*(e^{j\omega})Y(e^{j\omega})d\omega &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\Sigma_n x[n]e^{-j\omega n}\right)^* \Sigma_m y[m]e^{-j\omega m}\,d\omega \\
&\overset{(1)}{=} \frac{1}{2\pi}\int_{-\pi}^{\pi} \Sigma_n x^*[n]e^{j\omega n}\Sigma_m y[m]e^{-j\omega m}\,d\omega \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} \Sigma_n\Sigma_m x^*[n]y[m]e^{j\omega(n-m)}d\omega \\
&\overset{(2)}{=} \frac{1}{2\pi}\Sigma_n\Sigma_m x^*[n]y[m]\int_{-\pi}^{\pi} e^{j\omega(n-m)}d\omega \\
&\overset{(3)}{=} \Sigma_n x^*[n]y[n],
\end{aligned}$$

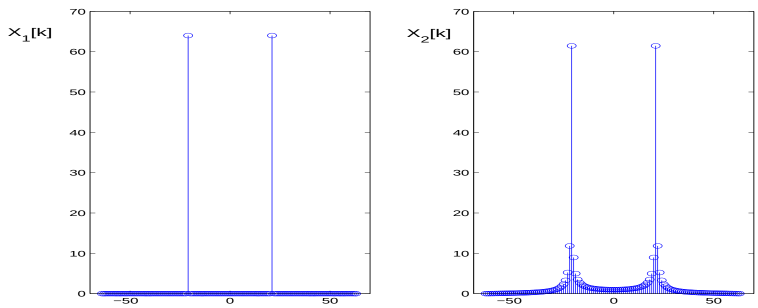
where (1) follows from the properties of the complex conjugate, (2) follows from swapping the integral and the sums and (3) from the fact that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{j\omega(n-m)}d\omega = \begin{cases} 1 & \text{if } m=n \\ 0 & \text{if } m\neq n. \end{cases}$$

(b) If $x[n] = y[n]$, then $\langle x[n], x[n]\rangle$ corresponds to the energy of the signal in the time domain and $\langle X(e^{j\omega}), X(e^{j\omega})\rangle$ to the energy of the signal in the frequency domain. In this case, the Plancherel-Parseval equality illustrates an energy conservation property from the time domain to the frequency domain. This property is known as the *Parseval theorem.*

Solution 4.8: Numerical computation of the DFT. The spectrum of the signal $x[n]$, for both frequencies, is given in the following figure that is obtained using the matlab commands given below.

```
>> N=128;fo1=21/128;fo2=21/127;
>> n=0:N-1;
>> x1=cos(2*pi*fo1*n);x2=cos(2*pi*fo2*n);
>> subplot(223),stem(n-N/2,fftshift(abs(fft(x1))))
>> subplot(224),stem(n-N/2,fftshift(abs(fft(x2))))
```



Since we take the cosine wave example, we expect to see just one sample at the frequency of the signal. This is the case in the left figure, where we have the DFT signal spectrum for the signal with $f_0 = 21/128$, and the 21st DFT coefficient represents the exact signal frequency. However, in the right figure, the frequency of the signal $f_0 = 21/127$ does not coincide with any DFT frequency component. The signal energy is spread over each of the DFT components. This is called frequency leakage. Therefore, we can conclude that if the signal period exactly fits the measurement time (number of samples), the frequency spectrum is correct, while if the period does not match the measurement time, the frequency spectrum is incorrect - it is broadened.

Solution 4.9: DTFT vs. DFT

(a) Note that $x[n]$ can also be expressed as:

$$x[n] = u[n] - u[n-a].$$

Using the DTFT shift property:

$$X(e^{jw}) = \frac{1}{1-e^{-jw}} + \frac{1}{2}\tilde{\delta}(w) - \frac{e^{-jaw}}{1-e^{-jw}} - \frac{e^{-jaw}}{2}\tilde{\delta}(w).$$

Note that $e^{-jaw}\tilde{\delta}(w) = \tilde{\delta}(w)$. Therefore,

$$X(e^{jw}) = \frac{1-e^{-jaw}}{1-e^{-jw}}.$$

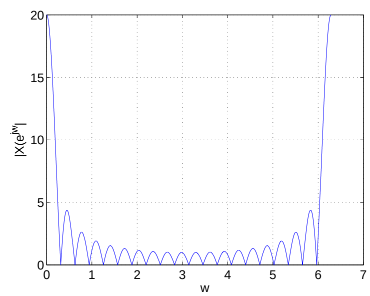


Figure 4.1: DTFT of $x[n]$.

(b) To visualize the magnitude of $X(e^{jw})$ using Matlab we generate 10000 points of one period of $|X(e^{jw})|$ (from 0 to 2π) for $a = 20$:

```
>> n=1:10000;
>> w=(n.*2*pi/max(n));
>> X=((1-exp(-j.*w.*20))./(1-exp(-j.*w)));
>> plot(w,abs(X));
```

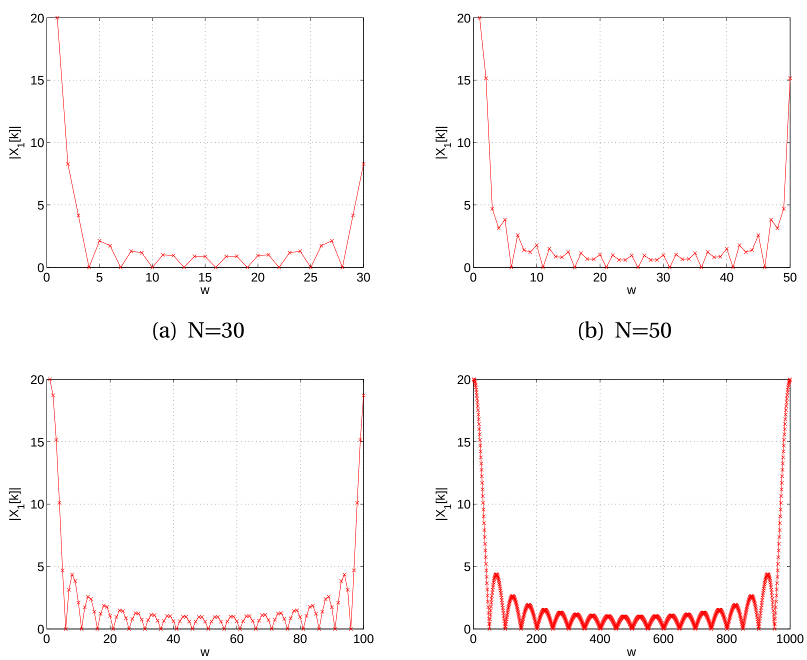
The result is shown in Fig. 4.1.

(c)

```
>> N=30;
>> x1=[ones(1,20),zeros(1,N-20)];
>> X1=fft(x1);
>> plot(abs(X1));
```

The result for different values of N is shown in Fig. 4.2.

In Fig. 4.2 we see that, as we increase the length N of $x_1[n]$, the DFT becomes closer and closer to the DTFT of $x[n]$. As we have seen in the course notes, the DFT and the DFS are formally identical, and as N grows, the DFS converges to the DTFT. Therefore, we can use Matlab to approximate the DTFT of any signal by the DFT of a finite sequence using a large enough length N.



(a) N=30 (b) N=50

(c) N=100 (d) N=1000

Figure 4.2: DFT of $x_1[n]$ for different values of N.

Chapter 5

Solution 5.1: Linearity and time-invariance (I). The system is not time-invariant. To see this consider the following signals:

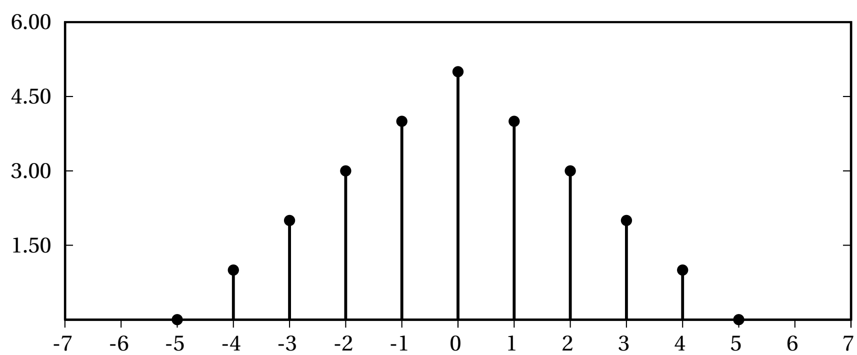
$$\begin{aligned} x[n] &= \delta[n] \\ y[n] &= \delta[n-1] \end{aligned}$$

We have $\mathscr{H}\{x[n]\} = w[n] = 0$ and, clearly, it is $y[n] = x[n-1]$; however, $\mathscr{H}\{y[n]\} = \delta[n-1] \neq w[n-1] = 0$.

Solution 5.2: Linearity and time-invariance (II). LTI system cannot change the frequency of a sinusoidal input, only its magnitude and phase. Since he input contains frequencies only at $\pm 0.4\pi$ while the output only at $\pm 0.5\pi$, the system cannot be LTI.

Solution 5.3: Finite-support convolution.

(a) The signal is the triangle:



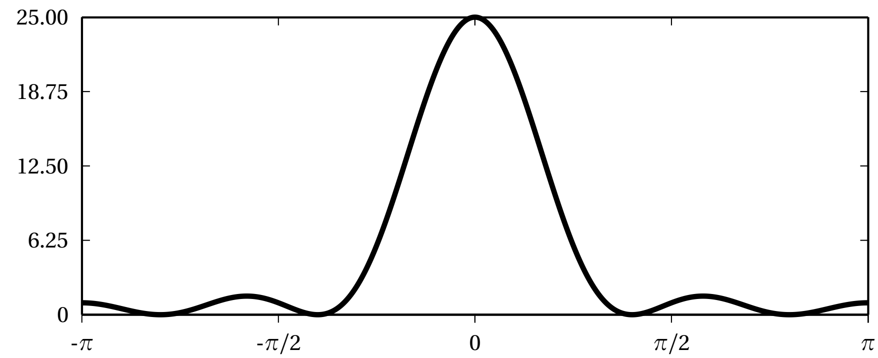
(b) We know (see page 74) that, in general,

$$H(e^{j\omega}) = \frac{\sin(\omega(M+0.5))}{\sin(\omega/2)};$$

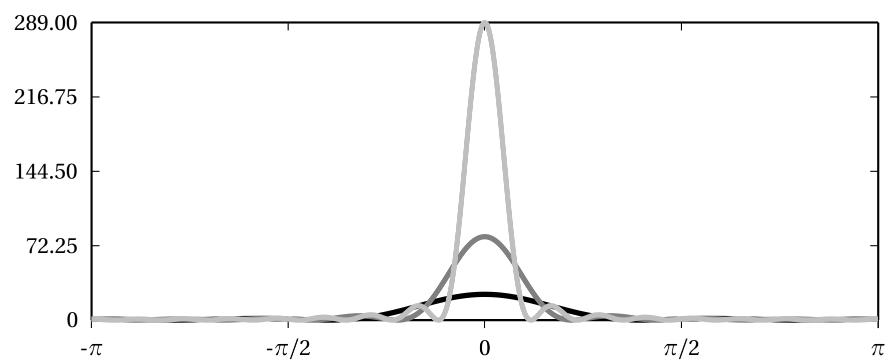
from the convolution theorem, it is simply

$$X(e^{j\omega}) = \left(\frac{\sin(\omega(M+0.5))}{\sin(\omega/2)}\right)^2;$$

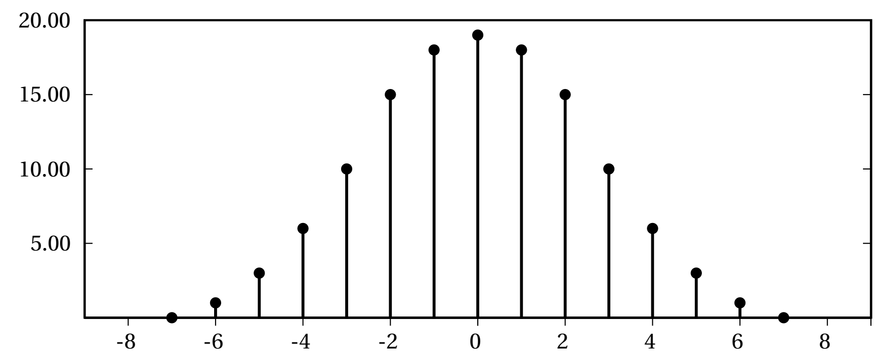
For $M = 2$ we have



(c) As M grows, the spectrum will exhibit more and more ripples and its peak will grow (it is always at $(2M+1)^2$):



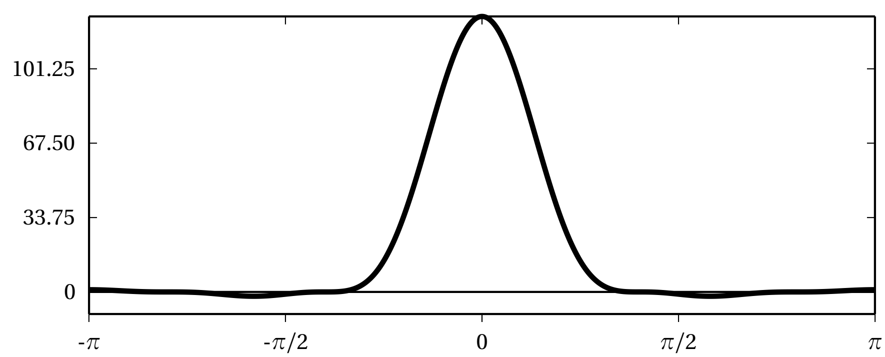
(d) The signal has a quadratic shape:



(e) Again, from the modulation theorem

$$X(e^{j\omega}) = \left(\frac{\sin(\omega(M+0.5))}{\sin(\omega/2)} \right)^3;$$

The DTFT is real and its plot is like this (for $M = 2$)



Solution 5.4: Convolution (I).

(a) The discrete-time sequence $x[n]$ can be written as the convolution of $x_1[n]$ and $x_2[n]$ defined as

$$x_1[n] = x_2[n] = \begin{cases} 1 & -(M-1)/2 \leq n \leq (M-1)/2 \\ 0 & \text{otherwise.} \end{cases}$$

In fact,

$$\begin{aligned} x_1[n] * x_2[n] &= \Sigma_k x_1[k] x_2[n-k] \\ &\stackrel{(1)}{=} \Sigma_k x_1[k] x_1[k-n] \\ &\stackrel{(2)}{=} x[n] \end{aligned}$$

where (1) follows from the fact that $x_1[n] = x_2[n]$ and from the symmetry of $x_1[n]$ and (2) noticing that the sum corresponds to the size of the overlapping area between $x_1[k]$ and its n-shifted version $x_1[k-n]$. When $|n| \geq M$ the two sequences do not overlap whereas the size of the overlapping area reaches its maximum M when $n = 0$.

Using Matlab, we can easily verify the above result for $M = 11$ using the following code:

```
>> M = 11;
>> x1 = ones(1,M);
>> x2 = x1;
>> x = conv(x1,x2);
>> stem([-M+1:M-1], x);
```

(b) Note that $x_1[n] = u[n+(M-1)/2] - u[n-(M+1)/2]$. We can thus compute its DTFT as

$$\begin{aligned} X_1(e^{j\omega}) &\stackrel{(1)}{=} \left(\frac{1}{1-e^{-j\omega}} + \frac{1}{2}\tilde{\delta}(\omega)\right)\left(e^{j\omega(M-1)/2} - e^{-j\omega(M+1)/2}\right) \\ &\stackrel{(2)}{=} \frac{e^{j\omega(M-1)/2} - e^{-j\omega(M+1)/2}}{1-e^{-j\omega}} = \frac{e^{-j\omega/2}(e^{j\omega M/2} - e^{-j\omega M/2})}{e^{-j\omega/2}(e^{j\omega/2} - e^{-j\omega/2})} \\ &= \frac{\sin(\omega M/2)}{\sin(\omega/2)} \end{aligned}$$

where (1) follows from the DTFT of $u[n]$ and (2) from the fact that

$$e^{jw(M-1)/2}\tilde{\delta}(w) = e^{-jw(M+1)/2}\tilde{\delta}(w) = \tilde{\delta}(w).$$

Using the convolution theorem, we can write

$$\begin{aligned} X(e^{jw}) &= X_1(e^{jw})X_2(e^{jw}) \\ &= X_1(e^{jw})X_1(e^{jw}) \\ &= \left(\frac{\sin(\omega M/2)}{\sin(\omega/2)}\right)^2. \end{aligned}$$

Solution 5.5: Convolution (II). We have that $(x[n])^2 = (1/2)(1+\cos(3n))$, while $y[n]$ is the impulse response of an ideal lowpass with cutoff frequency $\pi/5$. Therefore:

$$(x[n])^2 * y[n] = 1/2.$$

Solution 5.6: System properties.

(a) $y[n] = x[-n]$

- **Linearity:** $\mathscr{H}\{ax_1[n]+bx_2[n]\} = ax_1[-n]+bx_2[-n] = a\mathscr{H}\{x_1[n]\}+b\mathscr{H}\{x_2[n]\}$. Therefore, $\mathscr{H}$ is linear.
- **Time-Invariance:** $\mathscr{H}\{x[n-n_0]\} = x[-n-n_0] \neq y[n-n_0]$. Therefore, $\mathscr{H}$ is not time invariant.
- **Stability:** If $|x[n]| \leq M$, then $|\mathscr{H}\{x[n]\}| \leq M$. Therefore, $\mathscr{H}$ is BIBO stable.
- **Causality:** $\mathscr{H}$ is not causal.
- **Impulse Response:** $\mathscr{H}$ is not LTI, therefore the response to the impulse does not characterize the system.

(b) $y[n] = e^{-j\omega n}x[n]$

- **Linearity:** $\mathscr{H}\{ax_1[n]+bx_2[n]\} = e^{-j\omega n}(ax_1[n]+bx_2[n]) = a\mathscr{H}\{x_1[n]\}+b\mathscr{H}\{x_2[n]\}$. Therefore, $\mathscr{H}$ is linear.
- **Time-Invariance:** $\mathscr{H}\{x[n-n_0]\} = e^{-j\omega n}x[n-n_0] = e^{j\omega n_0}y[n-n_0]$. Therefore, $\mathscr{H}$ is not time invariant (except for $\omega = 0$).
- **Stability:** If $|x[n]| \leq M$, then $|\mathscr{H}\{x[n]\}| = |x[n]| \leq M$. Therefore, $\mathscr{H}$ is BIBO stable.
- **Causality:** $\mathscr{H}$ is causal.
- **Impulse Response:** $\mathscr{H}$ is not LTI, therefore the response to the impulse does not characterize the system.

(c) $y[n] = \sum_{k=n-n_0}^{n+n_0} x[k]$

- **Linearity:** $\mathscr{H}\{ax_1[n]+bx_2[n]\} = \sum_{k=n-n_0}^{n+n_0}(ax_1[k]+bx_2[k]) = a\mathscr{H}\{x_1[n]\}+b\mathscr{H}\{x_2[n]\}$. Therefore, $\mathscr{H}$ is linear.

- **Time-Invariance:** $\mathscr{H}\{x[n-n_0]\} = \sum_{k=n-n_0}^{n+n_0} x[k-n_0] = \sum_{k=n-2n_0}^{n} x[k] = y[n-n_0]$. Therefore, $\mathscr{H}$ is time invariant.
- **Stability:** If $|x[n]| \le M$, then $\mathscr{H}\{x[n]\} \le |2n_0+1|M$. Therefore, $\mathscr{H}$ is BIBO stable
- **Causality:** $\mathscr{H}$ is not causal.
- **Impulse Response:** If $x[n] = \delta[n]$, $y[n] = h[n]$:

$$h[n] = \begin{cases} 1 & \text{if } |n| \le |n_0|, \\ 0 & \text{otherwise.} \end{cases}$$

(d) $y[n] = ny[n-1] + x[n]$, such that if $x[n] = 0$ for $n < n_0$, then $y[n] = 0$ for $n < n_0$. Since $\mathscr{H}$ is recursive, we can not use the same technique as before. Note that all inputs $x[n]$ can be expressed as a linear combination of delayed impulses: $x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$. Therefore, to show that $\mathscr{H}$ is linear or time invariant, we can restrict the input to delayed impulses.

- **Linearity:** If $x[n] = \delta[n]$, we can obtain $y[n]$ by recursion:

  $$h[n] = y[n] = n!u[n].$$

  If $x[n] = a\delta[n] + b\delta[n]$:

  $$y[n] = (a+b)n!u[n].$$

  Therefore, $\mathscr{H}$ is linear.
- **Time-Invariance:** consider $x[n] = \delta[n-1]$. It is easy to check that $\mathscr{H}\{\delta[n-1]\} \ne h[n-1]$. Therefore, $\mathscr{H}$ is not time invariant.
- **Stability:** The system is non stable since $\mathscr{H}\{\delta[n]\} \to \infty$.
- **Causality:** $\mathscr{H}$ is causal.
- **Impulse Response:** $\mathscr{H}$ is not LTI, therefore the response to the impulse does not characterize the system.

Solution 5.7: Ideal filters. Consider a lowpass filter $h_{lp}[n]$ with bandwidth ω_b. If we consider the sequence

$$h = 2\cos(\omega_0 n)h_{lp}[n]$$

the Modulation theorem tells us that its Fourier transform is

$$H(e^{j\omega}) = H_{lp}(e^{j(\omega-\omega_0)}) + H_{lp}(e^{j(\omega+\omega_0)}) = H_{bp}(e^{j\omega})$$

Therefore the impulse response of the bandpass filter is

$$h_{bp}[n] = 2\cos(\omega_0 n) h_{lp}[n] = 2\cos(\omega_0 n)\,\frac{\omega_b}{2\pi}\,\mathrm{sinc}\left(\frac{\omega_b}{2\pi} n\right)$$

Solution 5.8: Zero-phase filtering.

(a) Consider the sequence $x[n] = \delta[n-1]$; we should have $\mathscr{R}\{x[n]\}[n] = \mathscr{R}\{\delta[n]\}[n-1]$ but instead it is

$$\begin{aligned} \mathscr{R}\{x[n]\}[n] &= x[-n] = \delta[-(n+1)] = \delta[n+1] \\ \mathscr{R}\{\delta[n]\}[n-1] &= \delta[n-1] \end{aligned}$$

(b) First of all recall that the DTFT of $x[-n]$ is $X(e^{-j\omega})$; if $x[n]$ is real, we also have $X(e^{j\omega}) = X^*(e^{-j\omega})$. In the frequency domain we therefore have:

   (a) $S(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega})$
   (b) $R(e^{j\omega}) = S(e^{-j\omega}) = H^*(e^{j\omega})X(e^{-j\omega})$ since $h[n]$ is real.
   (c) $W(e^{j\omega}) = H(e^{j\omega})R(e^{j\omega}) = |H(e^{j\omega})|^2 X(e^{-j\omega})$
   (d) $Y(e^{j\omega}) = W(e^{-j\omega}) = |H(e^{j\omega})|^2 X(e^{j\omega})$

   Therefore the chain of transformations defines an LTI filter $\mathscr{G}$ with frequency response $G(e^{j\omega}) = |H(e^{j\omega})|^2$. The corresponding impulse response is simply

$$g[n] = h[n] * h[-n]$$

   What is interesting to note here is that, even though $\mathscr{R}$ is not time invariant, we can combine time variant operators into an overall time-invariant transformation.

(c) $G(e^{j\omega})$ is a real function, therefore its phase is zero.

Solution 5.9: Nonlinear signal processing.

(a) $\mathscr{H}\{\delta[n]\} = \delta[n]$; but $\mathscr{H}\{a\delta[n]\} = a^2\delta[n] \neq a\mathscr{H}\{\delta[n]\}$.

(b) Let $y[n] = \mathscr{H}\{x[n]\}$; let $w[n] = x[n-n_0]$; $\mathscr{H}\{w[n]\} = w^2[n] = x^2[n-n_0] = y[n-n_0]$. QED.

(c) First of all, $y[n] = \cos^2(\omega_0 n) = (1+\cos(2\omega_0 n))/2$ from the well-known trigonometric identity. So $y[n]$ contains a sinusoid at *double* the original frequency (but be careful: double in the 2π-periodic sense: if ω_0 is larger than $\pi/2$, then $2\omega_0$ will wrap around the $[-\pi,\pi]$ interval).

If $\omega_0 = 3\pi/8$, then $y[n] = (1+\cos((3\pi/4)n))/2$; since $\mathscr{G}$ is a highpass with cutoff frequency $\pi/2$, it will kill the frequency components below $\pi/2$ and therefore it will kill the constant. The only component that passes through is the cosine at $3\pi/4$. The final output is therefore $v[n] = \frac{1}{2}\cos((3\pi/4)n)$.

(d) If $\omega_0 = 7\pi/8$, then $2\omega_0 = 7\pi/4 > \pi$. We can therefore bring back the frequency into the $[-\pi,\pi]$ interval. We have that $7\pi/4 = 2\pi - \pi/4$ and therefore $\cos((7\pi/4)n) = \cos((2\pi - \pi/4)n) = \cos((\pi/4)n)$. So in the end $y[n] = (1+\cos((\pi/4)n))/2$. Now the frequency of the cosine is below $\pi/2$ and therefore $v[n] = 1+\cos((\pi/4)n)$. Note that, as for most nonlinear systems, the frequency of the input sinusoid is different from the frequency of the output sinusoids: sinusoids are no longer eigenfunctions!

Chapter 6

Solution 6.1: Interleaving.

(a) We have clearly:

$$\begin{aligned} X(z) &= \sum_{n=-\infty}^{\infty} h[n]z^{-2n} + g[n]z^{-(2n+1)} \\ &= H(z^2) + z^{-1}G(z^2) \end{aligned}$$

(b) The ROC is determined by the zeros of the transform. Since the sequence is two sided, the ROC is a ring bounded by two poles z_L and z_R such that $|z_L| < |z_R|$ and no other pole has magnitude between $|z_L|$ and $|z_R|$. Consider $H(z)$; if z_0 is a pole of $H(z)$, $H(z^2)$ will have two poles at $\pm z^{1/2}$; however, the square root preserves the monotonicity of the magnitude and therefore no new poles will appear between the circles $|z| = \sqrt{|z_L|}$ and $|z| = \sqrt{|z_R|}$. Therefore the ROC for $H(z^2)$ is the ring $|z_L| < |z| < |z_R|$. The ROC of the sum $H(z^2) + z^{-1}G(z^2)$ is the intersection of the ROCs, and so

$$\text{ROC} = 0.8 < |z| < 2.$$

Solution 6.2: Properties of the z-transform.

(a) Let $H(z) = \Sigma_n h[n]z^{-n}$. We have that

$$\begin{aligned} \frac{d}{dz}H(z) &= \frac{d}{dz}\left(\Sigma_n h[n]z^{-n}\right) \\ &= \Sigma_n(-n)h[n]z^{-n-1} \\ &= -z^{-1}\Sigma_n n h[n]z^{-n} \end{aligned}$$

and the relation follows directly.

(b) We have that

$$\alpha^n u[n] \stackrel{Z}{\longleftrightarrow} \frac{1}{1-\alpha z^{-1}}.$$

Using (a) we find

$$n\alpha^n u[n] \stackrel{Z}{\longleftrightarrow} -z\frac{d}{dz}\left(\frac{1}{1-\alpha z^{-1}}\right) = \frac{\alpha z^{-1}}{(1-\alpha z^{-1})^2}.$$

Thus,

$$(n+1)\alpha^{n+1} u[n+1] \stackrel{Z}{\longleftrightarrow} z\frac{\alpha z^{-1}}{(1-\alpha z^{-1})^2}$$

and

$$(n+1)\alpha^n u[n+1] \stackrel{Z}{\longleftrightarrow} \frac{1}{(1-\alpha z^{-1})^2}.$$

The relation follows by noticing that

$$(n+1)\alpha^n u[n+1] = (n+1)\alpha^n u[n]$$

since when $n = -1$ both sides are equal to zero.

(c) The system is causal since the ROC corresponds to the outside of a circle of radius α (or equivalently since the impulse response is zero when $n < 0$). The system is stable when the unit circle lies inside the ROC, i.e. when $|\alpha| \leq 1$.

(d) When $\alpha = 0.8$, the angular frequency of the pole is $\omega = 0$. Thus the filter is lowpass. When $\alpha = -0.8$, $\omega = \pi$ and the filter is highpass.

Solution 6.3: Stability.

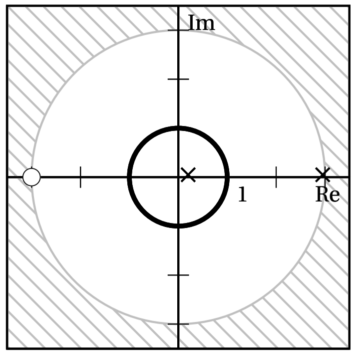


Figure 6.1: Pole-zero plot and ROC

(a) The transfer function can be obtained by taking the z-transform of the CCDE:

$$Y(z)(1-3.25z^{-1}+0.75z^{-2})=X(z)(z^{-1}+3z^{-2})$$

which leads to

$$H(z)=\frac{Y(z)}{X(z)}=\frac{z^{-1}+3z^{-2}}{1-3.25z^{-1}+0.75z^{-2}}=\frac{z+3}{(z-0.25)(z-3)}$$

Since the system is causal, the convergence region is $|z|>3$. We can see that the pole at $z=3$ that is outside of the unit circle and therefore the system is unstable. (Figure 6.1).

(b) the z-transform of the output signal is:

$$\begin{aligned} Y(z) &= H(z)X(z) \\ &= \frac{z^{-1}(1+3z^{-1})}{(1-0.25z^{-1})(1-3z^{-1})}(1-3z^{-1}) \\ &= \frac{z^{-1}+3z^{-2}}{1-0.25z^{-1}}. \end{aligned}$$

From $Y(z)$ we can see that the unstable pole at $z=3$ is cancelled and only the pole at $z=0.25$ is left. Since the system is causal, a stable output can be obtained with a suitable input signal. if the unstable pole is canceled by the input signal.

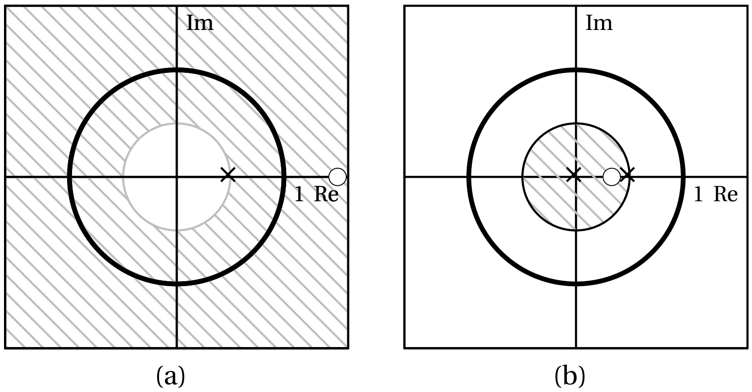


Figure 6.2: Pole-zero plot and ROC

Solution 6.4: Pole-zero plot and stability (I). The transfer function can be rewritten as

$$H(z) = \frac{1 - 2.25z^{-2}}{(1-0.5z^{-1})(1+1.5z^{-1})} = \frac{1-1.5z^{-1}}{1-0.5z^{-1}}$$

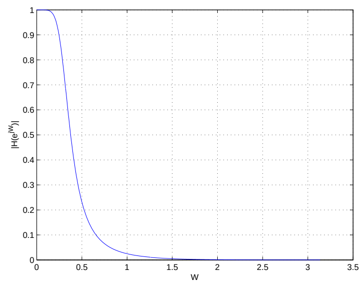
There is a pole in 0.5 and a zero in 1.5; the system is therefore stable, with the pole-zero plot and ROC as in Figure 6.2-(a).

Solution 6.5: Pole-zero plot and stability (II). The transfer function can be rewritten as

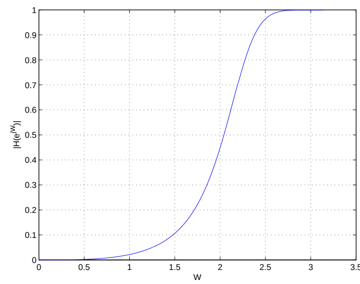
$$H(z) = 1.5z^{-1}\frac{1-z^{-1}}{1-(1/3)z^{-1}}$$

so that the zeros are $z_1 = 0$ and $z_2 = 1/3$ and the only pole is in $p_1 = 0.5$ (of course there's also a pole in zero, but it doesn't affect stability so we don't consider it). The pole-zero plot and the ROC are shown in Figure 6.2-(b) (the system is anticausal). The system is not stable.

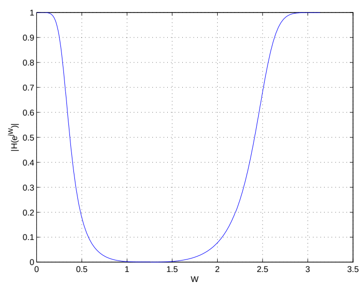
Solution 6.6: Pole-zero plot and magnitude response. To obtain the frequency response of a filter, we analyze the z-transform over the unit circle, that is, for $z = e^{j\omega}$. Figure 6.3 shows the magnitude response of the three filters:



(a) Diagram 1



(b) Diagram 2



(c) Diagram 3

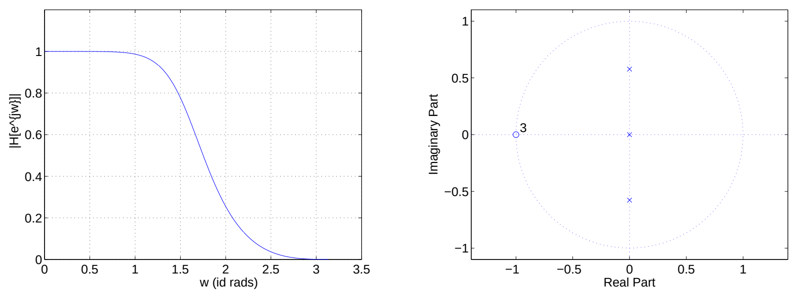
Figure 6.3: Zeros and Poles Diagrams

(a) The first filter is a low-pass filter. Note that there are three poles located in low frequency (near $\omega = 0$), while there is a zero located in high frequency ($\omega = \pi$).

(b) The second filter is just the opposite. The zero is located in low frequency, while the influence of the three poles is maximum in high frequency ($\omega = \pi$). Therefore, it is a high-pass filter.

(c) In the third system, there are poles which affect low and high frequency and two zeros close to $w = \pi/2$. Therefore, this system is a band-stop filter.

Solution 6.7: Z-transform and magnitude response. We will use Matlab to solve this exercise.



(a) Frequency Response (b) Poles and Zeros

Figure 6.4: Filter Descriptions.

(a) To compute the frequency response $H(e^{j\omega})$ we can write:

```
>> B=[1/6,1/2,1/2,1/6];
>> A=[1,0,1/3];
>> [H,W]=freqz(B,A,1000);
>> plot(W,abs(H));
```

Fig. 6.4(a) shows the frequency response between 0 and π. Clearly, $H(e^{j\omega})$ is a "low-pass" filter.

(b) To compute and plote the zeros and poles:

```
>> zplane(B,A)
```

The result is shown in Fig. 6.4(b). Given that the system is causal, the ROC extends outward from the outermost pole. Therefore, the ROC includes the unitary circle and the system is stable.

(c) We can generate $x[n]$ as

```
>> x=[zeros(1,50),ones(1,128-50)]
>> X=fft(x);
>> plot(abs(X))
```

and we can compute $y[n]$ by filtering $x[n]$:

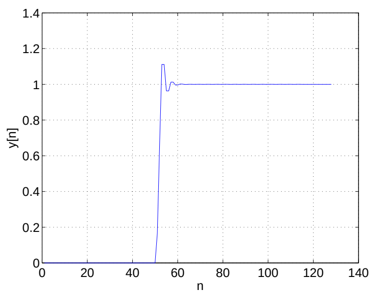


Figure 6.5: Filtered signal.

```
>> y=filter(B,A,x);
>> plot(y);
```

$y[n]$ is plotted in Fig. 6.5.

(d) $y[n]$ is a low-pass filtered version of $x[n]$, where the high frequencies have been removed. Note that the sharp transition between zero and one in the input has been smeared in time by the lowpass filter.

Solution 6.8: DFT and z-transform. Let $X(z)$ be the z-transform of the finite-support extension of $\mathbf{x}$:

$$X(z) = \sum_{n=0}^{N-1} x[n] z^{-n}$$

It is easy to see that the z-transform of the finite-support extension of $\mathbf{x}_r$ is simply:

$$X_r(z) = z^{-(N-1)} X(z^{-1});$$

for example, if $\mathbf{x} = [a\ b\ c]^T$:

$$\begin{aligned} X(z) &= a + bz^{-1} + cz^{-2} \\ X_r(z) &= c + bz^{-1} + az^{-2} = z^{-2}(cz^2 + bz + a) = z^{-2} X(z^{-1}) \end{aligned}$$

Therefore we have

$$\begin{aligned}
X_r[k] &= X_r(z)|_{z=W_N^{-k}} \\
&= z^{-(N-1)}X(z^{-1})|_{z=W_N^{-k}} \\
&= W_N^{(N-1)k}X(W_N^{k}) \\
&= W_N^{-k}X(W_N^{-(N-k)}) \\
&= \begin{cases} X[0] & \text{for } k=0 \\ W_N^{-k}X[N-k] & \text{for } k=1,\dots,N-1 \end{cases}
\end{aligned}$$

In other words, the DFT of the time-reversed signal $\mathbf{x}_r$ is the reversed DFT of $\mathbf{x}$ (reversed in the circular sense) and scaled by the weights W_N^{-k}.

Solution 6.9: A CCDE. Taking the z-transform of both sides,

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{z^{-1}+0.25z^{-2}}$$

whose pole-zero plot and ROC are shown in Figure 6.6, assuming the system is "causal". Note that the current value of the output ($y[n]$) depends structurally on the one-step-ahead value of the input $x[n+1]$ and therefore the filter is not strictly causal. It can be implemented offline by processing the whole input and then by delaying the output by one. One easy way to understand the concept is by computing the impulse response; if we write

$$H(z) = z\,\frac{1}{1+0.25z^{-1}}$$

we notice that the second term in the above expression is the z-transform of the sequence $a[n] = (-0.25)^n\, u[n]$ and that the leading z term is just an advance operator. Therefore

$$h[n] = \begin{cases} 0 & \text{for } n < -1 \\ (-2.5)^{n+1} & \text{for } n \geq -1 \end{cases}$$

which is nonzero for $n = -1$, i.e. not strictly causal.

Solution 6.10: Inverse transform. We have:

$$X(z) = 2z + 7 + 3z^{-1}$$

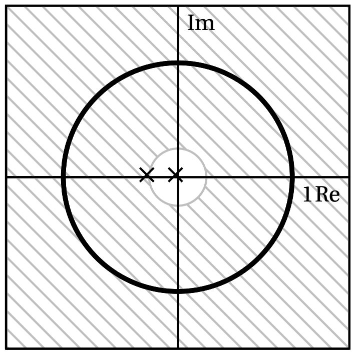


Figure 6.6: Pole-zero plot and ROC

and therefore

$$x[n] = \begin{cases} 2 & \text{for } n = -1 \\ 7 & \text{for } n = 0 \\ 3 & \text{for } n = 1 \\ 0 & \text{otherwise} \end{cases}$$

Solution 6.11: Signal transforms. To efficiently evaluate $X(z)$ on the unit circle first note that $X(z)$ is "symmetrical" around the missing term z^{-2} so that we can write

$$X(z) = z^{-2}(z^1 + z^2 + z^{-2} + z^{-1});$$

then remember that

$$[z^{-m} + z^m]_{z=e^{j\omega}} = 2\cos(m\omega).$$

Therefore,

$$X(e^{j\omega}) = 2e^{-j2\omega}[\cos\omega + \cos 2\omega]$$

whose magnitude is shown in Figure 6.7.
Finally, from the definition,

$$y[n] = [1 \quad 1 \quad 0 \quad 1]^T.$$

Therefore, after a few simple calculations,

$$Y[k] = [3 \quad 1 \quad -1 \quad 1]^T$$

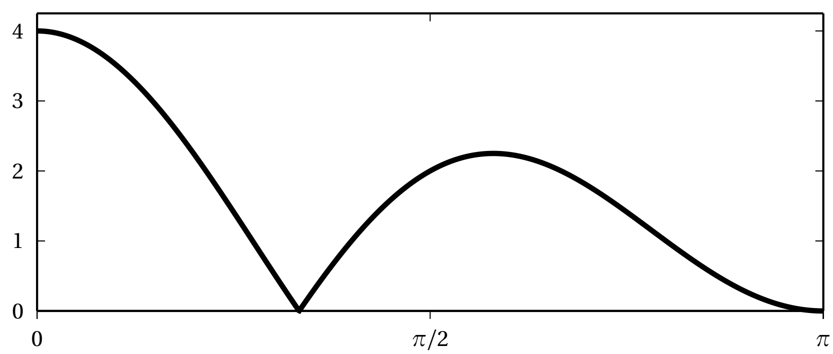


Figure 6.7: Magnitude frequency response

Chapter 7

Solution 7.1: Discrete-time systems and stability.

(a)

$$\begin{aligned}
y_0[n] &= x[n] + y_0[n-1] \\
y_1[n+1] &= y_1[n] + x[n] &&\Rightarrow y_1[n] = y_1[n-1] + x[n-1] \\
y[n] &= y_0[n] - y_1[n] &&\Rightarrow y[n] = (y_0[n-1] - y_1[n-1]) + x[n] - x[n-1] \\
& && \quad = y[n-1] + x[n] - x[n-1]
\end{aligned}$$

(b)

$$\begin{aligned}
H_0(z) &= \frac{1}{1-z^{-1}} \\
H_1(z) &= \frac{z^{-1}}{1-z^{-1}} \\
H(z) &= \frac{1-z^{-1}}{1-z^{-1}} = 1
\end{aligned}$$

(c) The system is BIBO stable since $y[n] = x[n]$.

(d) The system is not internally stable (the subsystems have poles on the unit circle.

(e) According to the CCDES,

$$\begin{aligned}
y_0[n] &= n\,u[n] \to \infty \\
y_1[n] &= (n-1)\,u[n] \to \infty \\
y[n] &= u[n]
\end{aligned}$$

(f) Although the transfer function is theoretically unity, the delay elements in the system, if implemented with finite precision, will at one point overflow since the output values of the two branches increase without limit. If binary arithmetic is used, the system may or may not work after an overflow according to the binary representation used. In general, it will not work.

Solution 7.2: Filter properties (I).

- False. The inverse filter is stable only if all the zeros of $G(z)$ are inside the unit circle; this is not true in general.
- False. The inverse filter is FIR only if $G(z)$ has no zeros; this is not true in general.
- True. If the filter is stable, the ROC of $G(z)$ includes the unit circle.
- True. The poles of the cascade double their multiplicity but remain inside the unit circle.

Solution 7.3: Filter properties (II).

- False. Zeros do not affect stability, therefore they can be anywhere for a stable system.
- False. The ROC includes the unit circle and extends outwards, therefore it includes all circles of radius greater than one, but not necessarily a circle of radius 0.5.
- True. Adding a zero does not affect stability.
- False. The system described does not necessarily have poles.

Solution 7.4: Fourier transforms and filtering.

(a) Starting at $n = 0$, a few values of the signal are

$$x[n] = \quad \ldots \quad -1, \quad 0, \quad 1, \quad 0, \quad -1, \quad 0, \quad 1, \quad 0, \quad -1, \quad 0, \quad 1, \quad 0, \quad \ldots$$

and it is easy to recognize that

$$x[n] = -\cos((\pi/2)n)$$

(b) $x[n]$ is 4-periodic, therefore the most appropriate Fourier representation is the DFS

$$\begin{aligned}\tilde{X}[k] &= \sum_{n=0}^{3} x[n] e^{-j\frac{2\pi}{4}nk} \\ &= -1 + e^{-j\pi k}, \quad k = 0,1,2,3 \\ &= [0 \quad -2 \quad 0 \quad -2]^T\end{aligned}$$

(c) We can then use the DFS-DTFT formula to obtain:

$$\begin{aligned}\tilde{X}(e^{j\omega}) &= \frac{1}{N}\sum_{k=0}^{N-1} X[k]\tilde{\delta}(\omega - \frac{2\pi}{N}k) \\ &= -\frac{1}{2}\tilde{\delta}(\omega - \frac{\pi}{2}) - \frac{1}{2}\tilde{\delta}(\omega - \frac{3\pi}{2})) \\ &= -\frac{1}{2}[\tilde{\delta}(\omega - \frac{\pi}{2}) + \tilde{\delta}(\omega + \frac{\pi}{2})]\end{aligned}$$

where in the last passage we have brought back the last delta onto the $[-\pi, \pi]$ interval. This is consistent with the definition of the DTFT of a cosine of frequency $\pi/2$

(d) The impulse response is clearly a sinc. We know that a response of the type

$$h_{lp}[n] = \frac{\omega_c}{\pi}\,\mathrm{sinc}\left(\frac{\omega_c}{\pi}n\right).$$

identifies a lowpass filter with cutoff frequency ω_c and passband magnitude of one. In our case we can rewrite the impulse response as:

$$h[n] = \frac{1}{\pi}\,\mathrm{sinc}\left(\frac{1}{\pi}n\right).$$

which indicates a lowpass filter with cutoff frequency of one radian. This filter will kill the deltas at frequencies above $\omega = 1$ and leave the rest unchanged. Since $\pi/2 > 1$, $y[n] = 0$.

Solution 7.5: FIR filters.

(a) First of all note that $1-2^{-k}=(2^k-1)/2^k$. With this we find that

$$\begin{aligned} z_1 &= e^{j\frac{1}{2}\pi} \\ z_2 &= e^{j\frac{3}{4}\pi} \\ z_3 &= e^{j\frac{7}{8}\pi} \\ z_4 &= e^{j\frac{15}{16}\pi} \end{aligned}$$

which are simply four points in the second quadrant on the unit circle.

(b) Each couple of complex-conjugate zeros contributes a factor of the form $(1-2z^{-1}\cos\theta+z^{-2})$ to the transfer function, where θ is the angle of the complex zero. We have in the end:

$$H(z)=(1+z^{-1})(1+z^{-2})(1-2\cos(\frac{3}{4}\pi)z^{-1}+z^{-2})$$

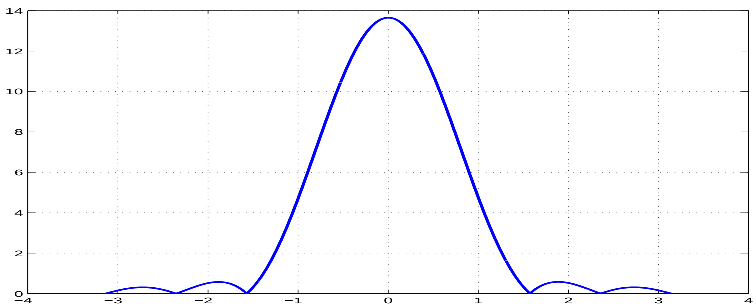
(c) $H(z)$ is a 5^{th} degree polynomial in z^{-1} and therefore it has at most 6 nonzero coefficients. The impulse response will have 6 nonzero taps.

(d) You don't even need Matlab to do this. First of all, the impulse response is real and therefore the magnitude of $H(e^{j\omega})$ is symmetric. Consider now the values of the frequency response at zero and π; these are computed from the z-transform for $z=1$ and $z=-1$ respectively; we have:

$$\begin{aligned} H(e^{j0}) &= H(1)=2\cdot 2\cdot 2(1-\cos(\frac{3}{4}\pi))\approx 13.6 \\ H(e^{j\pi}) &= H(-1)=0 \end{aligned}$$

Next, you need to consider that $H(z)$ is zero on the unit circle at z_1 and z_2, i.e. at $\omega=\pi/2$ and $\omega=3\pi/4$. Now you can plot the magnitude:

(e) First of all, is the filter linear phase? You can compute the coefficient of the transfer function and verify that $h[n]=1,2.4142,3.4142,3.4142,2.4142,1$ for $n=0,\ldots,5$. In a simpler way, you can simply notice that $H(z^{-1})=$



$z^5 H(z)$ and therefore the filter is linear phase, symmetric. The filter has an even number of taps and therefore it is Type II.

Because of the zero in π and the large value in zero, the filter is lowpass. However, it is not equiripple since the magnitude at the peak of the first sidelobe in the stopband is higher than the peak of the second sidelobe.

The filter is clearly not a good filter: the transition band is very large, it is not flat in the passband and the magnitude is rather large in the stopband.

Solution 7.6: Linear-phase FIR filter structure.

$z_1 = z_0^*$

$z_2 = 1/z_0$

$z_3 = 1/z_0^*$

$z_4 = 1$

$z_5 = -1$

Solution 7.7: FIR filters analysis (I).

(a) The filter is clearly symmetric around $n = 4$ and therefore it is linear phase. It has an odd number of taps ($M = 9$) and therefore it is Type I.

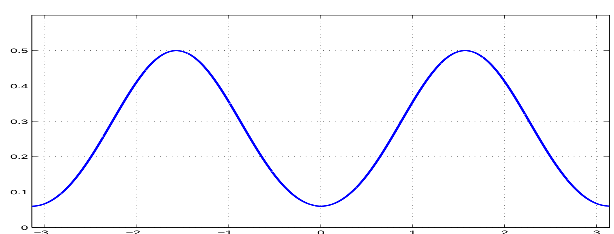
(b) The frequency response of the filter will be of the form $H(e^{j\omega}) = A(e^{j\omega})e^{-j\frac{M-1}{2}\omega}$, with $A(e^{j\omega}) \in \mathbb{R}$. The phase is therefore $\varphi(\omega) = -((M-1)/2)\omega$ and the group delay is constant and equal to $(M-1)/2 = 4$ samples.

(c) For the filter to be optimal in the Parks-McClellan sense, its magnitude response must exhibit at least $L+2$ alternations between zero and π, where $L = (M-1)/2 = 4$. Two alternations occur at the passband edge and at the stopband edge, and the others must occur at local maxima of the magnitude response. In other words, in our case the magnitude response should exhibit at least 4 local maxima distributed between the passband and stopband. We can see from the plot, however, that there are only two local maxima, one in $\omega = 0$ and another one in $\omega \approx 2.3$. The filter is therefore not optimal.

(d) We can easily see that

$$g[n] = h[n]\cos((\pi/2)n).$$

The filter $G(z)$ is therefore a modulated version of the original filter. The modulation shifts the lowpass response over to $\pi/2$ and the resulting filter is passband and the magnitude response looks like this:



(e) The transfer function of the whole system is simply $F(z) = 1 - H(z)$. From

$$\begin{aligned} F(z) &= 1 - 0.005 - 0.03z^{-1} - 0.11z^{-2} - 0.22z^{-3} - 0.27z^{-4} - \\ &\quad -0.22z^{-5} - 0.11z^{-6} - 0.03z^{-7} - 0.005z^{-8} \end{aligned}$$

we derive the finite impulse response

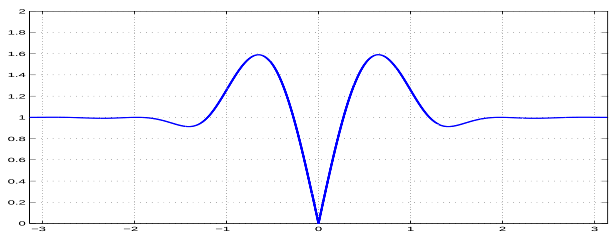
$$f[n] = \{0.995, -0.03, -0.11, -0.22, -0.27, -0.22, -0.11, -0.03, -0.005\}$$

which is *not* symmetric and therefore is not linear phase.

(f) Since $h[n]$ is linear phase, we can write $H(e^{j\omega}) = A(e^{j\omega})e^{-j4\omega}$ with $A(e^{j\omega}) \in \mathbb{R}$. The resulting squared magnitude response is therefore

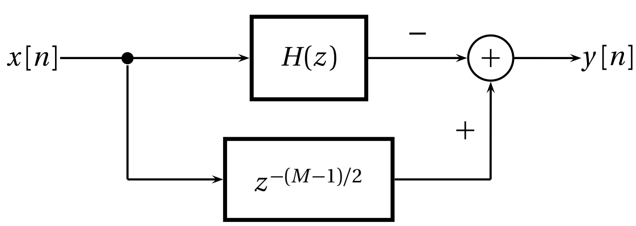
$$|F(e^{j\omega})|^2 = |1 - A(e^{j\omega})\,e^{-4j\omega}|^2 = 1 + A^2(e^{j\omega}) - 2A(e^{j\omega})\cos(4\omega)$$

which is not what was intended. Indeed the magnitude looks like this:



This is a very poor highpass filter.

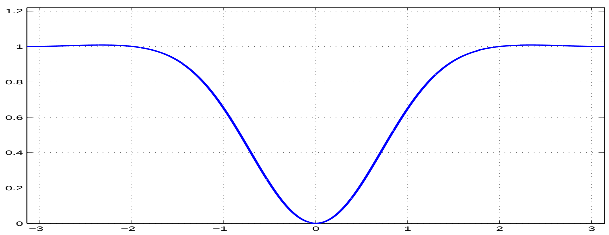
(g) The correct design must take into account the delay introduced by $H(z)$:



which we can easily do since M is an odd integer ($M = 9$) and therefore the delay value is an integer number (four samples). The resulting magnitude response is:

$$|F(e^{j\omega})| = |e^{-4j\omega} - A(e^{j\omega})\,e^{-4j\omega}|^2 = |1 - A(e^{j\omega})| = |1 - |H(e^{j\omega})||$$

which is what we want and which is plotted in the following figure:



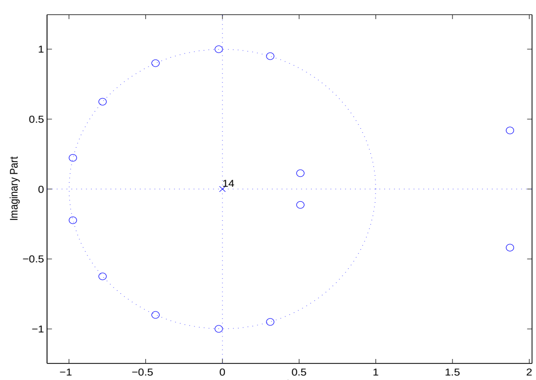
Solution 7.8: FIR filters analysis (II).

(a) We can write

$$H(e^{j\omega}) = e^{-j\omega\frac{N-1}{2}} H_r(e^{j\omega})$$

(b) The filter is not extraripple and, by inspection, it has $L+2=9$ alternations. Therefore the number of taps is $M=2L+1=15$.

(c) The filter has 14 zeros and no poles. Because of the symmetry constraints for Type-I FIRs, four of the zeros are at $z_0, z_0^*, 1/z_0$, and $1/z_0^*$. The other 10 zeros correspond to the zero crossings of $H_r(e^{j\omega})$ and can be easily estimated by inspection. The resulting pole-zero plot is shown below.



(d) It is $H_{1,r}(e^{j\omega}) = H_r(e^{j(\omega-\pi)})$. We can expand this as the transform of the noncausal impulse response $h_r[n]$

$$\begin{aligned} H_{1,r}(e^{j\omega}) &= \sum_{n=-L}^{L} h_r[n] e^{j(\omega-\pi)n} \\ &= \sum_{n=-L}^{L} (-1)^n \, h_r[n] e^{j\omega n} \end{aligned}$$

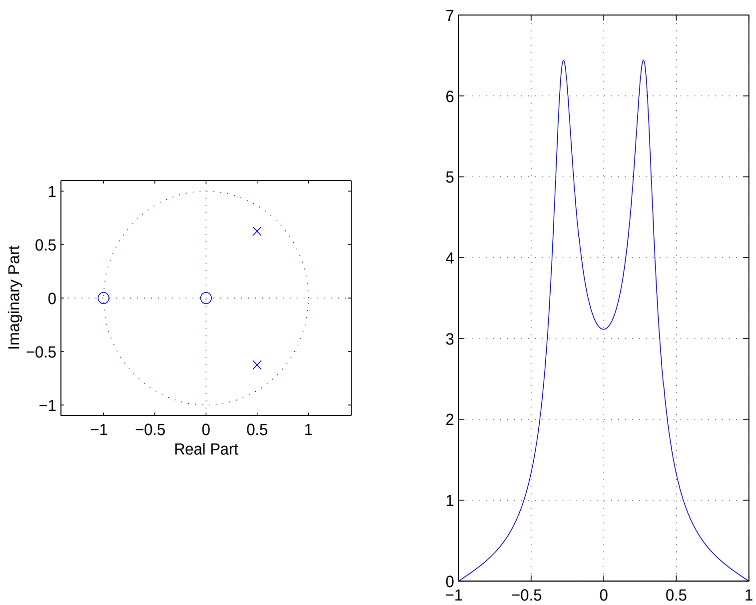
so that $h_1[n] = (-1)^n h[n]$.

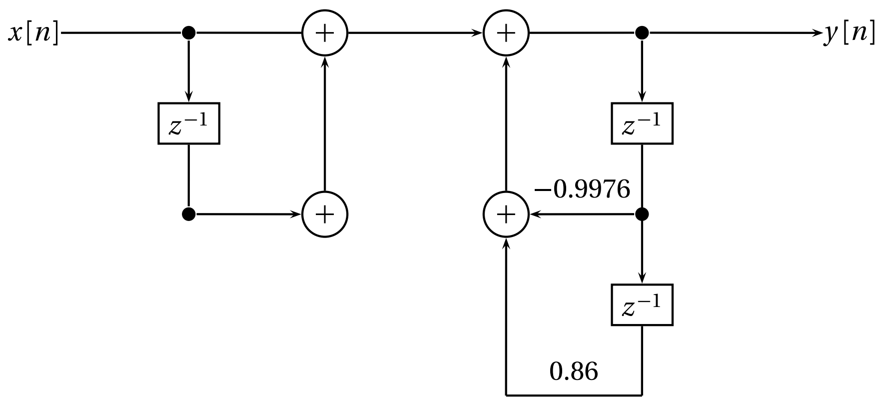
(e) The new filter is a Type-I highpass.

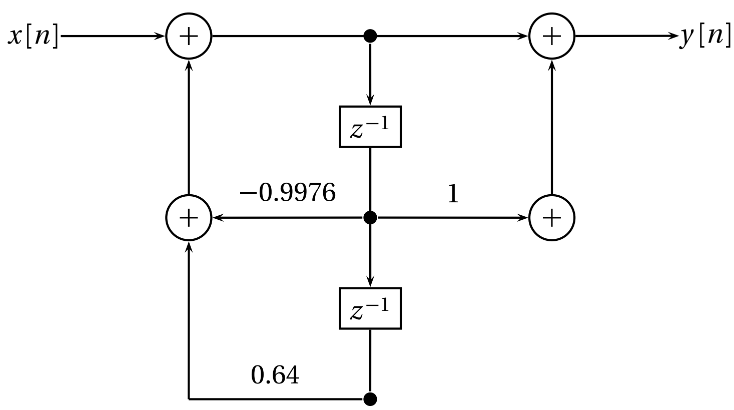
Solution 7.9: IIR filtering. The transfer function can be rewritten as:

$$H(z) = \frac{1+z^{-1}}{(1-pz^{-1})(1-p^*z^{-1})}$$

with $p = 0.8e^{j\omega_0}$, $\omega_0 = 2\pi/7$.







The first five approximate (i.e. with $a_1 \approx 1$) values for the input are

$$y[n] = 1, 4, 5.36, 2.8, -0.6304$$

as one can easily compute from the approximate difference equation

$$y[n] = x[n] + x[n-1] + y[n-1] - 0.64y[n-2]$$

Solution 7.10: Generalized Linear Phase Filters

(a) $H(e^{j\omega})$ can easily be expressed as

$$H(e^{j\omega}) = 2je^{-j\omega/2}\left(\frac{e^{j\omega/2} - e^{-j\omega/2}}{2j}\right)$$
$$= 2\sin(\omega/2)e^{-j(\omega/2-\pi/2)}.$$

Thus $H(z)$ is a generalized linear phase filter with group delay $d = 1/2$ and phase factor $\alpha = \pi/2$.

(b) The filter is of type IV since it has an even number of taps (2), is antisymmetric, has a fractional group delay and a $\pi/2$ phase factor.

(c) The filter impulse response is given by

$$h[n] = \delta[n] - \delta[n-1].$$

Thus,

$$\Sigma_n h[n]\sin(\omega(n-d)+\alpha) = \sin(-\omega/2+\pi/2) - \sin(\omega/2+\pi/2) = 0$$

for all ω.

(d) We have that

$$H(e^{j\omega}) = \Sigma_n h[n]e^{-j\omega} = \Sigma_n h[n]\cos(\omega n) - j\Sigma_n h[n]\sin(\omega n)$$

$$H(e^{j\omega}) = |H(e^{j\omega})|e^{-j(wd-\alpha)} = |H(e^{j\omega})|\cos(\alpha-\omega d) + j|H(e^{j\omega})|\sin(\alpha-\omega d).$$

Thus,

$$\tan(\alpha-\omega d) = \frac{\sin(\alpha-\omega d)}{\cos(\alpha-\omega d)} = \frac{-\Sigma_n h[n]\sin(\omega n)}{\Sigma_n h[n]\cos(\omega n)}$$

and

$$\Sigma_n h[n](\sin(\alpha-\omega d)\cos(\omega n) + \cos(\alpha-\omega d)\sin(\omega n)) = 0.$$

Using trigonometric identities we obtain,

$$\Sigma_n h[n]\sin(\omega(n-d)+\alpha) = 0$$

for all ω.

Solution 7.11: Echo cancellation. The transfer function of the echo system is

$$H(z) = 1 - \alpha z^{-D};$$

for $D = 12$ and $\alpha = 0.1$ the transfer function has 12 zeros at the roots of $z^{12} = 0.1$ which are 12 points on the circle of radius 0.826 and at angles

$$\measuredangle z_n = \frac{2\pi}{12} n$$

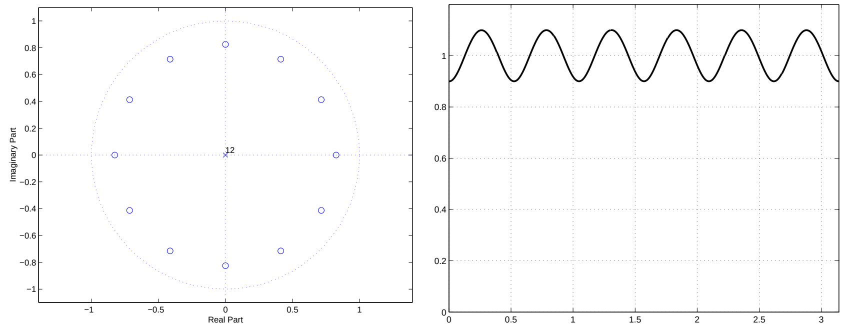
as shown in Figure 7.1(a). The magnitude response is plotted in Figure 7.1(b); note the dips in correspondence to each zero.
The echo cancelling system should remove the echo so that the overall transfer function is at most a simple delay. To do so, a candidate is the inverse transfer function

$$G(z) = \frac{1}{H(z)} = \frac{1}{1 - \alpha z^{-D}}$$

which is an IIR filter. Note that each zero in $H(z)$ becomes a pole for $G(z)$. For $D = 12$ and $\alpha = 0.1$ the poles are inside the unit circle so the system is stable. The pole-zero plot and magnitude response can be easily derived from those for $H(z)$.
The practical difficulty in implementing the system is the usual problem which affects all cancellation systems: because of the limited numerical precision of digital systems it may be hard or impossible to place the poles of the echo cancelling system *exactly* on top of the zeros of the natural echo. If the poles are close but not exactly on the zeros, the echo is attenuated but not eliminated. The structure used to implement the filter is also important: a cascade of second order stages will allow for a finer tuning of the zero positions as opposed to a simple transversal implementation.

Solution 7.12: FIR filter design (I). Type-III FIRs have a zero at $\omega = 0$, therefore they are not suitable for lowpass design.

Solution 7.13: FIR filter design (II). The Hilbert filter has an antisymmetric impulse response so that it can be approximated by a a Type-III linear phase filter. Type-III filters, however, have a zero at $\pm\pi$ and therefore there



(a) Pole-zero plot (b) Magnitude response

Figure 7.1: Echo cancellation.

will be a non-negligible magnitude error at the band edges. The wider the band of the input signals, the more taps will be necessary to achieve the approximation of an allpass behavior.

Chapter 8

Solution 8.1: Filtering a random process (I). The autocorrelation of the output is $r_y[n] = h[n] * h[-n] * r_x[n]$. We have that $h[n] * h[-n]$ is the usual triangle function with $h[-1] = h[1] = 1$, $h[0] = 2$ and zero elsewhere. Therefore

$$\begin{aligned} r_y[-3] &= 0 \\ r_y[-2] &= 0 \\ r_y[-1] &= \sigma^2 \\ r_y[0] &= 2\sigma^2 \\ r_y[1] &= \sigma^2 \\ r_y[2] &= 0 \\ r_y[3] &= 0 \end{aligned}$$

Solution 8.2: Filtering a random process (II). From the properties of the cross correlation, $r_{XY}[n] = h[n] * r_X[n] = \sigma^2 h[n]$ and therefore:

$$\begin{aligned}
r_{XY}[-3] &= 0 \\
r_{XY}[-2] &= 0 \\
r_{XY}[-1] &= 0 \\
r_{XY}[0] &= \sigma^2(1-\lambda) \\
r_{XY}[1] &= \sigma^2(1-\lambda)\lambda \\
r_{XY}[2] &= \sigma^2(1-\lambda)\lambda^2 \\
r_{XY}[3] &= \sigma^2(1-\lambda)\lambda^3
\end{aligned}$$

Solution 8.3: Power Spectral Density.

(a) We first note that $Y[n]$ can be easily expressed as the output of a LTI system with impulse response $h[n]$ defined as

$$h[n] = \delta[n] + \beta\delta[n-1].$$

Thus, the power spectrum density of $Y[n]$ is given by

$$P_Y(e^{j\omega}) = |H(e^{j\omega})|^2 P_X(e^{j\omega}).$$

We have that

$$\begin{aligned}
H(e^{j\omega}) &= 1 + \beta e^{-j\omega} \\
|H(e^{j\omega})|^2 &= \left(1 + \beta e^{-j\omega}\right)\left(1 + \beta e^{-j\omega}\right)^* \\
&= 1 + \beta e^{j\omega} + \beta e^{-j\omega} + \beta^2 \\
&= 1 + \beta^2 + 2\beta\cos(\omega)
\end{aligned}$$

and

$$\begin{aligned}
P_X(e^{j\omega}) &= \sum_n R_X[n]e^{-j\omega n} \\
&= \sigma^2 \sum_n \alpha^{|n|} e^{-j\omega n} \\
&= \sigma^2 \left(\sum_{n=-\infty}^{0} \alpha^{-n} e^{-j\omega n} + \sum_{n=0}^{\infty} \alpha^n e^{-j\omega n} - 1 \right) \\
&= \sigma^2 \left(\frac{1}{1-\alpha e^{j\omega}} + \frac{1}{1-\alpha e^{-j\omega}} - 1 \right) \\
&= \sigma^2 \frac{1-\alpha^2}{(1-\alpha e^{j\omega})(1-\alpha e^{-j\omega})} \\
&= \sigma^2 \frac{1-\alpha^2}{1+\alpha^2-2\alpha\cos(\omega)}.
\end{aligned}$$

Thus,

$$P_Y(e^{j\omega}) = \sigma^2(1-\alpha^2)\frac{1+\beta^2+2\beta\cos(\omega)}{1+\alpha^2-2\alpha\cos(\omega)}.$$

(b) To have a white noise, the samples of $Y[n]$ must uncorrelated, i.e. the power spectrum $P_Y(e^{j\omega})$ must be constant. Thus, $\beta = -\alpha$.

Solution 8.4: Filtering a sequence of independent random variables. To generate the random outcomes, we use the Matlab function `randn` that generates a normalized Gaussian random matrix (zero-mean, variance equal to 1). Type `help randn` for details.

(a) $X[1],\ldots,X[N]$ can be obtained numerically as:

```
% signal's length
N=100;
% Generate the Gaussian vector x with variance sigma^2=3
% Notice that the variance of the signal aU[n] is a^2
x=sqrt(3)*randn(1,N);
```

(b) To obtain $Z[1],\ldots,Z[N]$ note that $Z[n]$ has the unit variance, therefore:

```
z=randn(1,N);
```

(c) There are different ways to perform the filtering but the simplest is to use the Matlab function `filter`:

```
% The filter h is given by:
h=[0,1/2,1/4,1/4];
% The function filter performs zero-padding by default.
% Thus, we simulate circular convolution by extending
% the signal periodically. The extension of length(h)-1
% is added to both sides of the signal
lh=length(h); x1=[x(N-lh+2:N),x,x(1:lh-1)];
% Now, filtering...
y2=filter(h,1,x1);
% And, finally, we have to cut the extended parts
% of the convolved signal
y2=y2(lh:N+lh-1);
```

(d) We simulate the PSD by computing the average of the square modulus of the Fourier transform. We approximate the true average using a large number of realizations.

```
function PSD=estimate_psd(N,M)
% The length of the input vector and the number of
% iterations are given as the input arguments
% Initialize the filter h and the sum of
% the square modulus of the Fourier transform
h=[0,1/2,1/4,1/4]; PSD=zeros(1,N);
% The loop iterates the computation of the square
% modulus of the Fourier transform of different
% realizations of the variables {Y[i]}. The result
% is added to the accumulator vector PSD.
for i=1:M
  % generate x
  x=sqrt(3)*randn(1,N);
  % filter it through h
  x1=[x(N-2:N),x,x(1:3)];
  y1=filter(h,1,x1);
  y1=y1(4:N+3);
  % generate e
  e=randn(1,N);
  % sum up
  y=y1+e;
  % compute the PSD for this iteration
  % and add it to the sum
  PSD=PSD+(abs(fft(y)).^2)/N;
end
% average PSD
PSD=PSD/M;
```

(e) Filtering $X[n]$ with $h[n]$ produces a random signal $U[n]$ for which

$$U[n] = \sum_{k\in\mathbb{Z}} X[n-k]h[k] = \frac{1}{2}X[n-1] + \frac{1}{4}X[n-2] + \frac{1}{4}X[n-3]$$

so that

$$Y[n] = U[n] + Z[n] = \frac{1}{2}X[n-1] + \frac{1}{4}X[n-2] + \frac{1}{4}X[n-3] + Z[n]$$

Remark that if X and Z are independent (which is our hypothesis) then, by construction, U and Z are also independent.

We can now easily compute the mean of $Y[n]$:

$$\mathrm{E}[Y[n]] = \mathrm{E}[\tfrac{1}{2}X[n-1] + \tfrac{1}{4}X[n-2] + \tfrac{1}{4}X[n-3] + Z[n]] = 0.$$

The covariance $\mathrm{E}[Y[n+k]Y[n]^*]$ is

$$\begin{aligned}\mathrm{E}[Y[n+k]Y[n]^*] =& \mathrm{E}[U[n+k]U[n]^* + Z[n+k]U[n]^* + U[n+k]Z[n]^* + Z[n+k]Z[n]^*]\\ =& \mathrm{E}[U[n+k]U[n]^*] + \mathrm{E}[Z[n+k]U[n]^*] + \mathrm{E}[U[n+k]Z[n]^*] + \mathrm{E}[Z[n+k]Z[n]^*]\end{aligned}$$

Since U and Z are independent, and $\mathrm{E}[U[n]] = \mathrm{E}[Z[n]] = 0$ for every n:

$$\mathrm{E}[Y[n+k]Y[n]^*] = \mathrm{E}[U[n+k]U[n]^*] + \mathrm{E}[Z[n+k]Z[n]^*]$$

Now, $\mathrm{E}[Z[n+k]Z[n]^*] = \delta[k]$; given that all the quantities are real ($U^* = U$) we have

$$\begin{aligned}\mathrm{E}[U[n+k]U[n]] =& \frac{1}{4}\mathrm{E}[X[n+k-1]X[n-1]] + \frac{1}{8}\mathrm{E}[X[n+k-2]X[n-1]] + \frac{1}{8}\mathrm{E}[X[n+k-3]X[n-1]]\\ &+ \frac{1}{8}\mathrm{E}[X[n+k-1]X[n-2]] + \frac{1}{16}\mathrm{E}[X[n+k-2]X[n-2]] + \frac{1}{16}\mathrm{E}[X[n+k-3]X[n-2]]\\ &+ \frac{1}{8}\mathrm{E}[X[n+k-1]X[n-3]] + \frac{1}{16}\mathrm{E}[X[n+k-2]X[n-3]] + \frac{1}{16}\mathrm{E}[X[n+k-3]X[n-3]]\end{aligned}$$

Since $\mathrm{E}[X[n+k-l]X[n-j]] = 3\delta[k-l+j]$:

$$\begin{aligned}\mathrm{E}[U[n+k]U[n]] =& \frac{3}{4}\delta[k] + \frac{3}{8}\delta[k-1] + \frac{3}{8}\delta[k-2]\\ &+ \frac{3}{8}\delta[k+1] + \frac{3}{16}\delta[k] + \frac{3}{16}\delta[k-1]\\ &+ \frac{3}{8}\delta[k+2] + \frac{3}{16}\delta[k+1] + \frac{3}{16}\delta[k]\end{aligned}$$

Finally

$$
\mathrm{E}[Y[n+k]Y[n]^*] = \begin{cases} \frac{9}{8}+1 & \text{if } k=0 \\ \frac{9}{16} & \text{if } k=-1,+1 \\ \frac{3}{8} & \text{if } k=-2,+2 \\ 0 & \text{if } k\leq -3 \text{ or } k\geq 3 \end{cases}
$$

While the input process $X[n]$ was uncorrelated, from this result we can remark that the filtered process $Y[n]$ is correlated (short range correlation).

The power spectral density is given by the DTFT of the covariance function, therefore

$$
\begin{aligned}
f_X(\omega) &= \frac{17}{8}+\frac{9}{16}(e^{-j\omega}+e^{j\omega})+\frac{3}{8}(e^{-2j\omega}+e^{2j\omega}) \\
&= \frac{17}{8}+\frac{9}{8}\cos(\omega)+\frac{3}{4}\cos(2\omega).
\end{aligned}
$$

Try now to execute the function with several different values of M. Notice how the PSD is getting closer to the theoretical one as M grows. Figure 8.1 shows the result of the simulation for $N = 100$ and several values of M and compares them to the ideal one.

```
PSD=estimate_psd(N,M)
omega=[0:2*pi/(N-1):2*pi];
PSD_theo = (17/8)+((9/8)*cos(omega))+((3/4)*cos(2*omega));
figure
hold on
grid on
plot(PSD,'r--');
plot(PSD_theo,'b-');
```

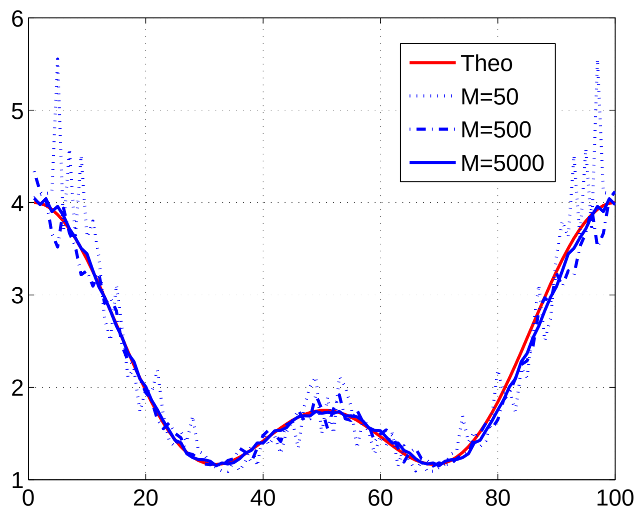


Figure 8.1: The ideal PSD and the estimations for $M = 50$, 500 and 5000.

Chapter 9

Solution 9.1: Zero-order hold.

(a) We have

$$\begin{aligned}
X_0(j\Omega) &= \int_{-\infty}^{\infty} x_0(t)e^{-j\Omega t}\,dt \\
&= \int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} x[n]\mathrm{rect}(t-n)e^{-j\Omega t}\,dt \\
&= \sum_{n=-\infty}^{\infty} x[n]\int_{-\infty}^{\infty} \mathrm{rect}(t-n)e^{-j\Omega t}\,dt \\
&= \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n}\int_{-1/2}^{1/2} e^{-j\Omega\tau}\,d\tau \\
&= \frac{\sin(\Omega/2)}{\Omega/2}X(e^{j\Omega}) \\
&= \mathrm{sinc}(\Omega/2\pi)X(e^{j\Omega}).
\end{aligned}$$

(b) Take for instance a discrete-time signal with a triangular spectrum such as in Figure 9.1-(a). We know that the sinc interpolation will give us a continuous-time signal which is strictly bandlimited to the $[-\Omega_N, \Omega_N]$ interval (with $\Omega_N = \pi/T_s = \pi$) and whose shape is exactly triangular; this is shown in Figure 9.1-(d). Conversely, the spectrum of the continuous-time signal interpolated by the zero-order hold is shown in Figure 9.1-(b). There are two main problems in the zero-order hold interpolation as compared to the sinc interpolation:

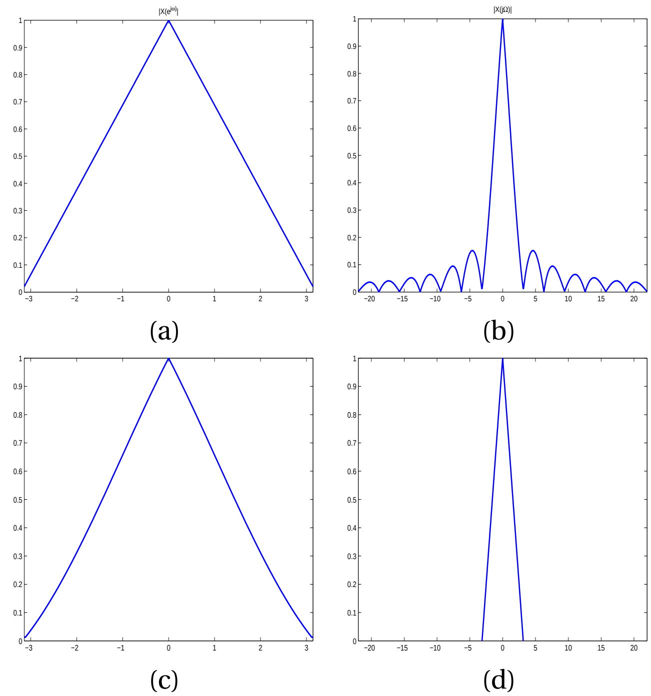


Figure 9.1: Zero Order Hold Exercise

- The zero-order hold interpolation is NOT bandlimited: the 2π-periodic replicas of the digital spectrum leak through in the continuous-time signal as high frequency components. This is due to the sidelobes of the interpolation function in the frequency domain (rect in time $\longleftrightarrow$ sinc in frequency) and it represents an undesirable high-frequency content which is typical of all local interpolation schemes.

- There is a distortion in the main portion of the spectrum (that between $-\Omega_N$ and Ω_N, with $\Omega_N = \pi$) due to the non-flat frequency response of the interpolation function. This is illustrated in detail in Figure 9.1-(c), which is simply a zoomed-in version of Figure 9.1-(b).

(c) Observe that $X(j\Omega)$ can be expressed as

$$X(j\Omega) = \begin{cases} X(e^{j\Omega}) & \text{if } \Omega \in [-\pi, \pi] \\ 0 & \text{otherwise,} \end{cases}$$

where $X(e^{j\Omega})$ is the DTFT of the sequence $x[n]$ evaluated at $\omega = \Omega$. So

$$X(j\Omega) = X(e^{j\Omega})\text{rect}\left(\frac{\Omega}{2\pi}\right) = X_0(j\Omega)\text{sinc}^{-1}\left(\frac{\Omega}{2\pi}\right)\text{rect}\left(\frac{\Omega}{2\pi}\right).$$

Hence

$$G(j\Omega) = \text{sinc}^{-1}\left(\frac{\Omega}{2\pi}\right)\text{rect}\left(\frac{\Omega}{2\pi}\right),$$

which is plotted in Figure 9.2 between $-\pi$ and π.

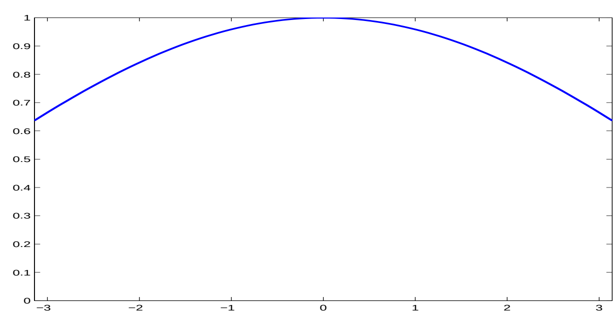


Figure 9.2: Zero Order Hold Exercise: $G(j\Omega)$

(d) A first solution is to compensate for the distortion introduced by $G(j\Omega)$ in the discrete-time domain. This is equivalent to pre-filtering $x[n]$ with a discrete-time filter of magnitude $1/G(e^{j\Omega})$ which can even be designed with the Parks-McClellan optimization technique. The advantages of this method is that digital filters such as this one are very easy to design and that the filtering can be done in the discrete-time domain. The disadvantage is that this approach does not eliminate or attenuate the high frequency leakage outside of the baseband.

Alternatively, one can cascade the interpolator with an analog lowpass filter to eliminate the leakage. The disadvantage is that it is hard to design an analog lowpass which can also compensate for the in-band distortion introduced by $G(j\Omega)$; such a filter will also introduce unavoidable phase distortion (no analog filter has linear phase).

Solution 9.2: A bizarre interpolator.

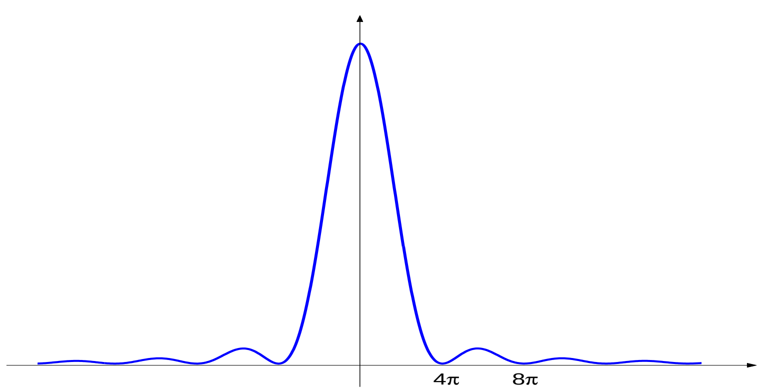
(a) As stated, $I(t) = r(t) * r(t)$, with $r(t) = \text{rect}(2t)$. The Fourier transform of $r(t)$ can be derived knowing that the transform of the rect function is the sinc, and that $r(t)$ has half the support of a normalized rect; alternatively, we can directly compute the easy integral:

$$R(j\Omega) = \int_{-1/4}^{1/4} e^{-j\Omega t}\, dt = \frac{1}{2}\text{sinc}\left(\frac{\Omega}{4\pi}\right)$$

From the convolution theorem,

$$I(j\Omega) = [R(j\Omega)]^2 = \frac{1}{4}\text{sinc}^2\left(\frac{\Omega}{4\pi}\right)$$

It appears that the Fourier transform of the interpolating function is real, and its sketch is as follows (note that the first zero is at $\Omega = 4\pi$):



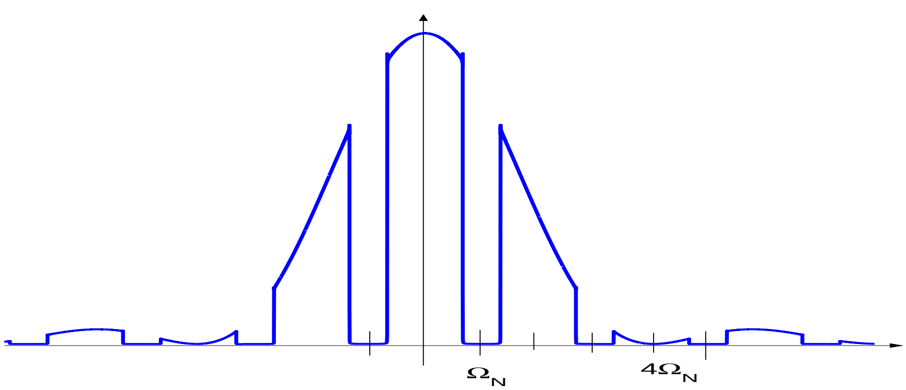
(b) From the interpolation formula $x(t) = \sum_n x[n] I((t - nT_s)/T_s)$ we have

$$
\begin{aligned}
X(j\Omega) &= T_s X(e^{j\Omega T_s}) I(j\Omega T_s) \\
&= \frac{T_s}{4} X(e^{j\Omega T_s}) \operatorname{sinc}^2\left(\frac{\Omega T_s}{4\pi}\right)
\end{aligned}
$$

So the Fourier transform of the interpolated signal is composed of the products of two parts (recall that, as usual, $\Omega_N = \pi/T_s$)):

- The $2\Omega_N$-periodic spectrum $X(e^{j\pi\Omega/\Omega_N})$
- The Fourier transform of the interpolating function. Please note that the first zero of $I(j\Omega T_s)$ is for $(\Omega T_s)/(4\pi) = 1$, i.e. for $\Omega = 4\Omega_N$

From this we can sketch the spectrum of the interpolated signal as follows



(c) There are two types of error, in-band and out-of-band:

- **In-band:** The spectrum between $[-\Omega_N, \Omega_N]$ (the baseband) is distorted by the non-flat response of the interpolating function over the baseband.
- **Out-of-band:** The periodic copies of $X(e^{j\pi\Omega/\Omega_N})$ outside of $[-\Omega_N, \Omega_N]$ are not eliminated by the interpolation filter, since it is not an ideal lowpass.

(d) We need to undo the linear distortion introduced by the nonflat response of the interpolation filter in the baseband. The idea is to have a modified spectrum $H(e^{j\omega})X(e^{j\omega})$ so that, in the $[-\Omega_N, \Omega_N]$, we have

$$X(j\Omega) = X(e^{j\Omega T_s}).$$

If we use $H(e^{j\omega})X(e^{j\omega})$ in the interpolation formula, we have

$$X(j\Omega) = \frac{T_s}{4} H(e^{j\Omega T_s})X(e^{j\Omega T_s})\,\mathrm{sinc}^2\left(\frac{\Omega T_s}{4\pi}\right)$$

so that

$$H(e^{j\Omega T_s}) = [\frac{T_s}{4}\mathrm{sinc}^2\left(\frac{\Omega T_s}{4\pi}\right)]^{-1}.$$

Therefore, the frequency response of the digital filter will be

$$H(e^{j\omega}) = \frac{4}{T_s}\mathrm{sinc}^{-2}\left(\frac{\omega}{4\pi}\right), \qquad -\pi \leq \omega \leq \pi$$

prolonged by 2π-periodicity over the entire frequency axis.

Solution 9.3: Another view of sampling. We have that

$$x_s(t) = x(t)\sum_{k=-\infty}^{\infty} \delta(t - kT_s)$$

and, by using the modulation theorem,

$$\begin{aligned} X_s(j\Omega) &= X(j\Omega) * P(j\Omega) \\ &= \int_{\mathbb{R}} X(j\tilde{\Omega})P(j(\Omega-\tilde{\Omega}))d\tilde{\Omega} = \frac{2\pi}{T_s}\int_{\mathbb{R}} X(j\tilde{\Omega})\sum_{k\in\mathbb{Z}}\delta\left(\Omega-\tilde{\Omega}-k\frac{2\pi}{T_s}\right)d\tilde{\Omega} \\ &= \frac{2\pi}{T_s}\sum_{k\in\mathbb{Z}}\int_{\mathbb{R}} X(j\tilde{\Omega})\delta\left(\Omega-\tilde{\Omega}-k\frac{2\pi}{T_s}\right)d\tilde{\Omega} = \frac{2\pi}{T_s}\sum_{k\in\mathbb{Z}} X\left(j\left(\Omega-k\frac{2\pi}{T_s}\right)\right). \end{aligned}$$

In other words, the spectrum of the delta-modulated signal is just the periodic repetition (with period $(2\pi/T_s)$ of the original spectrum. If the latter is bandlimited to (π/T_s) there will be no overlap and therefore $x(t)$ can be obtained simply by lowpass filtering $x_s(t)$ (in the continuous-time domain).

Solution 9.4: Aliasing can be good!

(a) According to our definition of bandlimited functions, the highest nonzero frequency is $2\Omega_0$ and therefore $x_c(t)$ is $2\Omega_0$-bandlimited for a total bandwidth of $4\Omega_0$. The maximum sampling period (i.e. the inverse of the *minimum* sampling frequency) which satisfies the sampling theorem is therefore $T_s = \pi/(2\Omega_0)$. Note however that the total support over which the (positive) spectrum is nonzero is the interval $[\Omega_0, 2\Omega_0]$ so that one could say that the total *effective* positive bandwidth of the signal is just Ω_0; this will be useful later.

(b) The digital spectrum will be the rescaled version of the periodized continuous-time spectrum

$$\tilde{X}_c(j\Omega) = \sum_{k=-\infty}^{\infty} X_c(j(\Omega - 2k\Omega_0)).$$

The general term $X_c(j\Omega - j2k\Omega_0)$ is nonzero only for

$$\Omega_0 \leq |\Omega - 2k\Omega_0| \leq 2\Omega_0 \quad \text{for } k \in \mathbb{Z}.$$

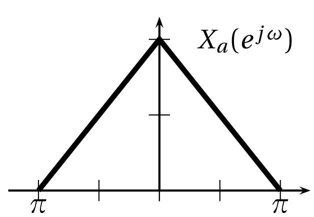
This translates to

$$\begin{array}{lcl} (2k+1)\Omega_0 & \leq \Omega \leq & (2k+2)\Omega_0 \\ (2k-2)\Omega_0 & \leq \Omega \leq & (2k-1)\Omega_0 \end{array}$$

which are non-overlapping intervals! Therefore, there will be no disruptive superpositions of the copies of the spectrum. The digital spectrum will be simply

$$X(e^{j\omega}) = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} X_c(j\frac{\omega}{T_s} - j\frac{2\pi k}{T_s})$$

and it will look like this (with 2π-periodicity, of course):



(c) Here's a possible scheme (verify that it works):

- Sinc-interpolate $x_a[n]$ with period T_s to obtain $x_b(t)$
- Multiply $x_b(t)$ by $\cos(2\Omega_0 t)$ in the continuous time domain to obtain $x_p(t)$ (i.e. modulate by a carrier at frequency (Ω_0/π) Hz).
- Bandpass filter $x_p(t)$ with an ideal bandpass filter with (positive) passband equal to $[\Omega_0, 2\Omega_0]$ to obtain $x_c(t)$.

(d) The effective *positive* bandwidth of such a signal is $\Omega_\Delta = (\Omega_1 - \Omega_0)$. Clearly, the sampling frequency must be at least equal to the effective total bandwidth so we have a first condition on the maximum allowable sampling period: $T_{\max} < \pi/\Omega_\Delta$.

Now, to make things simpler, assume that the upper frequency Ω_1 is a multiple of the bandwidth, i.e. $\Omega_1 = M\Omega_\Delta$ for some integer M (in the previous case, it was $M = 2$). In this case, the argument we made in the previous point can be easily generalized: if we pick $T_s = \pi/\Omega_\Delta$ and sample we have that

$$\tilde{X}_c(j\Omega) = \sum_{k=-\infty}^{\infty} X_c(j(\Omega - 2k\Omega_\Delta)).$$

The general term $X_c(j\Omega - j2k\Omega_\Delta)$ is nonzero only for

$$\Omega_0 \leq |\Omega - 2k\Omega_\Delta| \leq \Omega_1 \quad \text{for } k \in \mathbb{Z}.$$

Since $\Omega_0 = \Omega_1 - \Omega_\Delta = (M-1)\Omega_\Delta$, this translates to

$$\begin{aligned} (2k + M - 1)\Omega_\Delta &\leq \Omega \leq (2k + M)\Omega_\Delta \\ (2k - M)\Omega_\Delta &\leq \Omega \leq (2k - M + 1)\Omega_\Delta \end{aligned}$$

which are again non-overlapping intervals.

If Ω_1 is *not* a multiple of the bandwidth, then the easiest thing to do is to change the lower frequency Ω_0 to a new frequency Ω_0' so that the new bandwidth $\Omega_1 - \Omega_0'$ divides Ω_1 exactly. In other words we set a new lower frequency Ω_0' so that it will be $\Omega_1 = M(\Omega_1 - \Omega_0')$ for some integer M; it is easy to see that

$$M = \left\lfloor \frac{\Omega_1}{\Omega_1 - \Omega_0} \right\rfloor.$$

since this is the maximum number of copies of the Ω_Δ-wide spectrum which fit *with no overlap* in the $[0, \Omega_0]$ interval. Note also that, if $\Omega_\Delta > \Omega_0$ we cannot hope to reduce the sampling frequency and we have to use normal sampling. This artificial change of frequency will leave a small empty "gap" in the new bandwidth $[\Omega_0', \Omega_1]$, but that's no problem. Now we can use the previous result and sample with $T_s = \pi/(\Omega_1 - \Omega_0')$ with no overlap. Since $(\Omega_1 - \Omega_0') = \Omega_1/M$, we have that, in conclusion, the maximum sampling period is

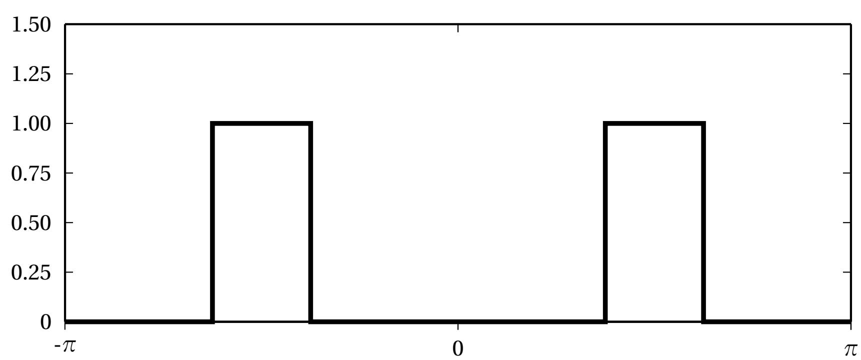
$$T_{\max} = \frac{\pi}{\Omega_1} \left\lfloor \frac{\Omega_1}{\Omega_1 - \Omega_0} \right\rfloor$$

i.e. we can obtain a sampling frequency reduction factor of $\lfloor \Omega_1/(\Omega_1 - \Omega_0) \rfloor$.

Solution 9.5: Digital processing of continuous-time signals.

(a) The digital frequencies are always $\omega = 2\pi(f/F_s)$ so that the digitized AM band resides in the $[0.5\pi, 0.6\pi]$ interval. Each 20KHz channel occupies a slice 0.01π-wide.

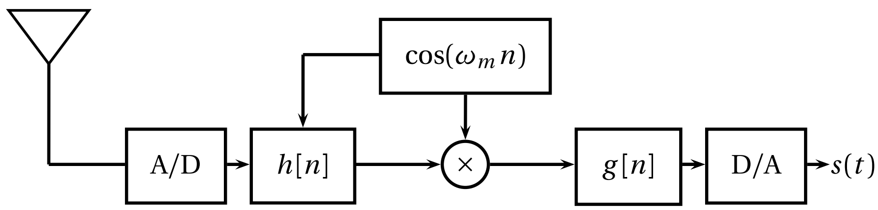
(b) the modulation moves the center frequency of the filter to $\pi/2$ so that the lowpass characteristic becomes as in the following figure. Note that the (positive) spectral support of the passband is $\pi/4$, i.e. twice the cutoff frequency.



(c) The bandwidth of a digitized AM channel is 0.01π so that we need a cutoff frequency $\omega_c = 0.005\pi$.

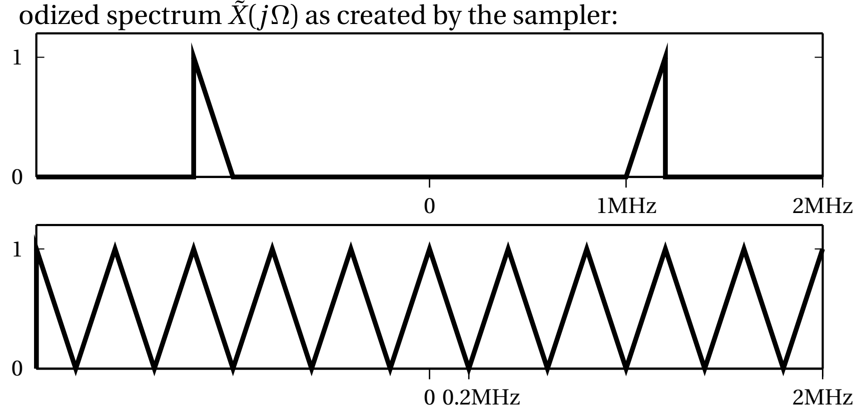
(d) The tuning frequencies are 0.505π, 0.545π and 0.595π respectively.

(e) The same sinusoidal oscillator can be used both to modulate the passband filter and to demodulated the extracted band to baseband; the demodulation centers the channel band around zero.



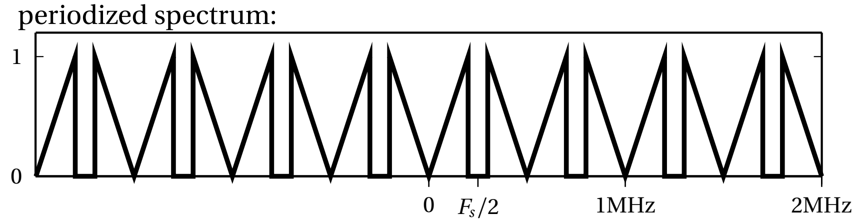
Here, $h[n]$ is used again after demodulation to remove the cross-modulation components. The D/A operates at the same frequency as the A/D, i.e. 4MHz; this is of course a waste, considering that the analog signal contains frequencies only up to 10KHz; clearly by using a suitable downsampler we could reduce such frequency as will be apparent after we study Chapter 11.

(f) Assume the spectrum is empty except for the AM band. Since this band starts at 1MHz and is 0.2MHz wide and since 1MHz is an integer multiple of 0.2MHz, we can use bandpass sampling, i.e. we can sample at a frequency as low as twice the bandwidth of the passband signal, i.e. 0.4MHz. There will be aliasing but the copies will not overlap with each other as can be readily seen from the figures below which represent the original AM spectrum $X(j\Omega)$ and the intermediate periodized spectrum $\tilde{X}(j\Omega)$ as created by the sampler:



This minimal frequency, however, will result in a swap of the positive and negative portions of the spectrum around baseband which greatly complicates the receiver.

In order to avoid this, we can sample at $F_s = 0.5MHz$, and the AM band will be correctly downshifted to baseband as seen here in the periodized spectrum:



After this, the scheme will proceed as before, with the following differences:

- the digitized AM band will occupy the $[0, 2\pi/3]$ positive band;
- each channel is $(\pi/15)$-wide;
- the tunable filter will have a cutoff frequency $\omega_c = \pi/30$
- the modulation frequency for both the tunable filter and the demodulation will start at $\pi/15$.

Note however that the spectrum is not empty outside of the AM band and therefore, in order to use bandpass sampling, we need to filter the signal *in the analog domain* with a sharp bandpass which kills everything outside the 1MHz-1.2MHz interval. This custom analog filter is precisely what one would like to avoid in a digital design.

Solution 9.6: Acoustic aliasing. Clearly a sampling rate $F_s = 8$ KHz is insufficient for a sinusoid at frequency $f = 10$ KHz, so there will be aliasing. The digital frequency after the sampler is $\omega_b = 2\pi(f/F_s) = 2.5\pi$. This frequency falls outside the $[-\pi, \pi]$ interval but, modulo 2π, it is equivalent to $\omega_b = 0.5\pi$. Therefore, the interpolated sinusoid will have a perceived frequency of $0.5 * (F_s/2) = 2$ KHz.

Solution 9.7: Interpolation subtleties. The Fourier transform of $x_c(t)$ is

$$X_c(j\Omega) = \int_0^{+\infty} e^{-t/T_s} e^{-j\Omega t} = \frac{T_s}{1 - jT_s\Omega}$$

which is not bandlimited. Therefore $x_c(t)$ cannot be an interpolated signal.

Solution 9.8: Time and frequency. No. The signal is time-limited and therefore it is not bandlimited. Consequently, there will always be a certain amount of aliasing in the sampled version regardless of how high the sampling frequency is.

Solution 9.9: Aliasing in time?

$$\begin{aligned}
\tilde{y}[n] &= \frac{2}{N}\sum_{k=0}^{N/2-1}\tilde{Y}[k]e^{j\frac{2\pi}{N/2}nk} \\
&= \frac{2}{N}\sum_{k=0}^{N/2-1}\tilde{X}[2k]e^{j\frac{2\pi}{N/2}nk} \\
&= \frac{2}{N}\sum_{k=0}^{N/2-1}\sum_{i=0}^{N-1}\tilde{x}[i]e^{-j\frac{2\pi}{N}(2k)i}\,e^{j\frac{2\pi}{N/2}nk} \\
&= \frac{2}{N}\sum_{i=0}^{N-1}\tilde{x}[i]\sum_{k=0}^{N/2-1}e^{j\frac{2\pi}{N/2}(n-i)k}
\end{aligned}$$

Now

$$\sum_{k=0}^{N/2-1}e^{j\frac{2\pi}{N/2}(n-i)k} = \begin{cases} N/2 & \text{if } (n-i) \text{ is a multiple of } (N/2) \\ 0 & \text{otherwise} \end{cases}$$

so that the only nonzero terms in the outer sum (that for index i) are those for $i=n$ and $i=n+N/2$. In the end

$$\tilde{y}[n] = \tilde{x}[n] + \tilde{x}[n+N/2].$$

Since $\tilde{x}[n]$ is N-periodic, this defines an $(N/2)$-periodic sequence obtained by summing two translated versions of $\tilde{x}[n]$. It's exactly like aliasing in the frequency domain: since we are not using enough DFS samples for the reconstruction, then the time-domain signal gets aliased in time.

Chapter 10

Solution 10.1: Quantization error (I). The quantized process $y[n] = Q\{x[n]\}$ is i.i.d. and therefore $P_y(e^{j\omega}) = \sigma_y^2$. The variance of the quantized process is that of a discrete random variable taking values over the set $\{-1, 0, 1\}$:

$$\begin{aligned}\sigma_y^2 &= (-1)^2 \cdot P[-1 \leq x < -0.5] + 0^2 \cdot P[-0.5 \leq x \leq 0.5] + 1^2 \cdot P[0.5 < x \leq 1] \\ &= 1/4 + 1/4\end{aligned}$$

and so

$$P_y(e^{j\omega}) = 1/2.$$

Solution 10.2: Quantization error (II). The error $e[n]$ at the output of the quantizer is an i.i.d. random process. Its average power is therefore:

$$\begin{aligned}\mathrm{E}[e^2[n]] &= \int_{-1}^{2} (x - Q\{x\})^2 f_x(x)\,dx \\ &= \frac{1}{3}\int_{-1}^{0} (x+1)^2\,dx + \frac{1}{3}\int_{0}^{2} (x-1)^2\,dx \\ &= \frac{1}{3}.\end{aligned}$$

where $f_x(x) = 1/3$ is the pdf of the input process (note that this is not the variance of the process since the process does not have zero mean). The power of the input process is

$$\mathrm{E}[x^2[n]] = \int_{-1}^{2} x^2 f_x(x)\,dx = 1$$

and therefore

$$\text{SNR} = 3 \approx 4.77\text{dB}.$$

Solution 10.3: More samples or more bits? The MSE introduced by quantizer A is

$$\text{MSE}_I = \frac{\sigma_x^2}{(256)^2}$$

and that is the total MSE for coding scheme I.
For scheme II, the downsampler does not introduce distortion because of the prefiltering, but the lowpass filtering introduces an error. Call $x'[n]$ the prefiltered signal before the downsampler; the MSE introduced by the filter is

$$\text{MSE}_{II,1} = \text{E}[|x[n] - x'[n]|^2] = \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2} \sigma_x^2 = \frac{\sigma_x^2}{2}$$

We approximate the downsampled signal as an i.i.d. signal with the same variance as the original, i.e. σ_x^2. Quantizer B will introduce an MSE of

$$\text{MSE}_{II,2} = \frac{\sigma_x^2}{(65536)^2}.$$

It is therefore apparent that Scheme I introduces an overall lower MSE. Indeed

$$\begin{aligned} \text{MSE}_I &\approx (1.5 \cdot 10^{-5})\sigma_x^2 \\ \text{MSE}_{II} = \text{MSE}_{II,1} + \text{MSE}_{II,2} &\approx (0.5 + 2.3 \cdot 10^{-10})\sigma_x^2 \gg \text{MSE}_I \end{aligned}$$

Chapter 11

Solution 11.1: Multirate identities. Let us denote the downsampling by 2 and upsampling by 2 operations by $D_2\{\cdot\}$ and $U_2\{\cdot\}$ respectively. The identities are best analyzed in the z-transform domain:

(a) Downsampling by 2 followed by filtering by $H(z)$ can be written as

$$\begin{aligned} Y(z) &= H(z)D_2\{X(z)\} \\ &= \frac{1}{2}H(z)\left(X(z^{1/2}) + X(-z^{1/2})\right). \end{aligned}$$

Filtering by $H(z^2)$ followed by downsampling by 2 can be written as

$$\begin{aligned} Y(z) &= D_2\{H(z^2)X(z)\} \\ &= \frac{1}{2}\left(H(z)X(z^{1/2}) + H(z)X(-z^{1/2})\right) \\ &= \frac{1}{2}H(z)\left(X(z^{1/2}) + X(-z^{1/2}))\right). \end{aligned}$$

The two operations are thus equivalent.

(b) Filtering by $H(z)$ followed by upsampling by 2 can be written as

$$\begin{aligned} Y(z) &= U_2\{H(z)X(z)\} \\ &= H(z^2)X(z^2). \end{aligned}$$

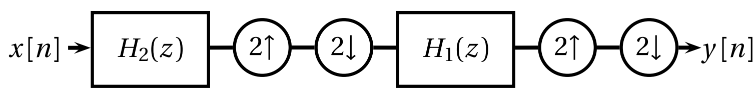
Upsampling by 2 followed by filtering by $H(z^2)$ can be written as

$$\begin{aligned} Y(z) &= H(z^2)U_2\{X(z)\} \\ &= H(z^2)X(z^2). \end{aligned}$$

The two operations are thus equivalent.

Solution 11.2: Multirate systems.

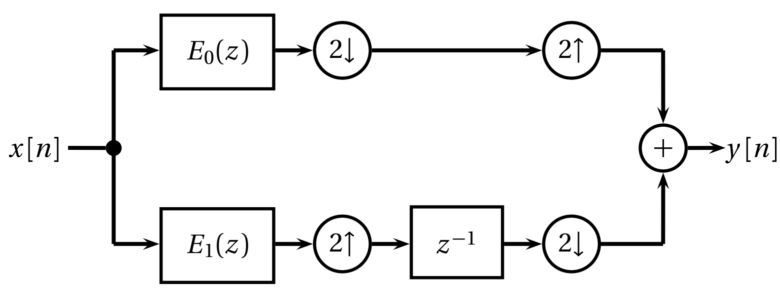
(a) Using the identities proven in the Exercise 11.1, the system can be redrawn as



We also know that an upsampler by N followed by a downsampler by N leave the signal unchanged and therefore the transfer function of the above system is simply

$$H(z) = \frac{Y(z)}{X(z)} = H_1(z)H_2(z).$$

(b) Again using the previous results, the system is equivalent to



The lower branch contains an upsampler followed by a delay and a downsampler. The output of such a system is easily seen to be 0. Thus only the upper branch remains and the final transfer function of the system is given by

$$\frac{Y(z)}{X(z)} = E_0(z).$$

(c) System 1 is described by the following equation:

$$\begin{aligned} Y(z) &= D_2\{H(z)G(z)U_2\{X(z)\}\} \\ &= D_2\{H(z)G(z)X(z^2)\} \\ &= \frac{1}{2}\left(H(z^{1/2})G(z^{1/2})X(z) + H(-z^{1/2})G(-z^{1/2})X(z)\right) \\ &= \frac{1}{2}\left(H(z^{1/2})G(z^{1/2}) + H(-z^{1/2})G(-z^{1/2})\right) X(z) \\ &= X(z). \end{aligned}$$

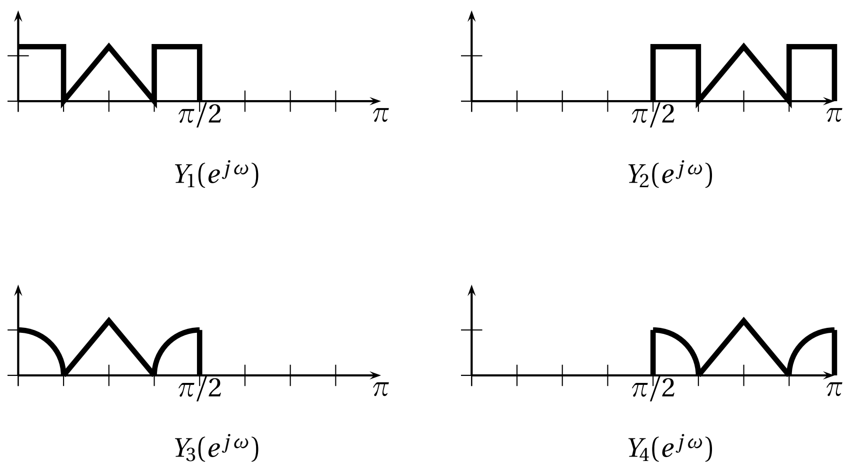
so that system 1 is unity.

System 2 is described by the following equation:

$$\begin{aligned} Y(z) &= D_2\{H(z)F(z)U_2\{X(z)\}\} \\ &= D_2\{H(z)F(z)X(z^2)\} \\ &= \frac{1}{2}\left(H(z^{1/2})F(z^{1/2})X(z) + H(-z^{1/2})F(-z^{1/2})X(z)\right) \\ &= \frac{1}{2}\left(H(z^{1/2})F(z^{1/2}) + H(-z^{1/2})F(-z^{1/2})\right) X(z) \\ &= 0. \end{aligned}$$

so that system 2 is zero.

Solution 11.3: Multirate Signal Processing. We have:



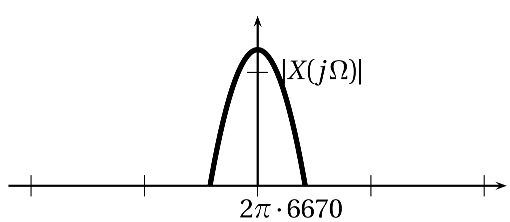
Solution 11.4: Digital processing of continuous-time signals.

(a) Playing the record at lower rpm slows the signal down by a factor 33/78. Therefore

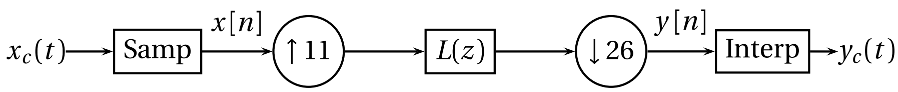
$$x(t) = s(\frac{33}{78}\,t) = s(\frac{11}{26}\,t)$$

(b) From the rescaling property of the Fourier transform

$$X(j\Omega) = \frac{26}{11}\,S(j\frac{26}{11}\Omega)$$



(c) We need to change the sampling rate so that, when $y[n]$ is interpolated at 44.1 KHz its spectrum is equal to $S(j\Omega)$. The rational sampling rate change factor is clearly 33/78 which is simply 11/26 after factoring. The processing scheme is as follows:



where $L(z)$ is a lowpass filter with cutoff frequency $\pi/26$ and gain $L_0 = 11/26$; both the sampler and interpolator work at $T_s = 1/44100$. We have:

$$
\begin{aligned}
X_c(j\Omega) &= \frac{26}{11} S(j\frac{26}{11}\Omega) \\
X(e^{j\omega}) &= \frac{1}{T_s} X_c(j\frac{\omega}{T_s}) \\
Y(e^{j\omega}) &= L_0 X(e^{j\frac{11}{26}\omega}) \\
&= \frac{11}{26}\frac{1}{T_s} X_c(j\frac{11}{26}\frac{\omega}{T_s}) \\
&= \frac{1}{T_s} S(j\frac{\omega}{T_s}) \\
Y_c(j\Omega) &= T_s Y(e^{j\Omega T_s}) \\
&= S(j\Omega)
\end{aligned}
$$

(d) The sampling rate change scheme stays the same except that now $45/78 = 15/26$. Therefore, the final upsampler has to compute more samples than in the previous scheme. The computational load of the sampling rate change is entirely dependent on the filter $L(z)$. If we upsample more before the output, we need to compute more filtered samples and therefore at 45rpm the scheme is less efficient.

Solution 11.5: Multirate is so useful! In the frequency domain we have:

- After the upsampler: $X_u(e^{j\omega}) = X(e^{jM\omega})$
- After the lowpass : $X_{lp}(e^{j\omega}) = X(e^{jM\omega})\,\mathrm{rect}(\omega/(2\pi/M))$
- After the delay: $X_d(e^{j\omega}) = e^{-jL\omega} X(e^{jM\omega})\,\mathrm{rect}(\omega/(2\pi/M))$

- After the downsampler: $Y(e^{j\omega}) = e^{-j(L/M)\omega}\, X(e^{j\omega})$ with no aliasing since $X_d(e^{j\omega})$ was bandlimited to π/M.

Therefore the net effect of the multirate processing scheme is that of a fractional delay with transfer function

$$H_d(e^{j\omega}) = e^{-j(L/M)\omega}.$$

For the transmission scheme example remember that, because of interpolation, we have that

$$S_c(j\Omega) = T_s\, S(e^{j\Omega T_s})$$

and therefore:

(b) $\hat{S}_c(j\Omega) = e^{-j\Omega t_0}\, S_c(j\Omega)$

(c) From the sampling theorem (no aliasing case)

$$\hat{S}(e^{j\omega}) = \frac{1}{T_s}\hat{S}_c(j\frac{\omega}{T_s}) = e^{-j\omega(t_0/Ts)}\, S(e^{j\omega})$$

(d) If $t_0 = 4.6\, T_s$, the fractional delay is 4.6 samples. We can compensate this by introducing an additional fractional delay of 0.4 samples so that the total delay becomes $D = 5$ samples. In order to do so, we simply need to set L and M so that $L/M = 0.4$ or, equivalently, $M = 2.5L$. The minimal choice for this so that M and L are integers is

$$\begin{aligned} M &= 5 \\ L &= 2 \end{aligned}$$

Solution 11.6: Multirate filtering. Call $x_u[n]$ the signal after the first upsampler and $x_f[n]$ the signal after the filter. In the z-transform domain we have

$$X_f(z) = H(z)X_u(z).$$

The maximum nonzero frequency of $X_f(z)$ is at most $\pi/10$, therefore for $M \leq 10$ the downsampler does not introduce aliasing. In other words,

$$Y(e^{j\omega}) = (1/M)X_f(e^{j\omega/M});$$

Since $X_f(e^{j\omega}) = H(e^{j\omega})X_u(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega M})$, we have

$$\begin{aligned} Y(e^{j\omega}) &= (1/M)H(e^{j\omega/M})X(e^{j\omega}) \\ &= H_d(e^{j\omega})X(e^{j\omega}) \end{aligned}$$

with

$$H_d(e^{j\omega}) = \begin{cases} 1/M & \text{for } |\omega| < M\pi/10 \\ 0 & \text{otherwise} \end{cases}$$

Therefore:

(a) For $M = 2$ the system corresponds to an ideal lowpass filter with cutoff frequency $\pi/5$ and gain $1/2$;

(b) For $M = 5$ the system corresponds to an ideal lowpass filter with cutoff frequency $\pi/2$ and gain $1/5$;

(c) For $M = 9$ the system corresponds to an ideal lowpass filter with cutoff frequency $9\pi/10$ and gain $9/10$;

(d) For $M = 10$ the system corresponds to multiplication by the gain factor $1/10$.

Solution 11.7: Oversampled sequences. Given that $X(e^{j\omega}) = 0$, $\frac{\pi}{3} \leq |\omega| \leq \pi$, $x[n]$ can be thought of as an oversampled signal that has been sampled at 3 times the Nyquist frequency. Therefore, we can downsample the signal without losing information.

(a) The idea is to determine whether n_0 is odd or even. Then, downsample $\hat{x}[n]$ so that n_0 is discarded and then upsample back the signal to the original frequency so that we recover $x[n_0]$.

(b) If the value of n_0 is not known, we need to determine whether n_0 is odd or even. $\hat{x}[n]$ can be expressed as:

$$\hat{x}[n] = x[n] - A\delta[n - n_0];$$
$$\hat{X}(e^{j\omega}) = X(e^{j\omega}) - Ae^{-j\omega n_0}$$

Now, if we compute this value at $\omega = \frac{\pi}{2}$ we have:

$$\hat{X}(e^{j\frac{\pi}{2}}) = X(e^{j\frac{\pi}{2}}) + A(-j)^{n_0}.$$

But given that $X(e^{j\frac{\pi}{2}}) = 0$ by hypothesis:

$$\hat{X}(e^{j\frac{\pi}{2}}) = A(-j)^{n_0}.$$

Therefore, If $\hat{X}(e^{j\frac{\pi}{2}})$ is real, n_0 is even and if it is imaginary, n_0 is odd.

(c) If there are k corrupted samples, the worst case is when the corrupted samples are consecutive. Then, we need to downsample $\hat{x}[n]$ by a factor of k and then upsample it back. To do that without losing information (aliasing), we need:

$$X(e^{j\omega}) = 0, \ \frac{\pi}{k} \leq |\omega| \leq \pi.$$

Chapter 12

Solution 12.1: Raised cosine. The raised cosine is an ideal filter since its frequency response is constant over finite intervals (See Example 6.2). Typically, it is approximated with a Type-I FIR; although a Type-II would ensure a zero at band edge, it would also introduce a half-sample delay.

Solution 12.2: Digital resampling. The ratio between CD and DVD sampling rates is, after removal of common factors,

$$\beta = \frac{160}{147}$$

which means we have to produce 160 DVD samples for every 147 CD samples. Therefore, we will need 160 fractional filters to cover all the intermediate interpolation values before a CD and DVD samples coincide again. This number is independent of the interpolator's length.

Solution 12.3: A quick design From the specifications, we can proceed according to the following steps:

- the maximum bandwidth of the signal is 3300 Hz;
- leaving room for the raised cosine transitions band (say $\beta = 0.125$), we can say that the usable bandwidth is less than 2933 Hz
- we take a bandwidth of 2800 Hz and therefore a maximum symbol rate of 2800 Baud
- Eq (12.3) requires the internal sampling frequency to fulfill

$$\begin{cases} F_s \geq 7200 \\ F_s = K \cdot 2800, \quad K \in \mathbb{N}. \end{cases}$$

which has a solution for $K = 3$ and $F_s = 8400$ Hz

- with this choice, the carrier is $\omega_c = 2\pi(300+3300)/(2F_s) = 3\pi/7$
- with $p_e = 10^{-6}$, the maximum number of reliable bits per symbol is $M = \log_2(1-(3\cdot 10^{2.8})/(2\ln(10^{-6}))) \approx 6.12$
- we can use a regular (fourfold symmetric) 64-point constellation
- the overall bitrate is $R = 16800$ bits per second.

Solution 12.4: The shape of a constellation. First of all, notice that for both 8-point constellations the minimum distance is exactly $d_{\min} = 1$, considering circular decision boundaries centered upon the constellation points. This explains the apparently odd $(1+\sqrt{3})$ distance.
If we compute the intrinsic power of the two constellation we have (exploiting symmetries)

$$\begin{aligned}\sigma^2_{\alpha,\text{reg}} &= \frac{1}{8}\cdot 4(2+10) = 6\\ \sigma^2_{\alpha,\text{irreg}} &= \frac{1}{8}\cdot 4(2+(1+\sqrt{3})^2) \approx 4.73\end{aligned}$$

In other words, the irregular constellation offers more than a 1 dB gain over the regular one. This gain can be translated into a reliability gain by increasing G_0 while the transmitted signal remains within the power constraint.
