Chebfun Guide

1st Edition
For Chebfun version 5

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Dedicated to the Chebfun developers of the past, present, and future.

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Preface

This guide is an introduction to the use of Chebfun, an open source software package that aims to provide "numerical computing with functions." Chebfun extends MATLAB's inherent facilities with vectors and matrices to functions and operators. For those already familiar with MATLAB, much of what Chebfun does will hopefully feel natural and intuitive. Conversely, for those new to MATLAB, much of what you learn about Chebfun can be applied within native MATLAB too.

Some of the chapters give clues about how Chebfun does its work, but that is not an emphasis of this guide. For the mathematical underpinnings of Chebfun, the best source is *Approximation Theory and Approximation Practice,* by Lloyd N. Trefethen (Society for Industrial and Applied Mathematics, 2013). For the algorithmic backstory, refer to the list of publications maintained at www.chebfun.org.

We gratefully acknowledge the Engineering and Physical Sciences Research Council of the UK, The MathWorks, Inc., and the European Research Council, whose generous support has helped the project grow and thrive. We also acknowledge the support of the University of Oxford and the University of Delaware.

Most especially we acknowledge and praise the many contributors to Chebfun. As of version 5, the code consists of over 100,000 lines produced over the course of the past twelve years. In addition to writing code, many individuals have contributed their time to the design, testing, review, and debugging of code written by others. For their development efforts, we thank Anthony Austin, Zachary Battles, Ásgeir Birkisson, Pedro Gonnet, Stefan Güttel, Hrothgar, Mohsin Javed, Georges Klein, Hadrien Montanelli, Ricardo Pachón, Rodrigo Platte, Mark Richardson, Alex Townsend, Grady Wright, and Kuan Xu. Others who have made key contributions include Jean-Paul Berrut, Folkmar Bornemann, Yuji Nakatsukasa, Vanni Noferini, Sheehan Olver, Joris Van Deun, and Marcus Webb.

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June 2014

Part I
Functions of one variable

1. Getting Started with Chebfun

Lloyd N. Trefethen, October 2009, latest revision June 2014

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1.1 What is a chebfun?

A chebfun is a function of one variable defined on an interval $[a, b]$. The syntax for chebfuns is almost exactly the same as the usual MATLAB syntax for vectors, with the familiar MATLAB commands for vectors overloaded in natural ways. Thus, for example, whereas `sum(f)` returns the sum of the entries when `f` is a vector, it returns a definite integral when `f` is a chebfun.

Chebfun with a capital C is the name of the software system.

The aim of Chebfun is to "feel symbolic but run at the speed of numerics". More precisely our vision is to achieve for functions what floating-point arithmetic achieves for numbers: rapid computation in which each successive operation is carried out exactly apart from a rounding error that is very small in relative terms [Trefethen 2007].

The implementation of Chebfun is based on the mathematical fact that smooth functions can be represented very efficiently by polynomial interpolation in Chebyshev points, or equivalently, thanks to the Fast Fourier Transform, by expansions in Chebyshev polynomials. For a simple function, 20 or 30 points often suffice, but the process is stable and effective even for functions complicated enough to require 1000 or 1,000,000 points. Chebfun makes use of adaptive procedures that aim to find the right number of points automatically so as to represent each function to roughly machine precision, that is, about 15 digits of relative accuracy. (Originally Chebfun stored function values at Chebyshev points; in Version 5 it switched to storing Chebyshev expansion coefficients.)

The mathematical foundations of Chebfun are for the most part well established by results scattered throughout the 20th century. A key early figure, for example, was Bernstein in the 1910s. Much of

the relevant material can be found collected in the Chebfun-based book *Approximation Theory and Approximation Practice*[Trefethen 2013].

Chebfun was originally created by Zachary Battles and Nick Trefethen at Oxford during 2002-2005 [Battles & Trefethen 2004]. Battles left the project in 2005, and soon four new members were added to the team: Ricardo Pachon (from 2006), Rodrigo Platte (from 2007), and Toby Driscoll and Nick Hale (from 2008). In 2009, Asgeir Birkisson and Mark Richardson also became involved, and other contributors included Pedro Gonnet, Joris Van Deun, and Georges Klein. Nick Hale served as Director of the project during 2010-2014. The Chebfun Version 5 rewrite was directed by Nick Hale during 2013-2014, and the team included Anthony Austin, Asgeir Birkisson, Toby Driscoll, Hrothgar, Mohsin Javed, Hadrien Montanelli, Alex Townsend, Nick Trefethen, Grady Wright, and Kuan Xu. Further information about Chebfun history is available at the Chebfun web site,

`http://www.chebfun.org`

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This Guide is based on Chebfun Version 5, released in June 2014. Chebfun is available at

`http://www.chebfun.org`

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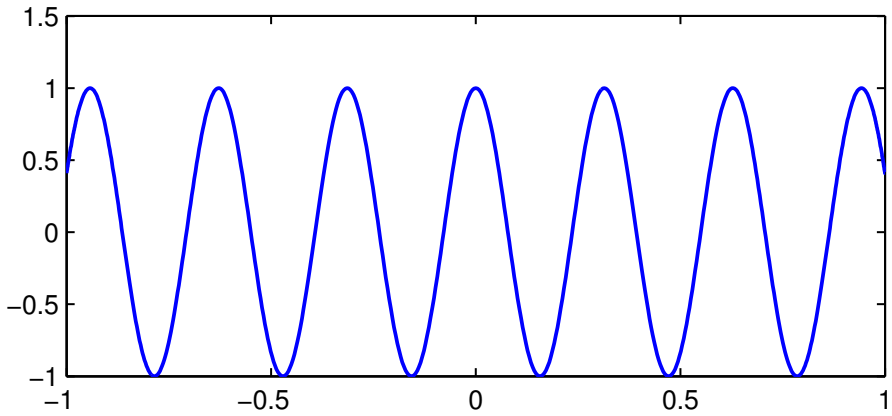
1.2 Constructing simple chebfuns

The `chebfun` command constructs a chebfun from a specification such as a string or an anonymous function. If you don't specify an interval, then the default interval $[-1, 1]$ is used. For example, the following command makes a chebfun corresponding to $\cos(20x)$ on $[-1, 1]$ and plots it.

```
f = chebfun('cos(20*x)');
plot(f)
```



From this little experiment, you cannot see that `f` is represented by a polynomial. One way to see this is to find the length of `f`:

```
length(f)
```

```
ans =
    61
```

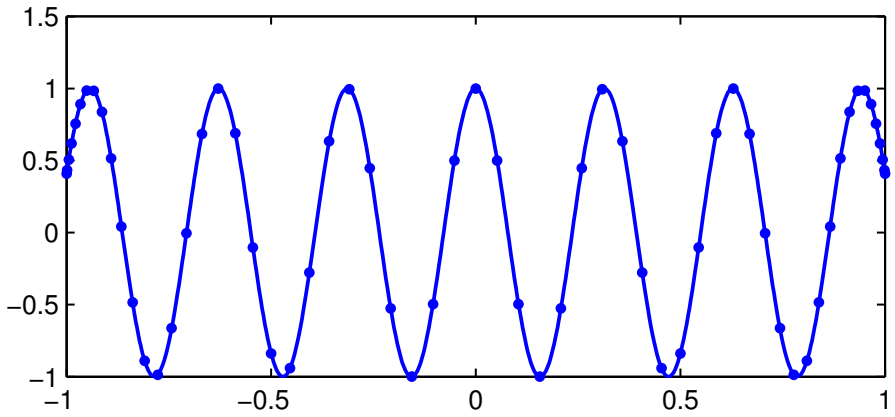
Another is to remove the semicolon that suppresses output:

```
  f
```

```
f =
   chebfun column (1 smooth piece)
       interval       length   endpoint values
[      -1,       1]       61      0.41     0.41
Epslevel = 2.183626e-15.  Vscale = 1.000000e+00.
```

These results tell us that `f` is represented by a polynomial interpolant through 61 Chebyshev points, i.e., a polynomial of degree 60. These numbers have been determined by an adaptive process. We can see the data points by plotting `f`|`with the |'.-'` option:

```
  plot(f,'.-')
```



The formula for $N+1$ Chebyshev points in $[-1,1]$ is

$$x(j) = -\cos(j\pi/N), \quad j = 0:N,$$

and in the figure we can see that the points are clustered accordingly near 1 and -1. Note that in the middle of the grid, there are about 5 points per wavelength, which is evidently what it takes to represent this cosine to 15 digits of accuracy. For intervals other than $[-1,1]$, appropriate Chebyshev points are obtained by a linear scaling.

The curve between the data points is the polynomial interpolant, which is evaluated by the barycentric formula introduced by Salzer [Berrut & Trefethen 2004, Salzer 1972]. This method of evaluating polynomial interpolants is stable and efficient even if the degree is in the millions [Higham 2004].

What is the integral of f from -1 to 1? Here it is:

```
  sum(f)
```

```
ans =
   0.091294525072763
```

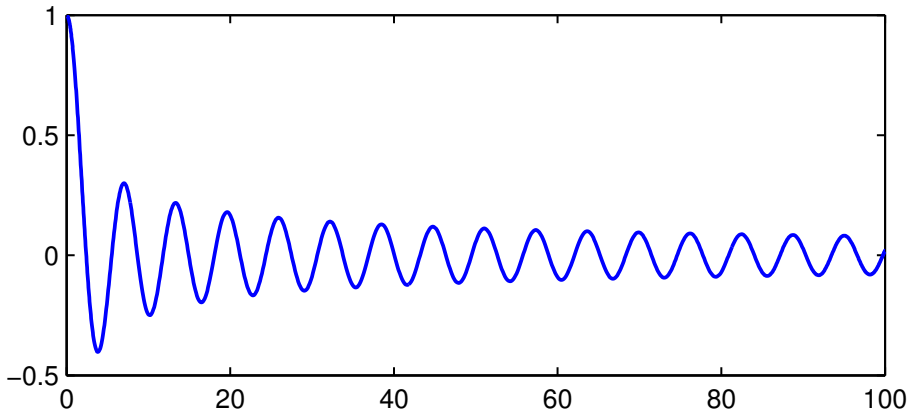
This number was computed by integrating the polynomial (Clenshaw-Curtis quadrature – see Section 2.1), and it is interesting to compare it to the exact answer from calculus:

```
exact = sin(20)/10
```

```
exact =
   0.091294525072763
```

Here is another example, now with the chebfun defined by an anonymous function instead of a string. In this case the interval is specified as $[0, 100]$.

```
g = chebfun(@(t) besselj(0,t),[0,100]);
plot(g), ylim([-.5 1])
```



The function looks complicated, but it is actually a polynomial of surprisingly small degree:

```
length(g)
```

```
ans =
    89
```

Is it accurate? Well, here are three random points in $[0, 100]$:

```
format long
x = 100*rand(3,1)
```

```
x =
  59.616523414856850
  87.537260012975111
  11.789355152760882
```

Let's compare the chebfun to the true Bessel function at these points:

```
exact = besselj(0,x);
error = g(x) - exact;
[g(x) exact error]
```

```
ans =
  -0.067323383971824  -0.067323383971825   0.000000000000000
   0.029772796120858   0.029772796120858   0.000000000000001
  -0.000506642933742  -0.000506642933741  -0.000000000000001
```

If you want to know the first 5 zeros of the Bessel function, here they are:

```
r = roots(g); r = r(1:5)
```

```
r =
   2.404825557695772
   5.520078110286314
   8.653727912911007
  11.791534439014285
  14.930917708487790
```

Notice that we have just done something nontrivial and potentially useful. How else would you find zeros of the Bessel function so readily? As always with numerical computation, we cannot expect the answers to be exactly correct, but they will usually be very close. In fact, these computed zeros are accurate to close to machine precision:

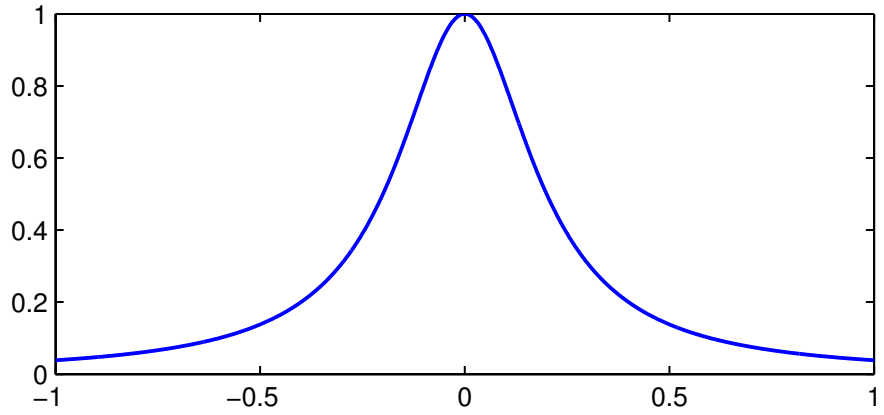
```
besselj(0,r)
```

```
ans =
   1.0e-14 *
   0.026212433453684
   0.111311651748761
   0.154471710674566
   0.071369727817872
  -0.092296985262750
```

Most often we get a chebfun by operating on other chebfuns. For example, here is a sequence that uses plus, times, divide, and power operations on an initial chebfun `x` to produce a famous function of Runge:

```
x = chebfun('x');
f = 1./(1+25*x.^2);
length(f)
clf, plot(f)
```

```
ans =
   181
```



1.3 Operations on chebfuns

There are more than 200 commands that can be applied to a chebfun. For a list of many of them you can type `methods`:

```
methods chebfun
```

```
Methods for class chebfun:

abs             cotd            jaccoeffs       range
acos            coth            join            rank
acosd           cov             jump            rdivide
acosh           csc             kron            real
acot            cscd            ldivide         reallog
acotd           csch            le              realpow
acoth           ctranspose      legcoeffs       realsqrt
acsc            cumprod         legpoly         rem
acscd           cumsum          length          remez
acsch           cylinder        log             repmat
airy            diff            log10           residue
all             dirac           log1p           restrict
and             disp            log2            roots
angle           display         logical         round
any             domain          loglog          sec
arcLength       ellipj          lt              secd
area            ellipke         lu              sech
asec            end             mat2cell        semilogx
asecd           epslevel        max             semilogy
asech           eq              mean            sign
asin            erf             measure         simplify
asind           erfc            merge           sin
asinh           erfcinv         mesh            sinc
atan            erfcx           min             sind
atan2           erfinv          minandmax       sinh
atan2d          exp             minus           size
atand           expm1           mldivide        sound
```

```
atanh            feval           mod             spy
besselh          fill            movie           sqrt
besseli          find            mrdivide        std
besselj          fix             mtimes          subsasgn
besselk          fliplr          ne              subspace
bessely          flipud          newDomain       subsref
bvp4c            floor           nextpow2        sum
bvp5c            fourcoeffs      norm            surf
cat              fred            normal          surface
ceil             ge              normest         surfc
cf               get             not             svd
cheb2cell        gmres           null            tan
cheb2quasi       gt              num2cell        tand
chebcoeffs       heaviside       or              tanh
chebellipseplot  horzcat         orth            times
chebfun          hscale          overlap         transpose
chebpade         hypot           pde15s          uminus
chebpoly         imag            permute         unwrap
chebtune         innerProduct    pinv            uplus
circconv         integral        plot            vander
comet            inv             plot3           var
comet3           isdelta         plotcoeffs      vertcat
complex          isempty         plus            volt
compose          isequal         poly            vscale
cond             isfinite        polyfit         waterfall
conj             ishappy         pow2            why
conv             isinf           power           xor
cos              isnan           prod
cosd             isreal          qr
cosh             issing          quantumstates
cot              iszero          quasi2cheb

Static methods:

interp1          ode15s          spline
lagrange         ode45           update
ode113           pchip
```

To find out what a command does, you can use `help`.

```
help chebfun/sum
```

```
 SUM   Definite integral of a CHEBFUN.
    SUM(F) is the integral of a column CHEBFUN F over its domain of definition.

    SUM(F, A, B), where A and B are scalars, integrates a column CHEBFUN F over
    [A, B], which must be a subdomain of F.domain:

                          B
                          /
```

```
                SUM(F) =  | F(t) dt.
                          /
                         A
```

```
    SUM(F, A, B), where A and B are CHEBFUN objects, returns a CHEBFUN S which
    satisfies

                        B(s)
                        /
                S(s) =  | F(t) dt.
                        /
                      A(s)

    SUM(F, DIM), where DIM is one of 1, 2, sums F over the dimension DIM. If F
    is a column CHEBFUN and DIM = 1 or if F is a row CHEBFUN and DIM = 2 then
    this integrates in the continuous dimension of F, as described above.
    Otherwise, SUM(F, DIM) sums across the columns (rows) of the column (row)
    CHEBFUN F.

  See also CUMSUM, DIFF.
```

Most of the commands in the list above exist in ordinary MATLAB; some exceptions are `domain`, `restrict`, `chebpoly`, and `remez`. We have already seen `length` and `sum` in action. In fact we have already seen `subsref` too, since that is the MATLAB command for (among other things) evaluating arguments in parentheses. Here is another example of its use:

```
f(0.5)
```

```
ans =
   0.137931034482759
```

Here for comparison is the true result:

```
1/(1+25/4)
```

```
ans =
   0.137931034482759
```

In this Runge function example, we have also implicitly seen `times`, `plus`, `power`, and `rdivide`, all of which have been overloaded from their usual MATLAB uses to apply to chebfuns.

In the next part of this tour we shall explore many of these commands systematically. First, however, we should see that chebfuns are not restricted to smooth functions.

1.4 Piecewise smooth chebfuns

Many functions of interest are not smooth but piecewise smooth. In this case a chebfun may consist of a concatenation of smooth pieces, each with its own polynomial representation. Each of the smooth

pieces is called a "fun". This enhancement of Chebfun was developed initially by Ricardo Pachon during 2006-2007, then also by Rodrigo Platte starting in 2007 [Pachon, Platte and Trefethen 2010]. Essentially funs are the "classic chebfuns" for smooth functions on $[-1,1]$ originally implemented by Zachary Battles in Chebfun Version 1.

Later we shall describe the options in greater detail, but for the moment let us see some examples. One way to get a piecewise smooth function is directly from the constructor, taking advantage of its capability of automatic edge detection. For example, in the default "splitting off" mode a function with a jump in its derivative produces a warning message,

```
f = chebfun('abs(x-.3)');
```

```
Warning: Function not resolved using 65537 pts. Have you tried 'splitting on'?
```

The same function can be successfully captured with splitting on:

```
f = chebfun('abs(x-.3)','splitting','on');
```

The `length` command reveals that `f` is defined by four data points, two for each linear interval:

```
length(f)
```

```
ans =
     4
```

We can see the structure of `f` in more detail by typing `f` without a semicolon:

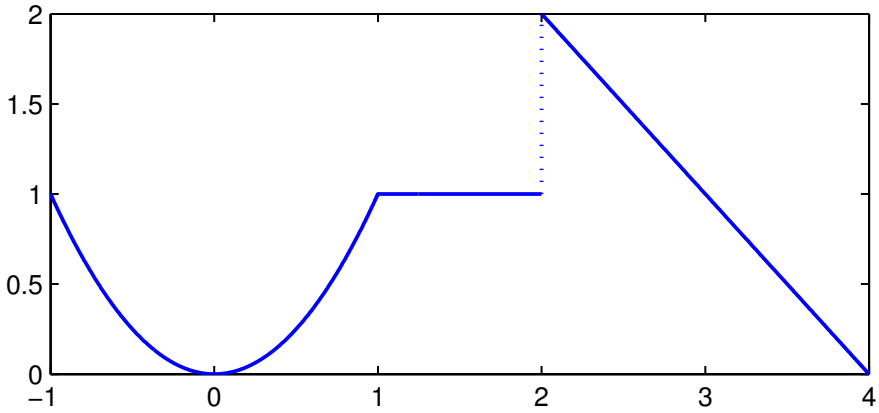
```
f
```

```
f =
   chebfun column (2 smooth pieces)
       interval       length   endpoint values
[      -1,     0.3]        2       1.3  2.2e-16
[     0.3,       1]        2   1.1e-16      0.7
Epslevel = 1.110223e-15.  Vscale = 1.300000e+00.  Total length = 4.
```

This output confirms that f consists of two funs, each defined by two points and two corresponding function values. The functions live on intervals defined by breakpoints at -1, 1, and a number very close to 0.3. The `Vscale` field is related to the maximum absolute value of `f` and `Epslevel` gives some rough information about its relative accuracy.

Another way to make a piecewise smooth chebfun is to construct it explicitly from various pieces. For example, the following command specifies three functions x^2, 1, and $4-x$, together with a vector of endpoints indicating that the first function applies on $[-1,1]$, the second on $[1,2]$, and the third on $[2,4]$:

```
f = chebfun({@(x) x.^2, 1, @(x) 4-x},[-1 1 2 4]);
plot(f)
```



We expect `f` to consist of three pieces of lengths 3, 1, and 2, and this is indeed the case:

```
f
```

```
f =
   chebfun column (3 smooth pieces)
       interval       length   endpoint values
[      -1,       1]        3         1        1
[       1,       2]        1         1        1
[       2,       4]        2         2 -1.1e-16
Epslevel = 1.110223e-15.  Vscale = 2.  Total length = 6.
```

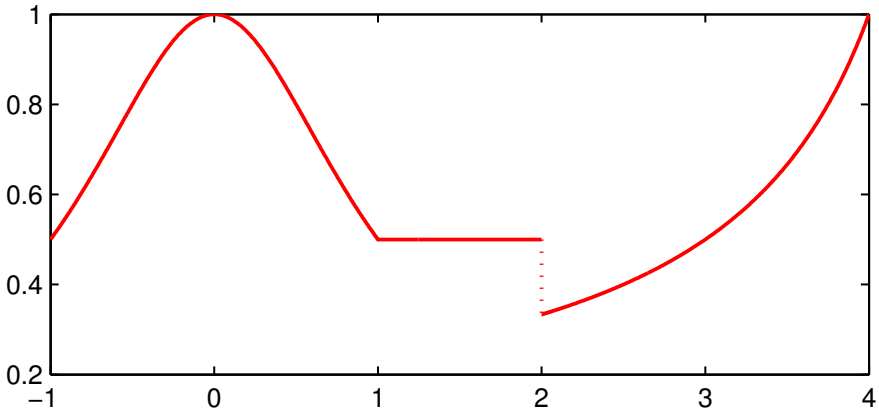
Our eyes see pieces, but to Chebfun, `f` is just another function. For example, here is its integral.

```
sum(f)
```

```
ans =
   3.666666666666667
```

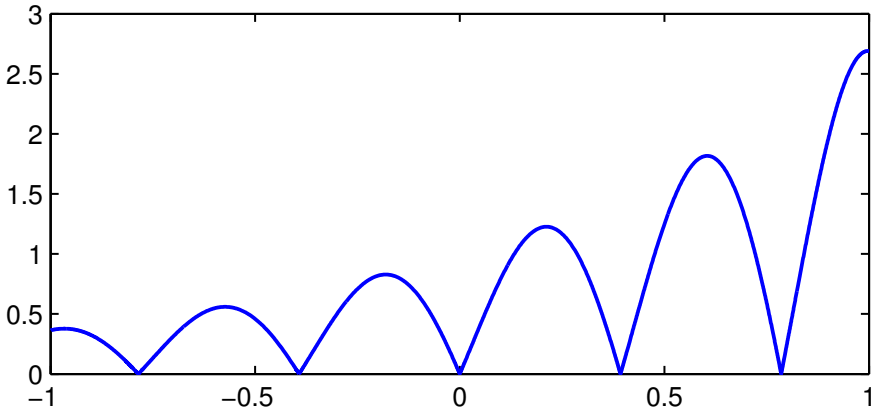
Here is an algebraic transformation of `f`, which we plot in another color for variety.

```
plot(1./(1+f),'r')
```



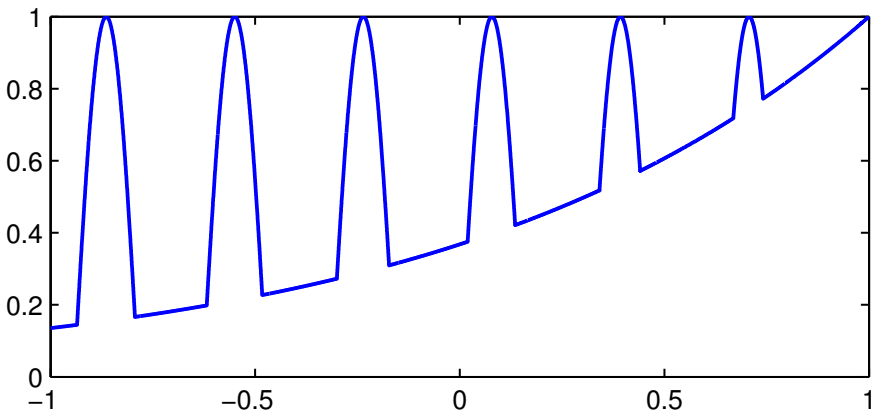
Some Chebfun commands naturally introduce breakpoints in a chebfun. For example, the `abs` command first finds zeros of a function and introduces breakpoints there. Here is a chebfun consisting of 6 funs:

```
f = abs(exp(x).*sin(8*x));
plot(f)
```



And here is an example where breakpoints are introduced by the `max` command, leading to a chebfun with 13 pieces:

```
f = sin(20*x);
g = exp(x-1);
h = max(f,g);
plot(h)
```



As always, `h` may look complicated to a human, but to Chebfun it is just a function. Here are its mean, standard deviation, minimum, and maximum:

```
mean(h)
```

```
ans =
   0.578242020778010
```

```
  std(h)
```

```
ans =
   0.280937455806246
```

```
  min(h)
```

```
ans =
   0.135335283236613
```

```
  max(h)
```

```
ans =
   1.000000000000000
```

A final note about piecewise smooth chebfuns is that the automatic edge detection or "splitting" feature, when it is turned on, may subdivide functions even though they do not have clean point singularities, and this may be desirable or undesirable depending on the application. For example, considering $\sin(x)$ over $[0, 1000]$ with splitting on, we end up with a chebfun with many pieces:

```
  tic, f = chebfun('sin(x)',[0 1000*pi],'splitting','on'), toc
```

```
f =
   chebfun column (32 smooth pieces)
       interval       length   endpoint values
[       0,      98]       87     3e-14    -0.71
[      98,   2e+02]       88     -0.71        1
[   2e+02, 2.9e+02]       88         1    -0.71
[ 2.9e+02, 3.9e+02]       87     -0.71  1.7e-13
[ 3.9e+02, 4.9e+02]       87   3.1e-13     0.71
[ 4.9e+02, 5.9e+02]       87      0.71       -1
[ 5.9e+02, 6.9e+02]       86        -1     0.71
[ 6.9e+02, 7.9e+02]       85      0.71 -7.6e-13
[ 7.9e+02, 8.8e+02]       86  -1.3e-12    -0.71
[ 8.8e+02, 9.8e+02]       86     -0.71        1
[ 9.8e+02, 1.1e+03]       86         1    -0.71
[ 1.1e+03, 1.2e+03]       86     -0.71 -3.2e-13
[ 1.2e+03, 1.3e+03]       85  -6.7e-13     0.71
[ 1.3e+03, 1.4e+03]       86      0.71       -1
[ 1.4e+03, 1.5e+03]       86        -1     0.71
[ 1.5e+03, 1.6e+03]       85      0.71 -1.6e-12
[ 1.6e+03, 1.7e+03]       86  -1.3e-12    -0.71
[ 1.7e+03, 1.8e+03]       86     -0.71        1
[ 1.8e+03, 1.9e+03]       86         1    -0.71
[ 1.9e+03,   2e+03]       85     -0.71  8.3e-13
[   2e+03, 2.1e+03]       85   2.2e-12     0.71
[ 2.1e+03, 2.2e+03]       86      0.71       -1
[ 2.2e+03, 2.3e+03]       86        -1     0.71
```

```
[ 2.3e+03, 2.4e+03]       87      0.71 -1.7e-12
[ 2.4e+03, 2.5e+03]       85   1.5e-12    -0.71
[ 2.5e+03, 2.6e+03]       86     -0.71        1
[ 2.6e+03, 2.7e+03]       84         1    -0.71
[ 2.7e+03, 2.7e+03]       85     -0.71 -3.2e-12
[ 2.7e+03, 2.8e+03]       85  -2.5e-13     0.71
[ 2.8e+03, 2.9e+03]       84      0.71       -1
[ 2.9e+03,   3e+03]       86        -1     0.71
[   3e+03, 3.1e+03]       85      0.71 -3.7e-12
Epslevel = 3.487391e-13.  Vscale = 1.000000e+00.  Total length = 2748.
Elapsed time is 0.637316 seconds.
```

In this case it is more efficient – and more interesting mathematically – to omit the splitting and construct one global chebfun:

```
tic, f2 = chebfun('sin(x)',[0 1000*pi]), toc
```

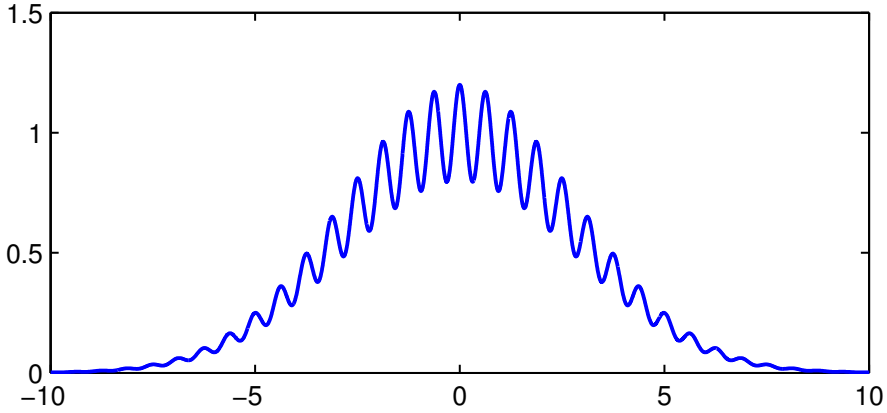
```
f2 =
   chebfun column (1 smooth piece)
       interval       length   endpoint values
[       0, 3.1e+03]     1684  -1.8e-14 -3.1e-13
Epslevel = 3.487867e-13.  Vscale = 9.999862e-01.
Elapsed time is 0.025475 seconds.
```

Splitting on and off are discussed further in Section 8.3.

1.5 Infinite intervals and infinite function values

A major change from Chebfun Version 2 to Version 3 was the generalization of chebfuns to allow certain functions on infinite intervals or which diverge to infinity; the initial credit for these innovations belongs to Nick Hale, Rodrigo Platte, and Mark Richardson. For example, here is a function on the whole real axis,

```
f = chebfun('exp(-x.^2/16).*(1+.2*cos(10*x))',[-inf,inf]);
plot(f)
```



and here is its integral:

```
sum(f)
```

```
ans =
   7.089815403621257
```

Here's the integral of a function on [1,inf]:

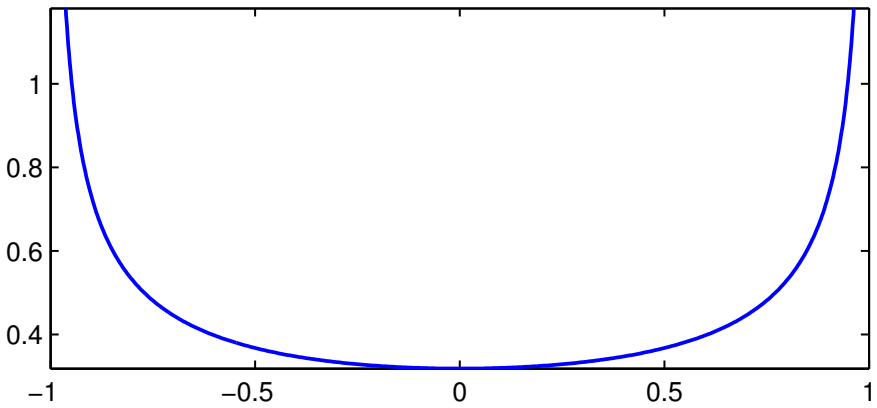
```
sum(chebfun('1./x.^4',[1 inf]))
```

```
ans =
   0.333333333305146
```

Notice that several digits of accuracy have been lost here. Be careful! – operations involving infinities in Chebfun are not always as accurate and robust as their finite counterparts.

Here is an example of a function that diverges to infinity, which we can capture with the `'exps'` flag; see Chapter 7 for details:

```
h = chebfun('(1/pi)./sqrt(1-x.^2)','exps',[-.5 -.5]);
plot(h)
```



In this case the integral comes out just right:

```
sum(h)
```

```
ans =
     1
```

For more on the treatment of infinities in Chebfun, see Chapter 9.

1.6 Periodic functions

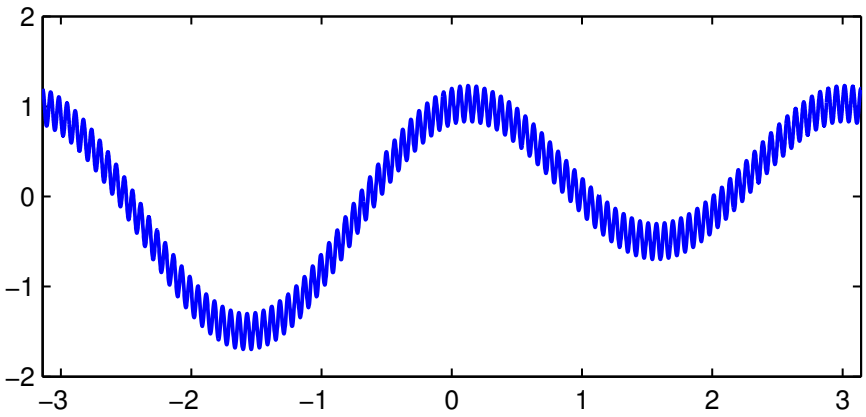
Until 2014, Chebfun used only nonperiodic representations, based on Chebyshev polynomials. Beginning with Version 5, there is a new capability of representing sufficiently smooth periodic functions by trigonometric polynomials instead, that is, Fourier series. Such an object is still called a chebfun,

but it is a periodic one. These features were added by Grady Wright in the first half of 2014, and will undoubtedly be developed further in the future.

For example, here is a periodic function on $[-\pi, \pi]$ represented in the usual way by a Chebyshev series.

```
ff = @(t) sin(t)/2 + cos(2*t) + 0.2*cos(100*t);
f = chebfun(ff,[-pi,pi]);
max(f)
plot(f)
```

```
ans =
   1.231249778347126
```



Its length, very roughly, is $100 \times \pi$,

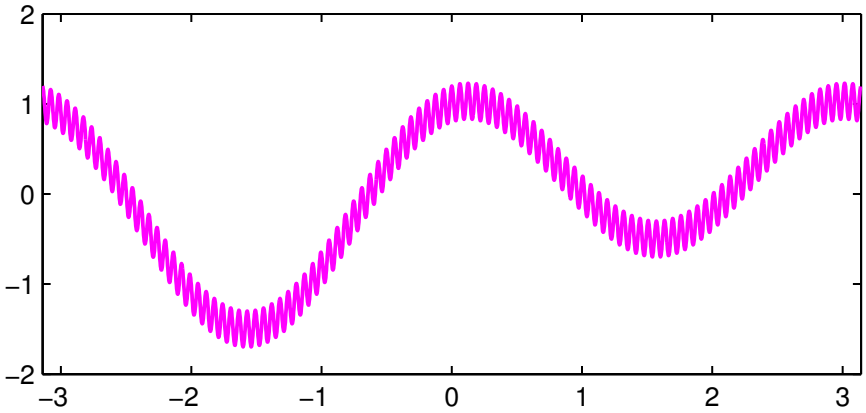
```
length(f)
```

```
ans =
   383
```

Here is the same function represented by a Fourier series:

```
f2 = chebfun(ff,[-pi,pi],'periodic')
max(f2)
plot(f2,'m')
```

```
f2 =
   chebfun column (1 smooth piece)
       interval       length   endpoint values periodic
[    -3.1,     3.1]      201       1.2      1.2
Epslevel = 3.739287e-15.  Vscale = 1.640723e+00.
ans =
   1.231249778347129
```



Its length is now only about 100×2 (exactly 201). This improvement by a factor of about $\pi/2$ is typical.

```
length(f2)
```

```
ans =
   201
```

We can confirm that the two functions agree like this:

```
norm(f-chebfun(f2,[-pi, pi]))
```

```
ans =
     1.303017177476116e-14
```

Readers may be interested to compare `plotcoeffs` applied to the first and second versions of f. Rather than display that here we shall turn to a simpler example involving a shorter Fourier series. Consider the function

```
f = chebfun('7 + sin(t) + exp(1)*cos(2*t)',[-pi,pi],'periodic')
```

```
f =
   chebfun column (1 smooth piece)
       interval       length   endpoint values periodic
[    -3.1,     3.1]        5       9.7      9.7
Epslevel = 6.661338e-16.  Vscale = 9.718282e+00.
```

Here are the coefficients of f as an expansion in sines and cosines:

```
[a,b] = fourcoeffs(f)
```

```
a =
   2.718281828459045   0.000000000000001   7.000000000000000
b =
                   0   1.000000000000000
```

Here they are as an expansion in complex exponentials:

```
c = fourcoeffs(f)
```

```
c =
  Column 1
  1.359140914229522 + 0.000000000000000i
  Column 2
  0.000000000000000 - 0.500000000000000i
  Column 3
  7.000000000000000 + 0.000000000000000i
  Column 4
  0.000000000000000 + 0.500000000000000i
  Column 5
  1.359140914229522 + 0.000000000000000i
```

Bookkeeping of Fourier coefficients can often be a headache. If these examples don't make the patterns clear, details can be found with `help fourcoeffs`.

For a mathematically less trivial example, here is the cosine expansion of a function whose Fourier series coefficients are known to be values of a Bessel function:

```
f = chebfun('exp(cos(t))',[-pi pi],'periodic');
[a,b] = fourcoeffs(f);
n = floor(length(f)/2);
exact = 2*besseli(n:-1:0,1); exact(end) = exact(end)/2;
disp('        computed            exact')
disp([a' exact'])
```

```
        computed            exact
   0.000000000000040   0.000000000000040
   0.000000000001039   0.000000000001039
   0.000000000024980   0.000000000024980
   0.000000000550590   0.000000000550590
   0.000000011036772   0.000000011036772
   0.000000199212481   0.000000199212481
   0.000003198436462   0.000003198436462
   0.000044977322954   0.000044977322954
   0.000542926311914   0.000542926311914
   0.005474240442094   0.005474240442094
   0.044336849848664   0.044336849848664
   0.271495339534077   0.271495339534077
   1.130318207984970   1.130318207984970
   1.266065877752008   1.266065877752008
```

1.7 Rows, columns, and quasimatrices

MATLAB doesn't only deal with column vectors: there are also row vectors and matrices. The same is true of Chebfun. The chebfuns shown so far have all been in column orientation, which is the default, but one can also take the transpose, compute inner products, and so on:

```
  x = chebfun(@(x) x)

x =
   chebfun column (1 smooth piece)
       interval       length   endpoint values
[      -1,       1]        2        -1        1
Epslevel = 1.110223e-15.  Vscale = 1.

  x'

ans =
   chebfun row (1 smooth piece)
       interval       length   endpoint values
[      -1,       1]        2        -1        1
Epslevel = 1.110223e-15.  Vscale = 1.

  x'*x

ans =
   0.666666666666667
```

One can also make matrices whose columns are chebfuns or whose rows are chebfuns, like this:

```
  A = [1 x x.^2]

A =
   chebfun column1 (1 smooth piece)
       interval       length   endpoint values
[      -1,       1]        3         1        1
Epslevel = 2.220446e-15.  Vscale = 1.
   chebfun column2 (1 smooth piece)
       interval       length   endpoint values
[      -1,       1]        3        -1        1
Epslevel = 1.110223e-15.  Vscale = 1.
   chebfun column3 (1 smooth piece)
       interval       length   endpoint values
[      -1,       1]        3         1        1
Epslevel = 2.220446e-15.  Vscale = 1.

  A'*A

ans =
   2.000000000000000  -0.000000000000000   0.666666666666667
  -0.000000000000000   0.666666666666667                   0
   0.666666666666667                   0   0.400000000000000
```

These are called *quasimatrices*, and they are discussed in Chapter 6.

1.8 Chebfun features not in this Guide

Some of Chebfun's most remarkable features haven't made it into this edition of the Guide. Here are some of our favorites:

o `leg2cheb` and `cheb2leg` for fast Legendre-Chebyshev conversions,

o `conv` and `circconv` for convolution,

o The `'equi'` flag to the Chebfun constructor for equispaced data,

o `polyfit` for least-squares fitting in the continuous context,

o `inv` for computing the inverse of a chebfun,

o `pde15s` for PDEs in one space and one time variable.

To learn about any of these options, try the appropriate `help` command. Just as a taster, here's a hint of how fast Chebfun can convert a ten-thousand coefficient Chebyshev expansion to Legendre coefficients and back again using an algorithm from [Hale & Townsend 2013]:

```
tic
ccheb = randn(10000,1);
cleg = cheb2leg(ccheb);
ccheb2 = leg2cheb(cleg);
norm(ccheb-ccheb2,inf)
toc
```

```
ans =
     3.428191064358543e-11
Elapsed time is 0.501795 seconds.
```

1.9 How this Guide is produced

This guide is produced in MATLAB using the `publish` command with a style sheet somewhat different from the usual; the output of `publish` is then processed by Markdown. To publish a chapter for yourself, make sure the chebfun guide directory is in your path and then type, for example, `open(publish('guide1'))`. The formatting may not be exactly right but it should certainly be intelligible.

1.10 References

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[Berrut & Trefethen 2005] J.-P. Berrut and L. N. Trefethen, "Barycentric Lagrange interpolation", *SIAM Review 46*, (2004), 501-517.

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[Salzer 1972] H. E. Salzer, "Lagrangian interpolation at the Chebyshev points cos(nu pi/n), nu = 0(1)n; some unnoted advantages", *Computer Journal* 15 (1972), 156-159.

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2. Integration and Differentiation

Lloyd N. Trefethen, November 2009, latest revision June 2014

Contents

2.1 `sum`

We have seen that the `sum` command returns the definite integral of a chebfun over its range of definition. The integral is normally calculated by an FFT-based version of Clenshaw-Curtis quadrature, as described first in [Gentleman 1972]. This formula is applied on each fun (i.e., each smooth piece of the chebfun), and then the results are added up.

Here is an example whose answer is known exactly:

```
f = chebfun(@(x) log(1+tan(x)),[0 pi/4]);
format long
I = sum(f)
Iexact = pi*log(2)/8
```

```
I =
   0.272198261287950
Iexact =
   0.272198261287950
```

Here is an example whose answer is not known exactly, given as the first example in the section "Numerical Mathematics in Mathematica" in *The Mathematica Book* [Wolfram 2003].

```
f = chebfun('sin(sin(x))',[0 1]);
sum(f)
```

```
ans =
   0.430606103120691
```

All these digits match the result 0.4306061031206906049... reported by Mathematica.

Here is another example:

```
F = @(t) 2*exp(-t.^2)/sqrt(pi);
f = chebfun(F,[0,1]);
I = sum(f)
```

```
I =
   0.842700792949715
```

The reader may recognize this as the integral that defines the error function evaluated at $t = 1$:

```
Iexact = erf(1)
```

```
Iexact =
   0.842700792949715
```

It is interesting to compare the times involved in evaluating this number in various ways. MATLAB's specialized `erf` code is the fastest:

```
tic, erf(1), toc
```

```
ans =
   0.842700792949715
Elapsed time is 0.000040 seconds.
```

Using MATLAB's various quadrature commands is understandably slower:

```
tol = 3e-14;
tic, I = quad(F,0,1,tol); t = toc;
  fprintf('  QUAD:  I = %17.15f  time = %6.4f secs\n',I,t)
tic, I = quadl(F,0,1,tol); t = toc;
  fprintf(' QUADL:  I = %17.15f  time = %6.4f secs\n',I,t)
tic, I = quadgk(F,0,1,'abstol',tol,'reltol',tol); t = toc;
  fprintf('QUADGK:  I = %17.15f  time = %6.4f secs\n',I,t)
```

```
  QUAD:  I = 0.842700792949715  time = 0.0204 secs
 QUADL:  I = 0.842700792949715  time = 0.0146 secs
QUADGK:  I = 0.842700792949715  time = 0.0253 secs
```

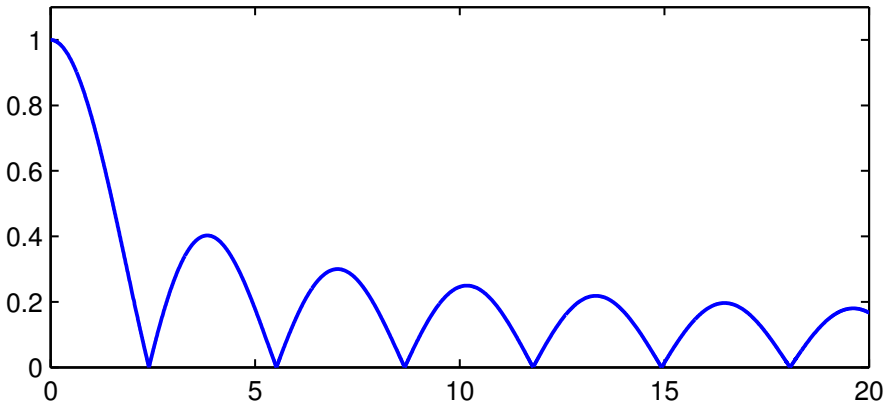
The timing for Chebfun comes out competitive:

```
tic, I = sum(chebfun(F,[0,1])); t = toc;
  fprintf('CHEBFUN:  I = %17.15f  time = %6.4f secs\n',I,t)
```

```
CHEBFUN:  I = 0.842700792949715  time = 0.0055 secs
```

Here is a similar comparison for a function that is more difficult, because of the absolute value, which leads with "splitting on" to a chebfun consisting of a number of funs.

```
F = @(x) abs(besselj(0,x));
f = chebfun(@(x) abs(besselj(0,x)),[0 20],'splitting','on');
plot(f), ylim([0 1.1])
```

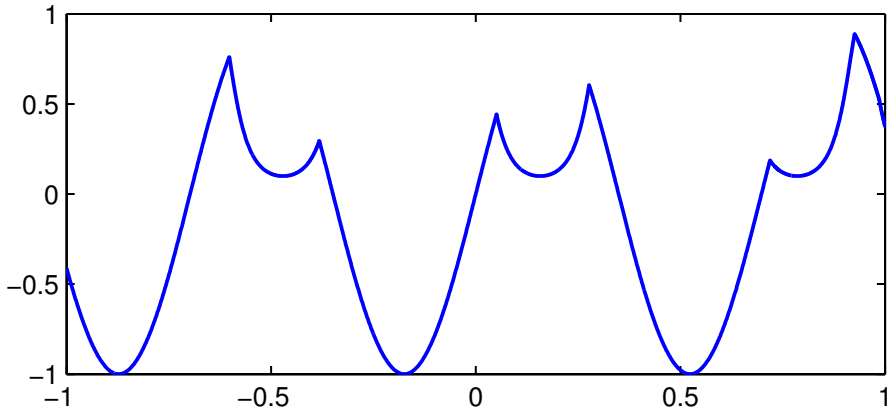



```
tol = 3e-14;
tic, I = quad(F,0,20,tol); t = toc;
  fprintf('   QUAD:  I = %17.15f  time = %5.3f secs\n',I,t)
tic, I = quadl(F,0,20,tol); t = toc;
  fprintf('  QUADL:  I = %17.15f  time = %5.3f secs\n',I,t)
tic, I = quadgk(F,0,20,'abstol',tol,'reltol',tol); t = toc;
  fprintf(' QUADGK:  I = %17.15f  time = %5.3f secs\n',I,t)
tic, I = sum(chebfun(@(x) abs(besselj(0,x)),[0,20],'splitting','on')); t = toc;
  fprintf('CHEBFUN:  I = %17.15f  time = %5.3f secs\n',I,t)
```

```
   QUAD:  I = 4.445031603001505  time = 0.077 secs
  QUADL:  I = 4.445031603001576  time = 0.046 secs
 QUADGK:  I = 4.445031603001578  time = 0.016 secs
CHEBFUN:  I = 4.445031603001567  time = 1.208 secs
```

This last example highlights the piecewise-smooth aspect of Chebfun integration. Here is another example of a piecewise smooth problem.

```
x = chebfun('x');
f = sech(3*sin(10*x));
g = sin(9*x);
h = min(f,g);
plot(h)
```



Here is the integral:

```
  tic, sum(h), toc
```

```
ans =
  -0.381556448850250
Elapsed time is 0.000860 seconds.
```

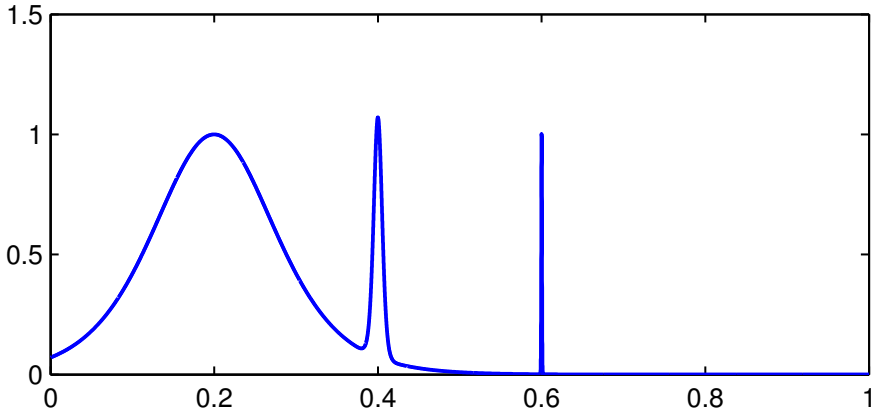
For another example of a definite integral we turn to an integrand given as example `F21F` in [Kahaner 1971]. We treat it first in the default mode of splitting off:

```
  ff = @(x) sech(10*(x-0.2)).^2 + sech(100*(x-0.4)).^4 + sech(1000*(x-0.6)).^6;
  f = chebfun(ff,[0,1])
```

```
f =
   chebfun column (1 smooth piece)
       interval       length   endpoint values
[       0,       1]    12318     0.071  4.5e-07
Epslevel = 4.547474e-13.  Vscale = 1.070659e+00.
```

The function has three spikes, each ten times narrower than the last:

```
  plot(f)
```



The length of the global polynomial representation is accordingly quite large, but the integral comes out correct to full precision:

```
length(f)
sum(f)
```

```
ans =
       12318
ans =
   0.210802735500549
```

With splitting on, Chebfun misses the narrowest spike, and the integral comes out too small:

```
f = chebfun(ff,[0,1],'splitting','on');
length(f)
sum(f)
```

```
ans =
   228
ans =
   0.209736068833883
```

We can fix the problem by forcing finer initial sampling in the Chebfun constructor with the `minSamples` flag:

```
f = chebfun(ff,[0,1],'splitting','on','minSamples',100);
length(f)
sum(f)
```

```
ans =
   373
ans =
   0.210802735500549
```

Now the integral is correct again, and note that the length of the chebfun is much smaller than with the original global representation. For more about `minSamples`, see Section 8.6.

As mentioned in Chapter 1 and described in more detail in Chapter 9, Chebfun has some capability of dealing with functions that blow up to infinity. Here for example is a familiar integral:

```
f = chebfun(@(x) 1./sqrt(x),[0 1],'blowup',2);
sum(f)
```

```
ans =
   2.000000000000000
```

Certain integrals over infinite domains can also be computed, though the error is often large:

```
f = chebfun(@(x) 1./x.^2.5,[1 inf]);
sum(f)
```

```
Warning: Result may not be accurate as the function decays slowly at infinity.
ans =
   0.666666674295143
```

Chebfun is not a specialized item of quadrature software; it is a general system for manipulating functions in which quadrature is just one of many capabilities. Nevertheless Chebfun compares reasonably well as a quadrature engine against specialized software. This was the conclusion of an Oxford MSc thesis by Phil Assheton [Assheton 2008], which compared Chebfun experimentally to quadrature codes including MATLAB's `quad` and `quadl`, Gander and Gautschi's `adaptsim` and `adaptlob`, Espelid's `modsim`, `modlob`, `coteda`, and `coteglob`, QUADPACK's `QAG` and `QAGS`, and the NAG Library's `d01ah`. In both reliability and speed, Chebfun was found to be competitive with these alternatives. The overall winner was `coteda` [Espelid 2003], which was typically about twice as fast as Chebfun. For further comparisons of quadrature codes, together with the development of some improved codes based on a philosophy that has something in common with Chebfun, see [Gonnet 2009]. See also "Battery test of Chebfun as an integrator" in the Quadrature section of the Chebfun Examples collection.

2.2 `norm`, `mean`, `std`, `var`

A special case of an integral is the `norm` command, which for a chebfun returns by default the 2-norm, i.e., the square root of the integral of the square of the absolute value over the region of definition. Here is a well-known example:

```
norm(chebfun('sin(pi*theta)'))
```

```
ans =
     1
```

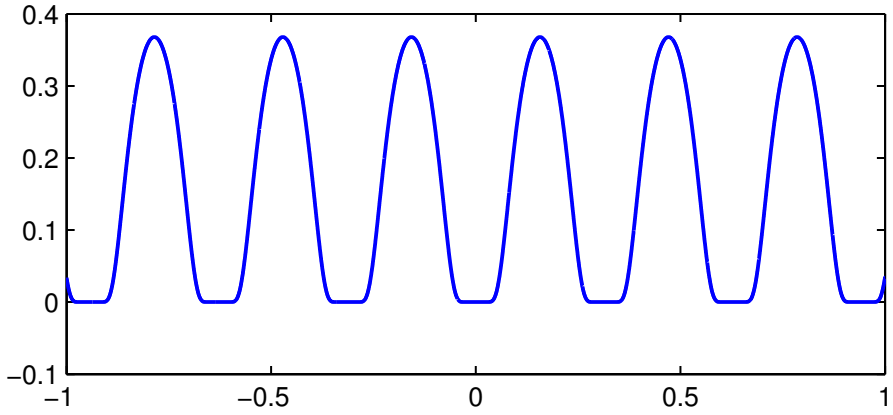
If we take the sign of the sine, the norm increases to $\sqrt{2}$:

```
norm(chebfun('sign(sin(pi*theta))','splitting','on'))
```

```
ans =
   1.414213562373095
```

Here is a function that is infinitely differentiable but not analytic.

```
f = chebfun('exp(-1./sin(10*x).^2)');
plot(f)
```



Here are the norms of `f` and its tenth power:

```
norm(f), norm(f.^10)
```

```
ans =
   0.292873834331035
ans =
     2.187941295308666e-05
```

2.3 cumsum

In MATLAB, `cumsum` gives the cumulative sum of a vector,

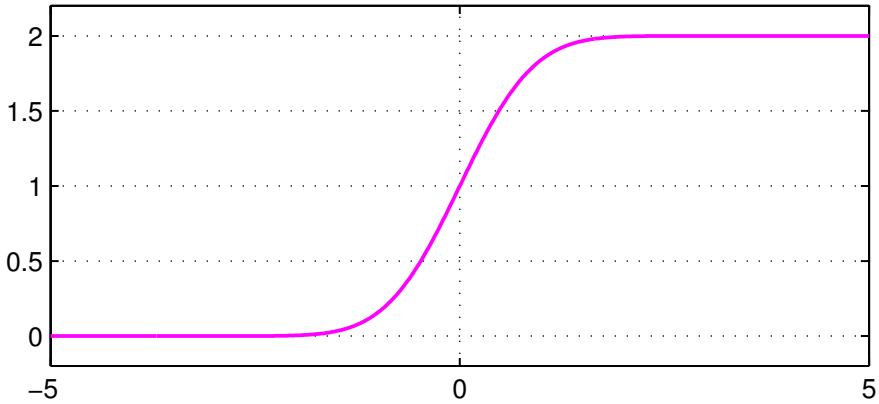
```
v = [1 2 3 5]
cumsum(v)
```

```
v =
     1     2     3     5
ans =
     1     3     6    11
```

The continuous analogue of this operation is indefinite integration. If `f` is a fun of length n, then `cumsum(f)` is a fun of length $n + 1$. For a chebfun consisting of several funs, the integration is performed on each piece.

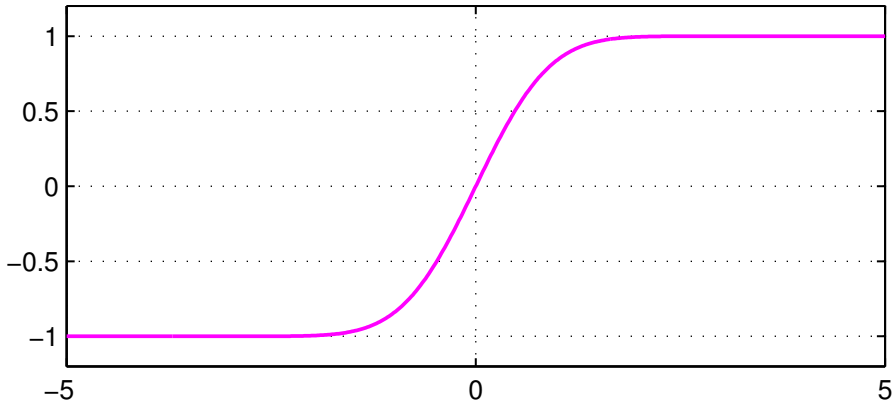
For example, returning to an integral computed above, we can make our own error function like this:

```
t = chebfun('t',[-5 5]);
f = 2*exp(-t.^2)/sqrt(pi);
fint = cumsum(f);
plot(fint,'m')
ylim([-0.2 2.2]), grid on
```



The default indefinite integral takes the value 0 at the left endpoint, but in this case we would like 0 to appear at $t = 0$:

```
fint = fint - fint(0);
plot(fint,'m')
ylim([-1.2 1.2]), grid on
```



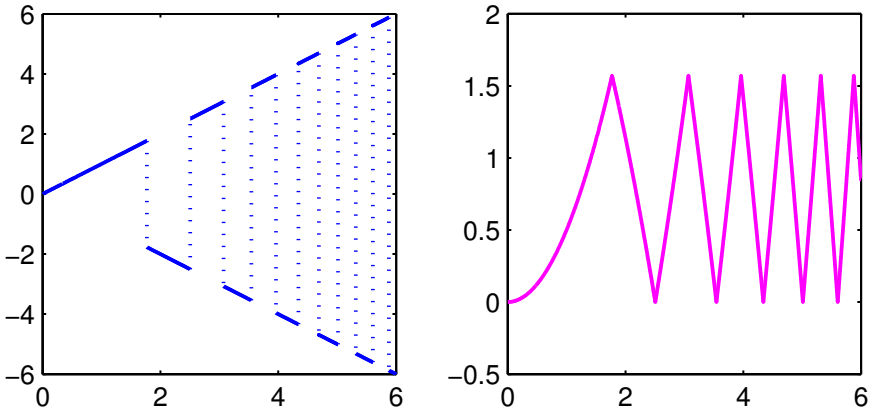
The agreement with the built-in error function is convincing:

```
[fint((1:5)') erf((1:5)')]
```

```
ans =
   0.842700792949715   0.842700792949715
   0.995322265018953   0.995322265018953
   0.999977909503002   0.999977909503001
   0.999999984582742   0.999999984582742
   0.999999999998463   0.999999999998463
```

Here is the integral of an oscillatory step function:

```
x = chebfun('x',[0 6]);
f = x.*sign(sin(x.^2)); subplot(1,2,1), plot(f)
g = cumsum(f); subplot(1,2,2), plot(g,'m')
```



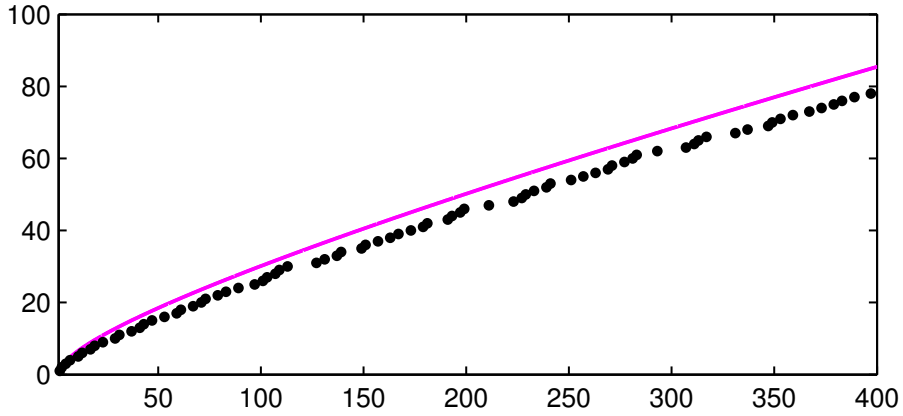
And here is an example from number theory. The logarithmic integral, $Li(x)$, is the indefinite integral from 0 to x of $1/\log(s)$. It is an approximation to $\pi(x)$, the number of primes less than or equal to x. To avoid the singularity at $x = 0$ we begin our integral at the point $\mu = 1.451...$ where $Li(x)$ is zero, known as Soldner's constant. The test value $Li(2)$ is correct except in the last digit:

```
mu = 1.45136923488338105;      % Soldner's constant
xmax = 400;
Li = cumsum(chebfun(@(x) 1./log(x),[mu xmax]));
lengthLi = length(Li)
Li2 = Li(2)
```

```
lengthLi =
   392
Li2 =
   1.045163780117242
```

(Chebfun has no trouble if `xmax` is increased to 10^5 or 10^{10}.) Here is a plot comparing $Li(x)$ with $\pi(x)$:

```
clf, plot(Li,'m')
p = primes(xmax);
hold on, plot(p,1:length(p),'.k')
```



The Prime Number Theorem implies that $\pi(x) \sim Li(x)$ as $x \to \infty$. Littlewood proved in 1914 that although $Li(x)$ is greater than $\pi(x)$ at first, the two curves eventually cross each other infinitely often. It is known that the first crossing occurs somewhere between $x = 10^{14}$ and $x = 2 \times 10^{316}$ [Kotnik 2008].

The `mean`, `std`, and `var` commands have also been overloaded for chebfuns and are based on integrals. For example,

```
mean(chebfun('cos(x).^2',[0,10*pi]))
```

```
ans =
   0.500000000000000
```

2.4 `diff`

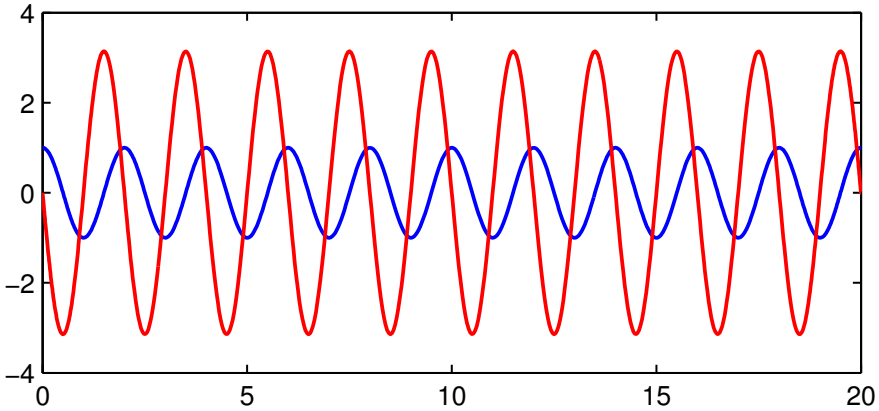
In MATLAB, `diff` gives finite differences of a vector:

```
v = [1 2 3 5]
diff(v)
```

```
v =
     1     2     3     5
ans =
     1     1     2
```

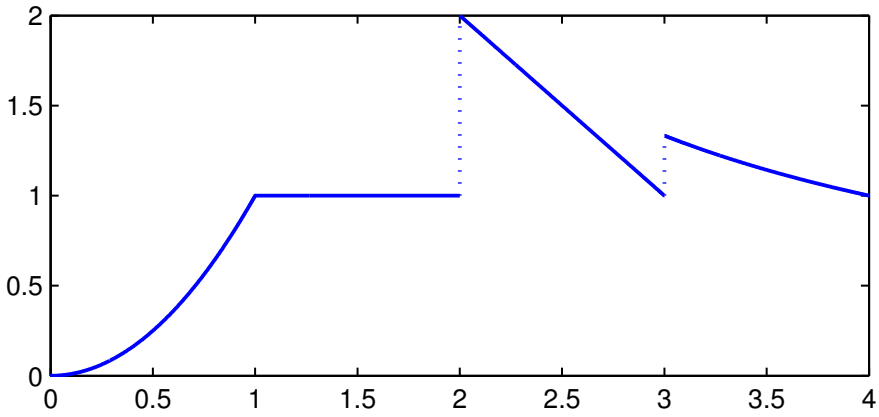
The continuous analogue of this operation is differentiation. For example:

```
f = chebfun('cos(pi*x)',[0 20]);
fprime = diff(f);
hold off, plot(f)
hold on, plot(fprime,'r')
```



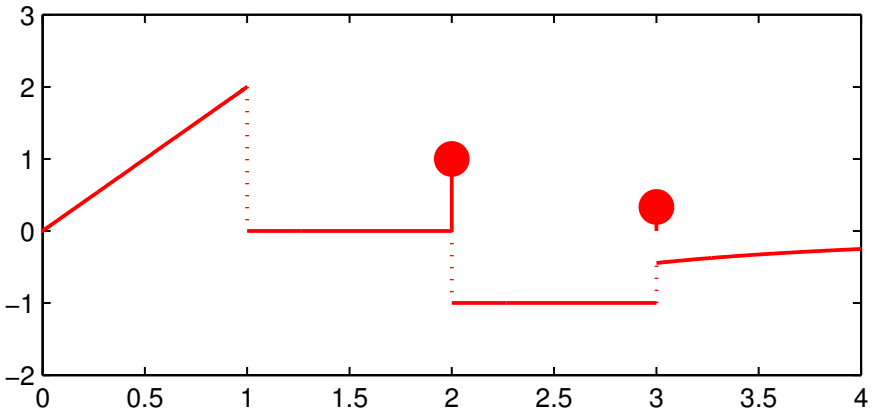
If the derivative of a function with a jump is computed, then a delta function is introduced. Consider for example this function defined piecewise:

```
f = chebfun({@(x) x.^2, 1, @(x) 4-x, @(x) 4./x},0:4);
hold off, plot(f)
```



Here is the derivative:

```
fprime = diff(f);
plot(fprime,'r'), ylim([-2,3])
```



The first segment of f' is linear, since f is quadratic here. Then comes a segment with $f' = 0$, since f is constant. At the end of this second segment appears a delta function of amplitude 1, corresponding to the jump of f by 1. The third segment has constant value $f' = -1$. Finally another delta function, this time with amplitude $1/3$, takes us to the final segment.

Thanks to the delta functions, `cumsum` and `diff` are essentially inverse operations. It is no surprise that differentiating an indefinite integral returns us to the original function:

```
norm(f-diff(cumsum(f)))
```

```
ans =
     2.387305176974604e-15
```

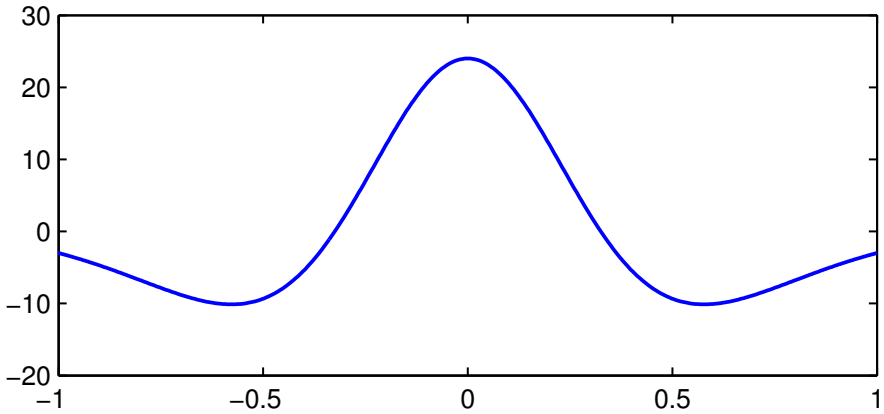
More surprising is that integrating a derivative does the same, as long as we add in the value at the left endpoint:

```
d = domain(f);
f2 = f(d(1)) + cumsum(diff(f));
norm(f-f2)
```

```
ans =
   1.000000000000000
```

Multiple derivatives can be obtained by adding a second argument to `diff`. Thus for example,

```
f = chebfun('1./(1+x.^2)');
g = diff(f,4); plot(g)
```



However, one should be cautious about the potential loss of information in repeated differentiation. For example, if we evaluate this fourth derivative at $x = 0$ we get an answer that matches the correct value 24 only to 11 places:

```
g(0)
```

```
ans =
  24.000000000055742
```

For a more extreme example, suppose we define a chebfun for $\exp(x)$ on $[-1, 1]$:

```
f = chebfun('exp(x)');
length(f)
```

```
ans =
    15
```

Differentiation is a notoriously ill-posed problem, and since f is a polynomial of low degree, it cannot help but lose information rather fast as we differentiate. In fact, differentiating 15 times eliminates the function entirely.

```
for j = 0:length(f)
  fprintf('%6d %19.12f\n', j,f(1))
  f = diff(f);
end
```

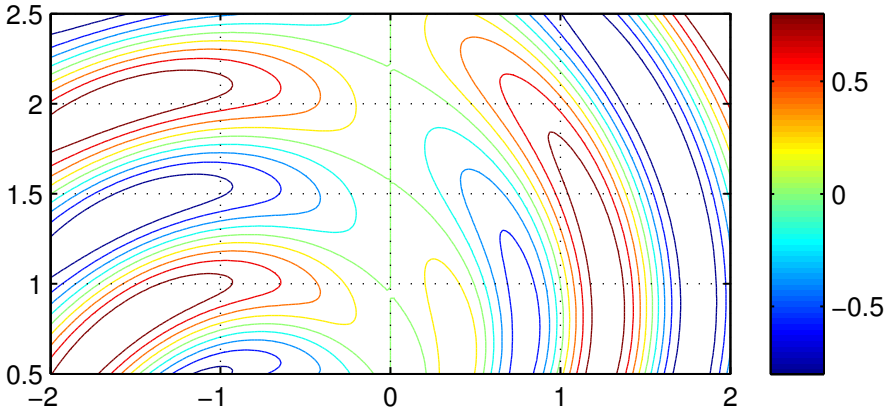
```
 0      2.718281828459
 1      2.718281828459
 2      2.718281828460
 3      2.718281828476
 4      2.718281828599
 5      2.718281824149
 6      2.718281637734
 7      2.718277739843
 8      2.718221404644
 9      2.717612947873
10      2.712588330520
11      2.680959565421
12      2.532851885655
13      2.043523780708
14      1.020835497184
15      0.000000000000
```

Is such behavior "wrong"? Well, that is an interesting question. Chebfun is behaving correctly in the sense mentioned in the second paragraph of Section 1.1: the operations are individually stable in that each differentiation returns the exact derivative of a function very close to the right one. The trouble is that because of the intrinsically ill-posed nature of differentiation, the errors in these stable operations accumulate exponentially as successive derivatives are taken.

2.5 Integrals in two dimensions

Chebfun can often do a pretty good job with integrals over rectangles. Here for example is a colorful function:

```
r = @(x,y) sqrt(x.^2+y.^2); theta = @(x,y) atan2(y,x);
f = @(x,y) sin(5*(theta(x,y)-r(x,y))).*sin(x);
x = -2:.02:2; y = 0.5:.02:2.5; [xx,yy] = meshgrid(x,y);
clf, contour(x,y,f(xx,yy),-1:.2:1)
axis([-2 2 0.5 2.5]), colorbar, grid on
```



Using 1D Chebfun technology, we can compute the integral over the box like this. Notice the use of the flag `vectorize` to construct a chebfun from a function only defined for scalar arguments.

```
Iy = @(y) sum(chebfun(@(x) f(x,y),[-2 2]));
tic, I = sum(chebfun(@(y) Iy(y),[0.5 2.5],'vectorize')); t = toc;
fprintf('CHEBFUN:  I = %16.14f  time = %5.3f secs\n',I,t)
```

```
CHEBFUN:  I = 0.02041246545700  time = 0.261 secs
```

Here for comparison is MATLAB's `dblquad/quadl` with a tolerance of 10^{-11}:

```
tic, I = dblquad(f,-2,2,0.5,2.5,1e-11,@quadl); t = toc;
fprintf('DBLQUAD/QUADL:  I = %16.14f  time = %5.3f secs\n',I,t)
```

```
DBLQUAD/QUADL:  I = 0.02041246545700  time = 2.625 secs
```

This example of a 2D integrand is smooth, so both Chebfun and `dblquad` can handle it to high accuracy.

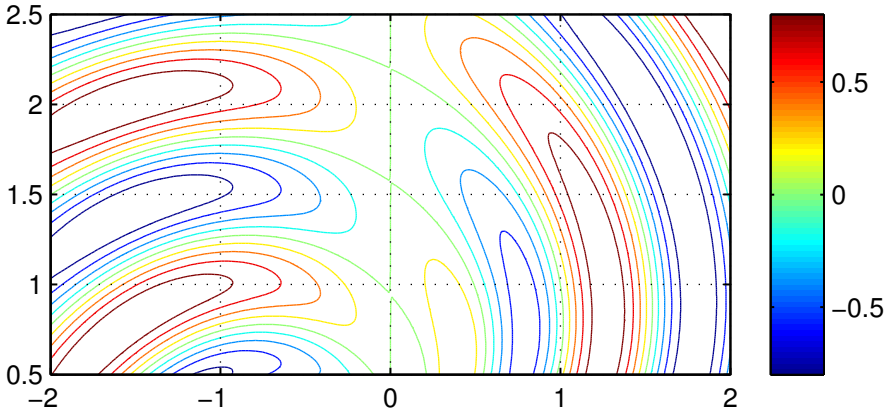
A much better approach for this problem, however, is to use Chebfun2, which is described in Chapters 11-15. With this method we can compute the integral quickly,

```
tic
f2 = chebfun2(f,[-2 2 0.5 2.5]);
sum2(f2)
toc
```

```
ans =
   0.020412465456999
Elapsed time is 0.220325 seconds.
```

and we can plot the function without the need for `meshgrid`:

```
contour(f2,-1:.2:1), colorbar, grid on
```



2.6 Gauss and Gauss-Jacobi quadrature

For quadrature experts, Chebfun contains some powerful capabilities due to Nick Hale and Alex Townsend [Hale & Townsend 2013]. To start with, suppose we wish to carry out 4-point Gauss quadrature over $[-1, 1]$. The quadrature nodes are the zeros of the degree 4 Legendre polynomial `legpoly(4)`, which can be obtained from the Chebfun command `legpts`, and if two output arguments are requested, `legpts` provides weights also:

```
  [s,w] = legpts(4)
```

```
s =
  -0.861136311594053
  -0.339981043584856
   0.339981043584856
   0.861136311594053
w =
  Columns 1 through 3
   0.347854845137454   0.652145154862546   0.652145154862546
  Column 4
   0.347854845137454
```

To compute the 4-point Gauss quadrature approximation to the integral of $\exp(x)$ from -1 to 1, for example, we could now do this:

```
  x = chebfun('x');
  f = exp(x);
  Igauss = w*f(s)
  Iexact = exp(1) - exp(-1)
```

```
Igauss =
   2.350402092156377
Iexact =
   2.350402387287603
```

There is no need to stop at 4 points, however. Here we use 1000 Gauss quadrature points:

```
  tic
  [s,w] = legpts(1000); Igauss = w*f(s)
  toc
```

```
Igauss =
   2.350402387287603
Elapsed time is 0.066921 seconds.
```

Even 100,000 points doesn't take very long:

```
  tic
  [s,w] = legpts(100000); Igauss = w*f(s)
  toc
```

```
Igauss =
   2.350402387287603
Elapsed time is 0.155497 seconds.
```

Traditionally, numerical analysts computed Gauss quadrature nodes and weights by the eigenvalue algorithm of Golub and Welsch [Golub & Welsch 1969]. However, the Hale-Townsend algorithms are both more accurate and much faster [Hale & Townsend 2013]. Closely related fast algorithms developed independently are presented in [Bogaert, Michiels & Fostier 2012].

For Legendre polynomials, Legendre points, and Gauss quadrature, use `legpoly` and `legpts`. For Chebyshev polynomials, Chebyshev points, and Clenshaw-Curtis quadrature, use `chebpoly` and `chebpts` and the built-in Chebfun commands such as `sum`. A third variant is also available: for Jacobi polynomials, Gauss-Jacobi points, and Gauss-Jacobi quadrature, see `jacpoly` and `jacpts`. These arise in integration of functions with singularities at one or both endpoints, and are used internally by Chebfun for integration of chebfuns with singularities (Chapter 9).

As explained in the help texts, all of these operators work on general intervals $[a, b]$, not just on $[-1, 1]$.

2.7 References

[Assheton 2008] P. Assheton, *Comparing Chebfun to Adaptive Quadrature Software*, dissertation, MSc in Mathematical Modelling and Scientific Computing, Oxford University, 2008.

[Bogaert, Michiels, and Fostier, "O(1) computation of Legendre polynomials and Legendre nodes and weights for parallel computing", *SIAM Journal on Scientific Computing*,34 (2012), C83-C101.

[Espelid 2003] T. O. Espelid, "Doubly adaptive quadrature routines based on Newton-Cotes rules", *BIT Numerical Mathematics*, 43 (2003), 319-337.

[Gentleman 1972] W. M. Gentleman, "Implementing Clenshaw-Curtis quadrature I and II", *Journal of the ACM*, 15 (1972), 337-346 and 353.

[Golub & Welsch 1969] G. H. Golub and J. H. Welsch, "Calculation of Gauss quadrature rules", *Mathematics of Computation*, 23 (1969), 221-230.

[Gonnet 2009] P. Gonnet, *Adaptive Quadrature Re-Revisited*, ETH dissertation no. 18347, Swiss Federal Institute of Technology, 2009.

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[Hale & Trefethen 2012] N. Hale and L. N. Trefethen, Chebfun and numerical quadrature, *Science in China*, 55 (2012), 1749-1760.

[Kahaner 1971] D. K. Kahaner, "Comparison of numerical quadrature formulas", in J. R. Rice, ed., *Mathematical Software*, Academic Press, 1971, 229-259.

[Kotnik 2008] T. Kotnik, "The prime-counting function and its analytic approximations", *Advances in Computational Mathematics*, 29 (2008), 55-70.

[Wolfram 2003] S. Wolfram, *The Mathematica Book*, 5th ed., Wolfram Media, 2003.

3. Rootfinding and Minima and Maxima

Lloyd N. Trefethen, October 2009, latest revision June 2014

Contents

3.1 `roots`

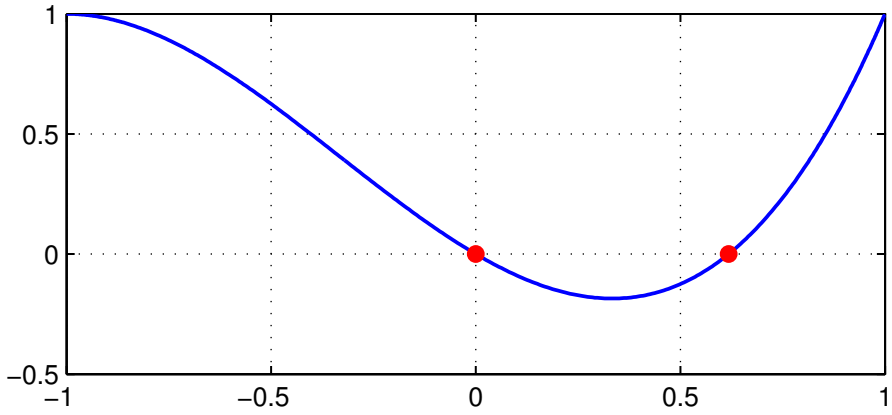
Chebfun comes with a global rootfinding capability – the ability to find all the zeros of a function in its region of definition. For example, here is a polynomial with two roots in $[-1,1]$:

```
x = chebfun('x');
p = x.^3 + x.^2 - x;
r = roots(p)
```

```
r =
                   0
   0.618033988749895
```

We can plot p and its roots like this:

```
plot(p), grid on
MS = 'markersize';
hold on, plot(r,p(r),'.r',MS,20)
```



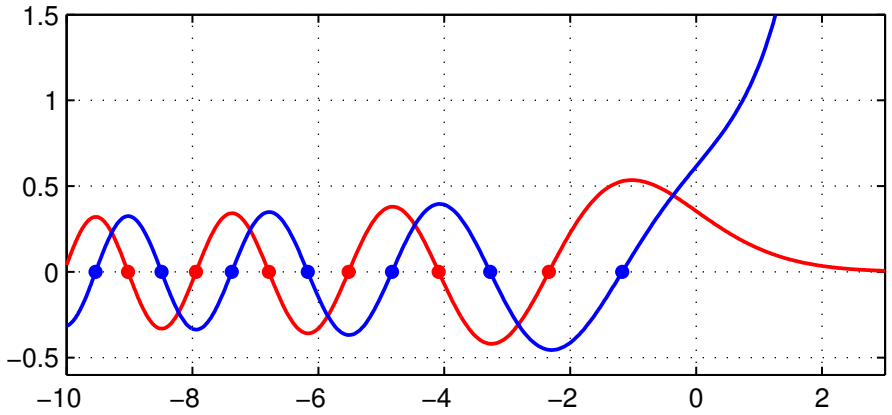
Of course, one does not need Chebfun to find roots of a polynomial. The MATLAB `roots` command works from a polynomial's coefficients and computes estimates of all the roots, not just those in a particular interval.

```
roots([1 1 -1 0])
```

```
ans =
                   0
  -1.618033988749895
   0.618033988749895
```

A more substantial example of rootfinding involving a Bessel function was considered in Sections 1.2 and 2.4. Here is a similar calculation for the Airy functions Ai and Bi, modeled after the page on Airy functions at WolframMathWorld.

```
Ai = chebfun(@(x) airy(0,x),[-10,3]);
Bi = chebfun(@(x) airy(2,x),[-10,3]);
hold off, plot(Ai,'r')
hold on, plot(Bi,'b')
rA = roots(Ai); plot(rA,Ai(rA),'.r',MS,16)
rB = roots(Bi); plot(rB,Bi(rB),'.b',MS,16)
axis([-10 3 -.6 1.5]), grid on
```



Here for example are the three roots of Ai and Bi closest to 0:

```
[rA(end-2:end) rB(end-2:end)]
```

```
ans =
  -5.520559828095558  -4.830737841662006
  -4.087949444130973  -3.271093302836356
  -2.338107410459772  -1.173713222709129
```

Chebfun finds roots by a method due to Boyd and Battles [Boyd 2002, Battles 2006]. If the chebfun is of degree greater than about 50, it is broken into smaller pieces recursively. On each small piece zeros are then found as eigenvalues of a "colleague matrix", the analogue for Chebyshev polynomials of a companion matrix for monomials [Specht 1960, Good 1961]. This method can be startlingly fast and accurate. For example, here is a sine function with 11 zeros:

```
f = chebfun('sin(pi*x)',[0 10]);
lengthf = length(f)
tic, r = roots(f); toc
r
```

```
lengthf =
    44
Elapsed time is 0.003508 seconds.
r =
                   0
   0.999999999999997
   1.999999999999999
   3.000000000000000
   4.000000000000000
   4.999999999999999
   6.000000000000001
   7.000000000000000
   8.000000000000004
   9.000000000000002
  10.000000000000000
```

A similar computation with 101 zeros comes out equally well:

```
f = chebfun('sin(pi*x)',[0 100]);
lengthf = length(f)
tic, r = roots(f); toc
fprintf('%22.14f\n',r(end-4:end))
```

```
lengthf =
   214
Elapsed time is 0.011363 seconds.
     96.00000000000000
     97.00000000000000
     98.00000000000001
     99.00000000000000
    100.00000000000000
```

And here is the same on an interval with 1001 zeros.

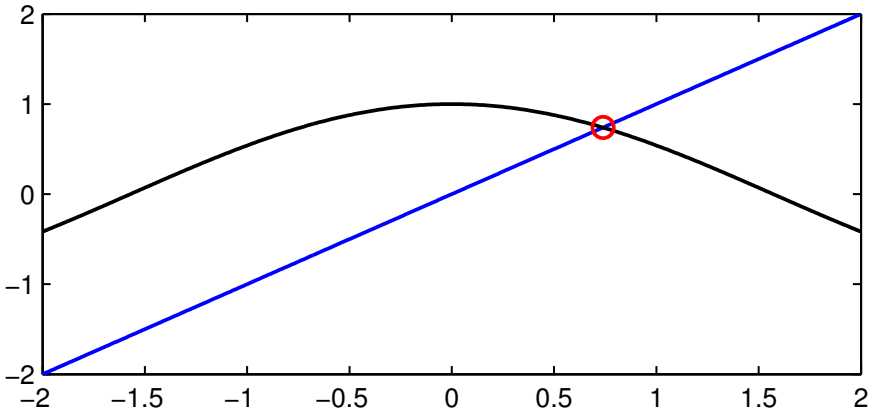
```
f = chebfun('sin(pi*x)',[0 1000]);
lengthf = length(f)
tic, r = roots(f); toc
fprintf('%22.13f\n',r(end-4:end))
```

```
lengthf =
        1684
Elapsed time is 0.184578 seconds.
     996.0000000000000
     996.9999999999999
     998.0000000000000
     999.0000000000000
    1000.0000000000000
```

With the ability to find zeros, we can solve a variety of nonlinear problems. For example, where do the curves x and $\cos(x)$ intersect? Here is the answer.

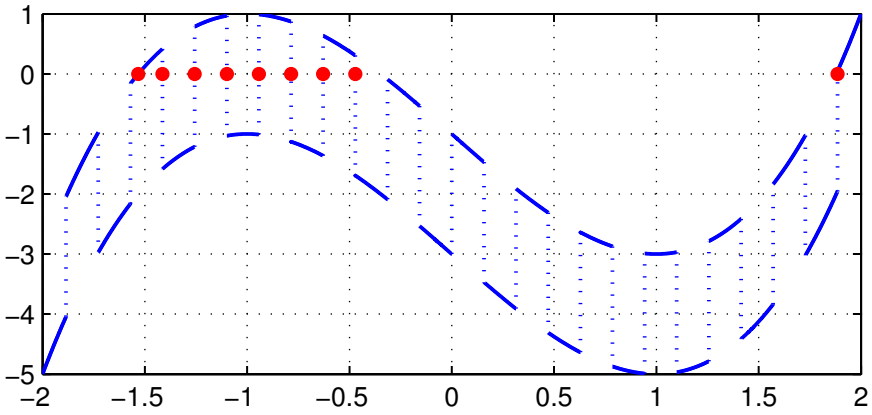
```
x = chebfun('x',[-2 2]);
hold off, plot(x)
f = cos(x);
hold on, plot(f,'k')
r = roots(f-x)
plot(r,f(r),'or',MS,8)
```

```
r =
   0.739085133215160
```



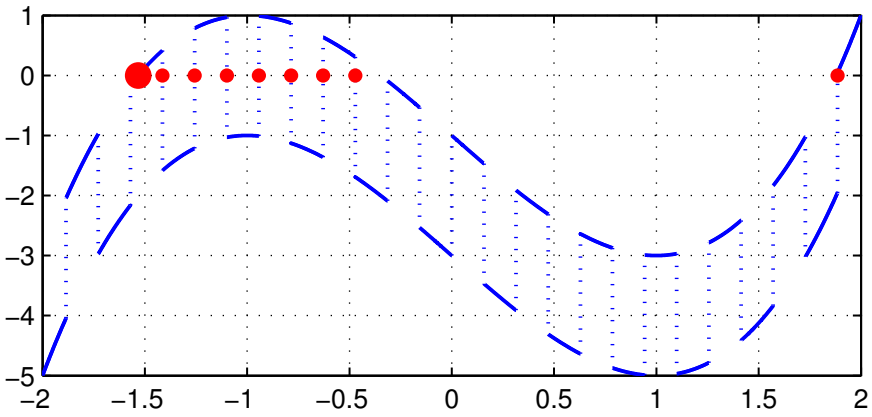
All of the examples above concern chebfuns consisting of a single fun. If there are several funs, then roots are included at jumps as necessary. The following example shows a chebfun with 26 pieces. It has nine zeros: one within a fun, eight at jumps between funs.

```
x = chebfun('x',[-2 2]);
f = x.^3 - 3*x - 2 + sign(sin(20*x));
hold off, plot(f), grid on
r = roots(f);
hold on, plot(r,0*r,'.r',MS,16)
```



If one prefers only the "genuine" roots, omitting those at jumps, they can be computed like this:

```
r = roots(f,'nojump');
plot(r,0*r,'.r',MS,30)
```



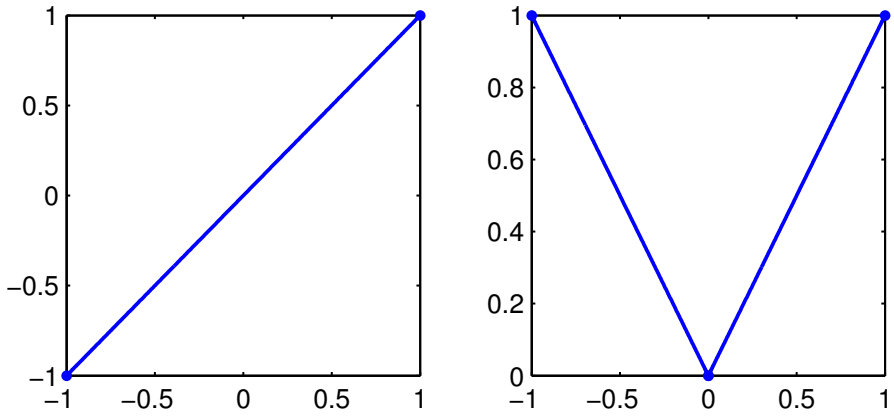
3.2 `min`, `max`, `abs`, `sign`, `round`, `floor`, `ceil`

Rootfinding is more central to Chebfun than one might at first imagine, because a number of commands, when applied to smooth chebfuns, must produce non-smooth results, and it is rootfinding that tells us where to put the discontinuities. For example, the `abs` command introduces breakpoints wherever the argument goes through zero. Here we see that `x` consists of a single piece, whereas `abs(x)` consists of two pieces.

```
x = chebfun('x')
absx = abs(x)
subplot(1,2,1), plot(x,'.-')
subplot(1,2,2), plot(absx,'.-')
```

```
x =
   chebfun column (1 smooth piece)
       interval       length   endpoint values
[      -1,       1]        2        -1        1
```

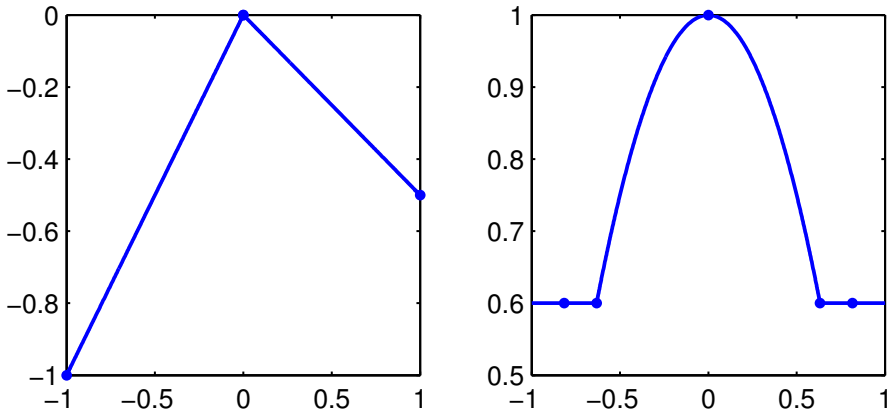
```
Epslevel = 1.110223e-15.  Vscale = 1.
absx =
   chebfun column (2 smooth pieces)
       interval       length   endpoint values
[      -1,       0]        2         1        0
[       0,       1]        2         0        1
Epslevel = 1.110223e-15.  Vscale = 1.  Total length = 4.
```



We saw this effect already in Section 1.4. Another similar effect shown in that section occurs with `min(f,g)` or `max(f,g)`. Here, breakpoints are introduced at points where `f-g` is zero:

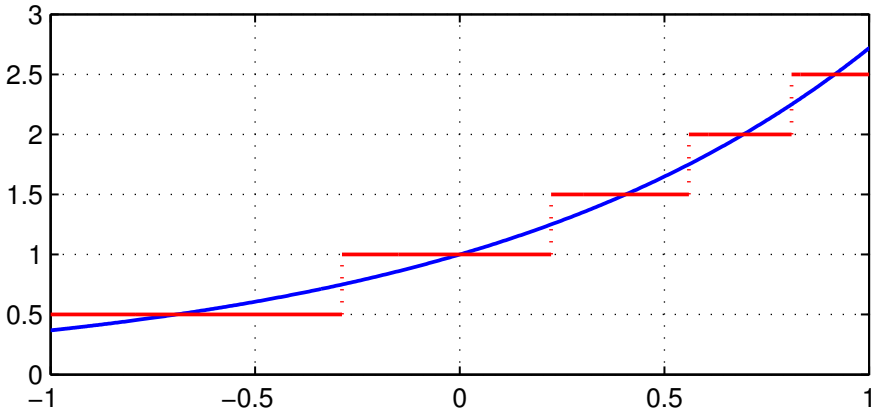
```
f = min(x,-x/2), subplot(1,2,1), plot(f,'.-')
g = max(.6,1-x.^2), subplot(1,2,2), plot(g,'.-'), ylim([.5,1])
```

```
f =
   chebfun column (2 smooth pieces)
       interval       length   endpoint values
[      -1,       0]        2        -1        0
[       0,       1]        2         0     -0.5
Epslevel = 1.332268e-15.  Vscale = 1.  Total length = 4.
g =
   chebfun column (3 smooth pieces)
       interval       length   endpoint values
[      -1,   -0.63]        1       0.6      0.6
[   -0.63,    0.63]        3       0.6      0.6
[    0.63,       1]        1       0.6      0.6
Epslevel = 2.220446e-16.  Vscale = 1.  Total length = 5.
```



The function `sign` also introduces breaks, as illustrated in the last section. The commands `round`, `floor`, and `ceil` behave like this too. For example, here is $\exp(x)$ rounded to nearest multiples of 0.5:

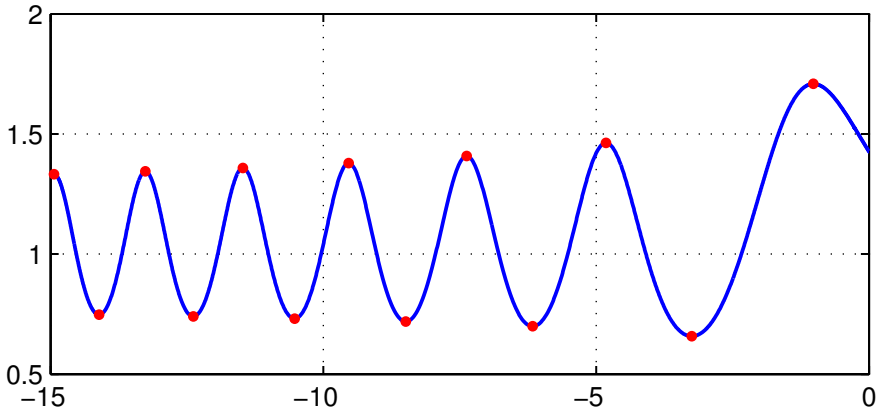
```
g = exp(x);
clf, plot(g)
gh = 0.5*round(2*g);
hold on, plot(gh,'r');
grid on
```



3.3 Local extrema

Local extrema of smooth functions can be located by finding zeros of the derivative. For example, here is a variant of the Airy function again, with all its extrema in its range of definition located and plotted.

```
f = chebfun('exp(real(airy(x)))',[-15,0]);
clf, plot(f)
r = roots(diff(f));
hold on, plot(r,f(r),'.r'), grid on
```



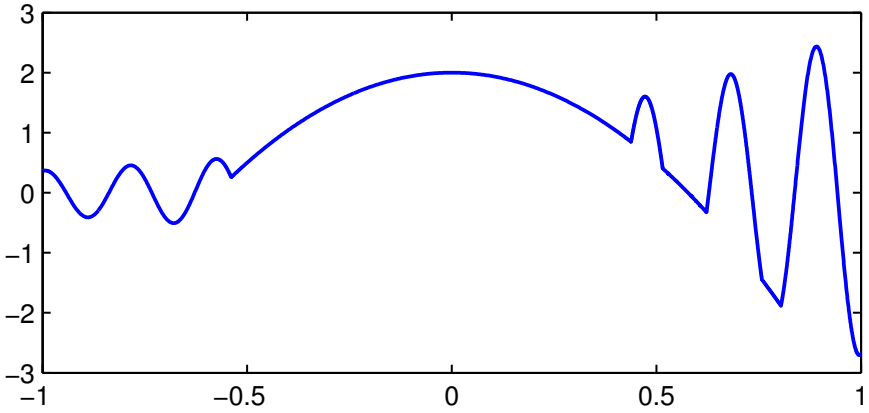
Chebfun users don't have to compute the derivative explicitly to find extrema, however. An alternative is to type

```
[ignored,r2] = minandmax(f,'local');
```

which returns both interior local extrema and also the endpoints of `f`. Similarly one can type `min(f,'local')` and `max(f,'local')`.

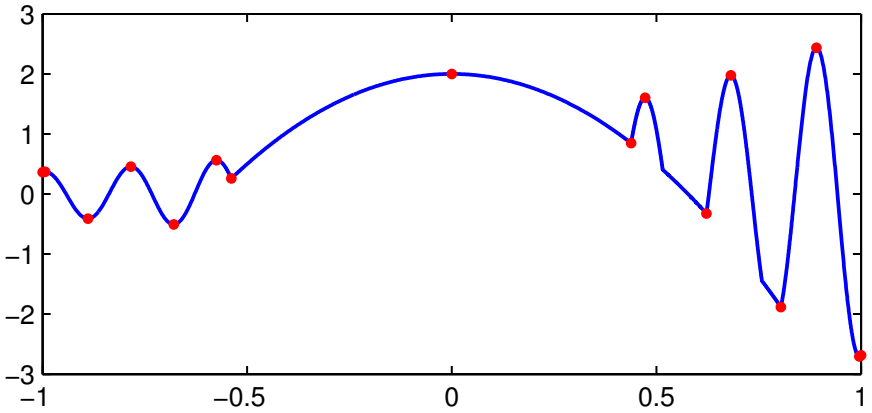
These methods will find non-smooth extrema as well as smooth ones, since these correspond to "zeros" of the derivative where the derivative jumps from one sign to the other. Here is an example.

```
x = chebfun('x');
f = exp(x).*sin(30*x);
g = 2-6*x.^2;
h = max(f,g);
clf, plot(h)
```



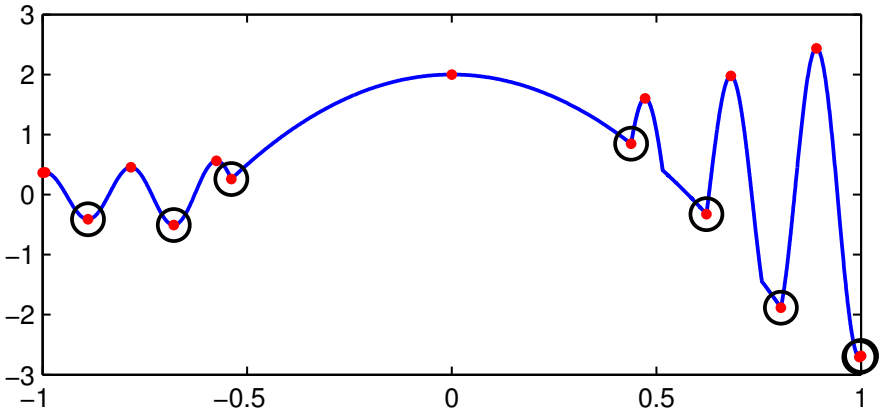
Here are all the local extrema, smooth and nonsmooth:

```
[ignored,extrema] = minandmax(h,'local');
hold on, plot(extrema,h(extrema),'.r')
```



Suppose we want to pick out the local extrema that are actually local minima. We can do that by hand by checking for the second derivative to be positive:

```
h2 = diff(h,2);
maxima = extrema(h2(extrema)>0);
plot(maxima,h(maxima),'ok',MS,12)
```

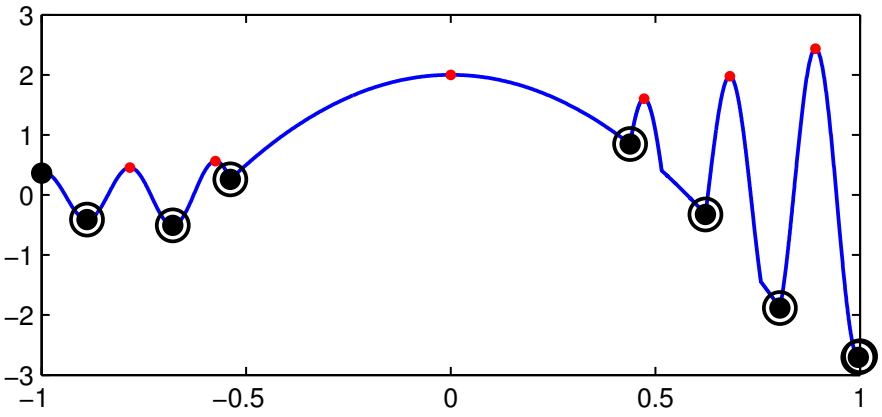


Or we could do it implicitly with `local`,

```
[maxval,maxpos] = min(h,'local')
plot(maxpos,maxval,'.k',MS,24)
```

```
maxval =
   0.363476521730995
  -0.410835468002949
  -0.506554704820389
   0.257807200650400
   0.848004324296445
  -0.324870829559027
  -1.882251804443052
  -2.705787848740123
maxpos =
  -1.000000000000000
  -0.889007218654500
```

```
-0.679567708415180
-0.538855701053878
 0.438177223602422
 0.622477687626771
 0.804389189016844
 0.995948373499376
```



3.4 Global extrema: max and min

If `min` or `max` is applied to a single chebfun, it returns its global minimum or maximum. For example:

```
f = chebfun('1-x.^2/2');
[min(f) max(f)]
```

```
ans =
   0.500000000000000   1.000000000000000
```

Chebfun computes such a result by checking the values of `f` at all endpoints and at zeros of the derivative.

As with standard MATLAB, one can find the location of the extreme point by supplying two output arguments:

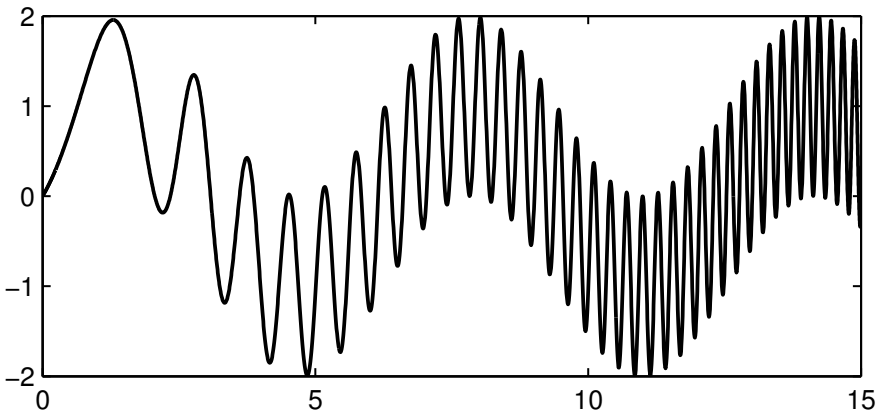
```
[minval,minpos] = min(f)
```

```
minval =
   0.500000000000000
minpos =
    -1
```

Note that just one position is returned even though the minimum is attained at two points. This is consistent with the behavior of standard MATLAB.

This ability to do global 1D optimization in Chebfun is rather remarkable. Here is a nontrivial example.

```
f = chebfun('sin(x)+sin(x.^2)',[0,15]);
hold off, plot(f,'k')
```



The length of this chebfun is not as great as one might imagine:

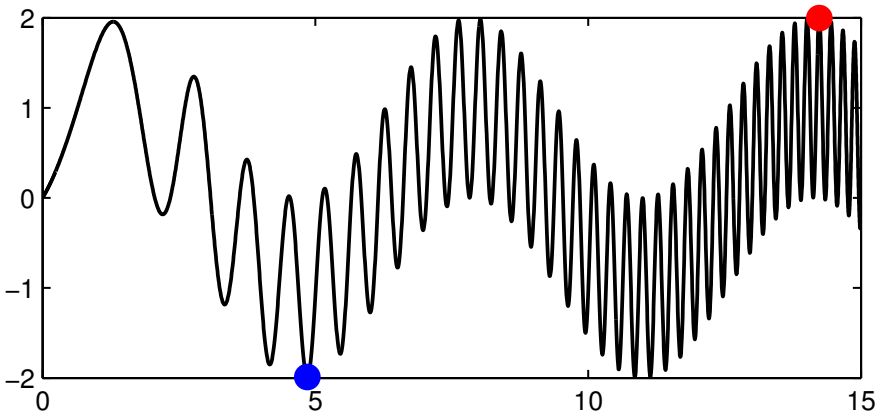
```
length(f)
```

```
ans =
   216
```

Here are its global minimum and maximum:

```
[minval,minpos] = min(f)
[maxval,maxpos] = max(f)
hold on
plot(minpos,minval,'.b',MS,30)
plot(maxpos,maxval,'.r',MS,30)
```

```
minval =
  -1.990085468159411
minpos =
   4.852581429906174
maxval =
   1.995232599437864
maxpos =
  14.234791972306914
```



For larger chebfuns, it is inefficient to compute the global minimum and maximum separately like this – each one must compute the derivative and find all its zeros. The alternative `minandmax` code mentioned above provides a faster alternative:

```
[extremevalues,extremepositions] = minandmax(f)
```

```
extremevalues =
  -1.990085468159411
   1.995232599437864
extremepositions =
   4.852581429906174
  14.234791972306914
```

3.5 `norm(f,1)` and `norm(f,inf)`

The default, 2-norm form of the `norm` command was considered in Section 2.2. In standard MATLAB one can also compute 1-, ∞-, and Frobenius norms with `norm(f,1)`, `norm(f,inf)`, and `norm(f,'fro')`. These have been overloaded in Chebfun, and in the first two cases, rootfinding is part of the implementation. (The 2- and Frobenius norms are equal for a single chebfun but differ for quasimatrices; see Chapter 6.) The 1-norm `norm(f,1)` is the integral of the absolute value, and Chebfun computes this by adding up segments between zeros, at which $|f(x)|$ will typically have a discontinuous slope. The ∞-norm is computed from the formula $\|f\|_\infty = \max(\max(f), -\min(f))$.

For example:

```
f = chebfun('sin(x)',[103 103+4*pi]);
norm(f,inf)
norm(f,1)
```

```
ans =
   1.000000000000004
ans =
   7.999999999999984
```

3.6 Roots in the complex plane

Chebfuns live on real intervals, and the funs from which they are made live on real subintervals. But a polynomial representing a fun may have roots outside the interval of definition, which may be complex. Sometimes we may want to get our hands on these roots, and the `roots` command makes this possible in various ways through the flags `'all'`, `'complex'`, and `'norecursion'`.

The simplest example is a chebfun that is truly intended to correspond to a polynomial. For example, the chebfun

```
f = 1+16*x.^2;
```

has no roots in $[-1, 1]$:

```
roots(f)
```

```
ans =
   Empty matrix: 0-by-1
```

We can extract its complex roots with the command

```
roots(f,'all')
```

```
ans =
  0.000000000000000 - 0.250000000000000i
  0.000000000000000 + 0.250000000000000i
```

The situation for more general chebfuns is more complicated. For example, the chebfun

```
g = exp(x).*f(x);
```

also has no roots in [-1,1],

```
roots(g)
```

```
ans =
   Empty matrix: 0-by-1
```

but it has plenty of roots in the complex plane:

```
roots(g,'all')
```

```
ans =
 -0.000000000000005 - 0.249999999999983i
 -0.000000000000005 + 0.249999999999983i
 -4.513839611821153 + 0.000000000000000i
 -4.304953574643823 - 1.512787764786624i
 -4.304953574643823 + 1.512787764786624i
 -3.665238911771629 - 2.987288348279861i
```

```
 -3.665238911771629 + 2.987288348279861i
 -2.550107949077733 - 4.375390939226574i
 -2.550107949077733 + 4.375390939226574i
 -0.861360687347932 - 5.603012177418021i
 -0.861360687347932 + 5.603012177418021i
  1.622627117319532 - 6.531305785733470i
  1.622627117319532 + 6.531305785733470i
  5.548850506877216 - 6.812852935529656i
  5.548850506877216 + 6.812852935529656i
```

Most of these are spurious. What has happened is that `g` is represented by a polynomial chosen for its approximation properties on $[-1, 1]$. Inevitably that polynomial will have roots in the complex plane, even if they have little to do with `g`. (See the discussion of the Walsh and Blatt-Saff theorems in Chapter 18 of [Trefethen 2013].)

One cannot expect Chebfun to solve this problem perfectly – after all, it is working on a real interval, not in the complex plane, and analytic continuation from the one to the other is well known to be an ill-posed problem. Nevertheless, Chebfun may do a pretty good job of selecting genuine complex (and real) roots near the interval of definition if you use the `'complex'` flag:

```
roots(g,'complex')
```

```
ans =
 -0.000000000000005 - 0.249999999999983i
 -0.000000000000005 + 0.249999999999983i
```

We will not go into detail here of how this is done, but the idea is that associated with any fun is a family of "Chebfun ellipses" in the complex plane, with foci at the endpoints, inside which one may expect reasonably good accuracy of the fun. Assuming the interval is $[-1, 1]$ and the fun has length L, the Chebfun ellipse associated with a parameter $\delta \ll 1$ is the region in the complex plane bounded by the image under the map $(z + 1/z)/2$ of the circle $|z| = r$, where r is defined by the condition $r^{-L} = \delta$. (See Chapter 8 of [Trefethen 2013].) The command `roots(g,'complex')` first effectively does `roots(g,'all')`, then returns only those roots lying inside a particular Chebfun ellipse – we take the one corresponding to delta equal to the square root of the Chebfun tolerance parameter `eps`.

One must expect complex roots of chebfuns to lose accuracy as one moves away from the interval of definition. Here's an example:

```
f = exp(exp(x)).*(x-0.1i).*(x-.3i).*(x-.6i).*(x-1i);
roots(f,'complex')
```

```
ans =
 -0.000000000001405 + 0.099999999998765i
  0.000000000832193 + 0.300000000573215i
 -0.000001488909431 + 0.599999277205987i
```

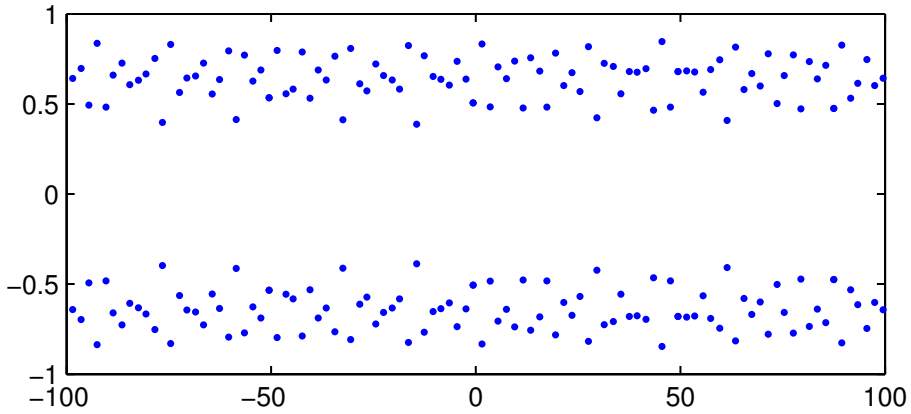
Notice that the first three imaginary roots are located with about 11, 8, and 5 digits of accuracy, while the fourth does not appear in the list at all.

Here is a more complicated example:

```
F = @(x) 4+sin(x)+sin(sqrt(2)*x)+sin(pi*x);
f = chebfun(F,[-100,100]);
```

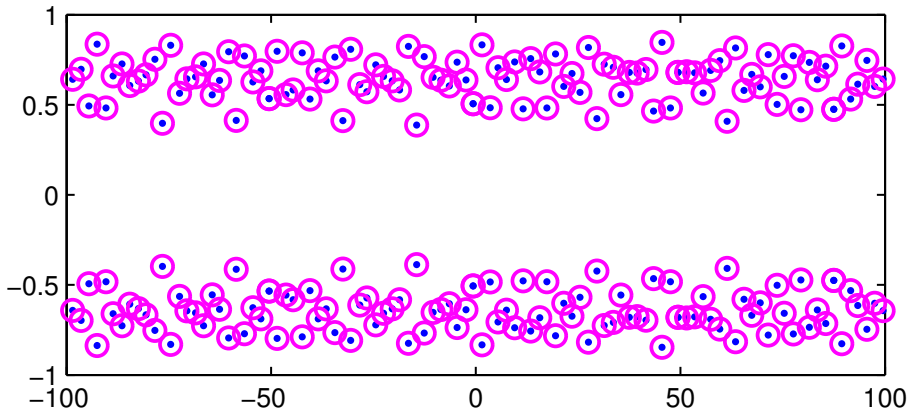
This function has a lot of complex roots lying in strips on either side of the real axis.

```
r = roots(f,'complex');
hold off, plot(r,'.',MS,8)
```



If you are dealing with complex roots of complicated chebfuns like this, it may be safer to add a flag `'norecursion'` to the roots call, at the cost of slowing down the computation. This turns off the Boyd-Battles recursion mentioned above, lowering the chance of missing a few roots near interfaces of the recursion. If we try that here we find that the results look almost the same as before in a plot:

```
r2 = roots(f,'complex','norecursion');
hold on, plot(r,'om',MS,8)
```



However, the accuracy has improved:

```
norm(F(r))
norm(F(r2))
```

```
ans =
     3.498622420314460e-08
```

```
ans =
     4.818930650722706e-09
```

To find poles in the complex plane as opposed to zeros, see Section 4.8. More advanced methods of rootfinding and polefinding are based on rational approximations rather than polynomials, an area where Chebfun has significant capabilities; see the next chapter of this guide, Chapter 28 of [Trefethen 2013], and [Webb 2013].

3.7 References

[Battles 2006] Z. Battles, *Numerical Linear Algebra for Continuous Functions*, DPhil thesis, Oxford University Computing Laboratory, 2006.

[Boyd 2002] J. A. Boyd, "Computing zeros on a real interval through Chebyshev expansion and polynomial rootfinding", *SIAM Journal on Numerical Analysis*, 40 (2002), 1666-1682.

[Good 1961] I. J. Good, "The colleague matrix, a Chebyshev analogue of the companion matrix", *Quarterly Journal of Mathematics*, 12 (1961), 61-68.

[Specht 1960] W. Specht, "Die Lage der Nullstellen eines Polynoms. IV", *Mathematische Nachrichten*, 21 (1960), 201-222.

[Trefethen 2013] L. N. Trefethen, *Approximation Theory and Approximation Practice*, SIAM, 2013.

[Webb 2011] M. Webb, "Computing complex singularities of differential equations with Chebfun", *SIAM Undergraduate Research Online*, 6 (2013),

`http://dx.doi.org/10.1137/12S011520`

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4. Chebfun and Approximation Theory

Lloyd N. Trefethen, November 2009, latest revision June 2014

Contents

4.1 Chebyshev series and interpolants

Chebfun is founded on the mathematical subject of approximation theory, and in particular, on Chebyshev series and interpolants. Conversely, it provides a simple environment in which to demonstrate these approximants and other approximation ideas.

The history of "Chebyshev technology" goes back to the 19th century Russian mathematician Pafnuty Chebyshev (1821-1894) and his mathematical descendants such as Zolotarev and Bernstein (1880-1968). These men realized that just as Fourier series provide an efficient way to represent a smooth periodic function, series of Chebyshev polynomials can do the same for a smooth nonperiodic function. A number of excellent textbooks and monographs have been published on approximation theory, including [Davis 1963], [Cheney 1966], [Meinardus 1967], [Lorentz 1986], and Powell [Powell, 1981], and in addition there are books devoted entirely to Chebyshev polynomials, including [Rivlin 1974] and [Mason & Handscomb 2003]. A Chebfun-based book on approximation theory and its computational applications is particularly relevant for Chebfun users [Trefethen 2013].

From the dates of publication above it will be clear that approximation theory flourished in the early computer era, and in the 1950s and 1960s a number of numerical methods were developed based on Chebyshev polynomials by Lanczos [Lanczos 1957], Fox [Fox & Parker 1966], Clenshaw, Elliott, Mason, Good, and others. The Fast Fourier Transform came in 1965 and Salzer's barycentric interpolation formula for Chebyshev points in 1972 [Salzer 1972]. Then in the 1970s Orszag and Gottlieb introduced spectral methods, based on the application of Chebyshev and Fourier technology to the solution of PDEs. The subject grew rapidly, and it is in the context of spectral methods that

Chebyshev techniques are particularly well known today [Boyd 2001], [Trefethen 2000], [Canuto et al. 2006/7].

We must be clear about terminology. We shall rarely use the term **Chebyshev approximation**, for that expression refers specifically to an approximation that is optimal in the minimax sense. Chebyshev approximations are fascinating, and in Section 4.6 we shall see that Chebfun makes it easy to compute them, but the core of Chebfun is built on the different techniques of polynomial interpolation in Chebyshev points and expansion in Chebyshev polynomials. These approximations are not quite optimal, but they are nearly optimal and much easier to compute.

By **Chebyshev points** we shall mean the set of points in $[-1, 1]$ defined by

$$x_j = -\cos(j\pi/N), \quad 0 \le j \le N,$$

where $N \ge 1$ is an integer. (If $N = 0$, we take $x_0 = 0$.) A fuller name is that these are **Chebyshev points of the second kind**. (Chebfun also enables computations based on Chebyshev points of the first kind; see Section 8.9.) Through any data values f_j at these points there is a unique polynomial interpolant $p(x)$ of degree $\le N$, which we call the **Chebyshev interpolant**. In particular, if the data are $f_j = (-1)^{n-j}$, then $p(x)$ is $T_N(x)$, the degree N Chebyshev polynomial, which can also be defined by the formula $T_N(x) = \cos(N \cos^{-1}(x))$. In Chebfun, the command `chebpoly(N)` returns a chebfun corresponding to T_N, and `poly` returns coefficients in the monomial basis $1, x, x^2, \ldots$. Thus we can print the coefficients of the first few Chebyshev polynomials like this:

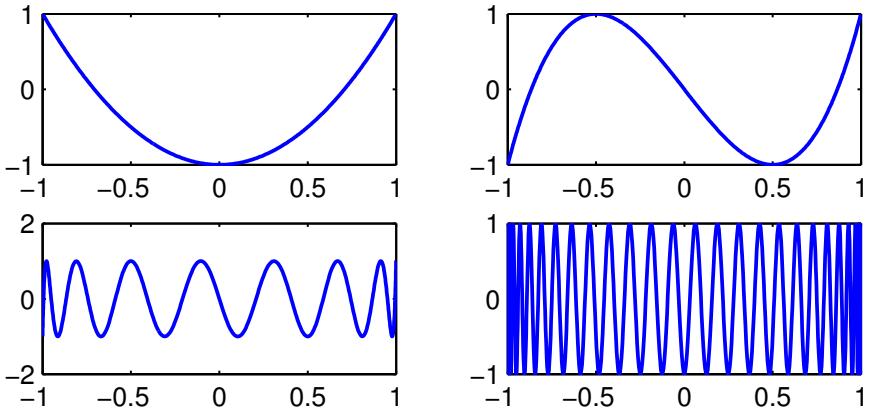
```
for N = 0:8
  disp(poly(chebpoly(N)))
end
```

```
   1
   1     0
   2     0    -1
   4     0    -3     0
   8     0    -8     0     1
  16     0   -20     0     5     0
  32     0   -48     0    18     0    -1
  64     0  -112     0    56     0    -7     0
 128     0  -256     0   160     0   -32     0     1
```

Note that that output of `poly` follows the pattern for MATLAB's standard `poly` command: it is a row vector, and the high-order coefficients come first. Thus, for example, the fourth row above tells us that $T_3(x) = 4x^3 - 3x$.

Here are plots of T_2, T_3, T_{15}, and T_{50}.

```
subplot(2,2,1), plot(chebpoly(2))
subplot(2,2,2), plot(chebpoly(3))
subplot(2,2,3), plot(chebpoly(15))
subplot(2,2,4), plot(chebpoly(50))
```



A **Chebyshev series** is an expansion

$$f(x) = \sum_{k=0}^{\infty} a_k T_k(x),$$

and the a_k are known as **Chebyshev coefficients**. So long as f is continuous and at least a little bit smooth (Lipschitz continuity is enough), it has a unique expansion of this form, which converges absolutely and uniformly, and the coefficients are given by the integral

$$a_k = \frac{2}{\pi} \int_{-1}^{1} \frac{f(x)T_k(x)dx}{\sqrt{1-x^2}}$$

except that for $k = 0$, the constant changes from $2/\pi$ to $1/\pi$. One way to approximate a function is to form the polynomials obtained by truncating its Chebyshev expansion,

$$f_N(x) = \sum_{k=0}^{N} a_k T_k(x).$$

This isn't quite what Chebfun does, however, since it does not compute exact Chebyshev coefficients. Instead Chebfun constructs its approximations via Chebyshev interpolants, which can also be regarded as finite series in Chebyshev polynomials for some coefficients c_k:

$$p_N(x) = \sum_{k=0}^{N} c_k T_k(x).$$

Each coefficient c_k will converge to a_k as $N \to \infty$ (apart from the effects of rounding errors), but for finite N, c_k and a_k are different. Chebfun versions 1-4 stored functions via their values at Chebyshev points, whereas version 5 switched to Chebyshev coefficients, but this hardly matters to the user, and both representations are exploited for various purposes internally in the system.

4.2 `chebcoeffs` and `poly`

We have just seen that the command `chebpoly(N)` returns a chebfun corresponding to the Chebyshev polynomial T_N. Conversely, if `f` is a chebfun, then `chebcoeffs(f)` is the vector of its Chebyshev coefficients. (Before Version 5, the command for this was `chebpoly`.) For example, here are the Chebyshev coefficients of x^3:

```
x = chebfun(@(x) x);
c = chebcoeffs(x.^3)
```

```
c =
  Columns 1 through 3
   0.250000000000000                   0   0.750000000000000
  Column 4
                   0
```

Like `poly`, `chebcoeffs` returns a row vector with the high-order coefficients first. Thus this computation reveals the identity $x^3 = (1/4)T_3(x) + (3/4)T_1(x)$.

If we apply `chebcoeffs` to a function that is not "really" a polynomial, we will usually get a vector whose first entry (i.e., highest order) is just above machine precision. This reflects the adaptive nature of the Chebfun constructor, which always seeks to use a minimal number of points.

```
chebcoeffs(sin(x))
```

```
ans =
  Columns 1 through 3
   0.000000000000039                   0  -0.000000000023960
  Columns 4 through 6
                   0   0.000000010498500                   0
  Columns 7 through 9
  -0.000003004651635                   0   0.000499515460422
  Columns 10 through 12
                   0  -0.039126707965337                   0
  Columns 13 through 14
   0.880101171489867                   0
```

Of course, machine precision is defined relative to the scale of the function:

```
chebcoeffs(1e100*sin(x))
```

```
ans =
   1.0e+99 *
  Columns 1 through 3
   0.000000000000385                   0  -0.000000000239601
  Columns 4 through 6
                   0   0.000000104985004                   0
  Columns 7 through 9
  -0.000030046516349                   0   0.004995154604224
  Columns 10 through 12
```

```
                  0  -0.391267079653368                  0
  Columns 13 through 14
   8.801011714898671                  0
```

By using `poly` we can print the coefficients of such a chebfun in the monomial basis. Here for example are the coefficients of the Chebyshev interpolant of $\exp(x)$ compared with the Taylor series coefficients:

```
cchebfun = flipud(chebcoeffs(exp(x))');
ctaylor = 1./gamma(1:length(cchebfun))';
disp('        chebfun              Taylor')
disp([cchebfun ctaylor])
```

```
        chebfun              Taylor
   1.266065877752008   1.000000000000000
   1.130318207984970   1.000000000000000
   0.271495339534077   0.500000000000000
   0.044336849848664   0.166666666666667
   0.005474240442094   0.041666666666667
   0.000542926311914   0.008333333333333
   0.000044977322954   0.001388888888889
   0.000003198436462   0.000198412698413
   0.000000199212481   0.000024801587302
   0.000000011036772   0.000002755731922
   0.000000000550590   0.000000275573192
   0.000000000024980   0.000000025052108
   0.000000000001039   0.000000002087676
   0.000000000000040   0.000000000160590
   0.000000000000001   0.000000000011471
```

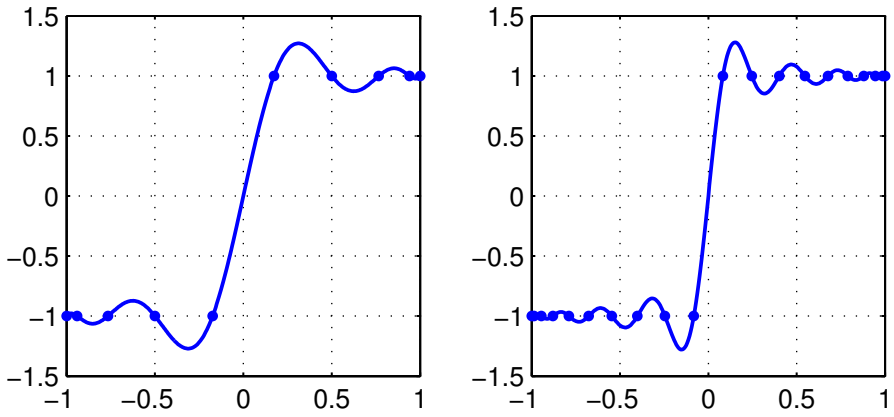
The fact that these differ is not an indication of an error in the Chebfun approximation. On the contrary, the Chebfun coefficients do a better job of approximating than the truncated Taylor series. If f were a function like $1/(1+25x^2)$, the Taylor series would not converge at all.

4.3 `chebfun(...,N)` and the Gibbs phenomenon

We can examine the approximation qualities of Chebyshev interpolants by means of a command of the form `chebfun(...,N)`. When an integer N is specified in this manner, it indicates that a Chebyshev interpolant is to be constructed of precisely length N rather than by the usual adaptive process.

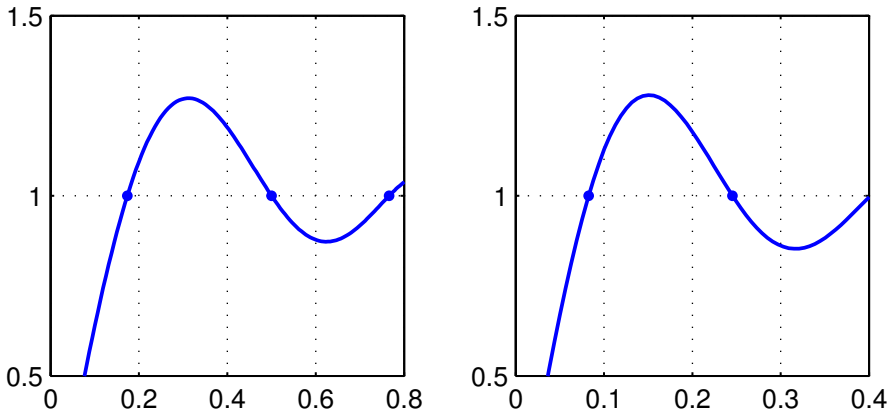
Let us begin with a function that cannot be well approximated by polynomials, the step function $\mathrm{sign}(x)$. To start with we interpolate it in 10 or 20 points, taking N to be even to avoid including a value 0 at the middle of the step.

```
f = chebfun('sign(x)',10);
subplot(1,2,1), plot(f,'.-'), grid on
f = chebfun('sign(x)',20);
subplot(1,2,2), plot(f,'.-'), grid on
```



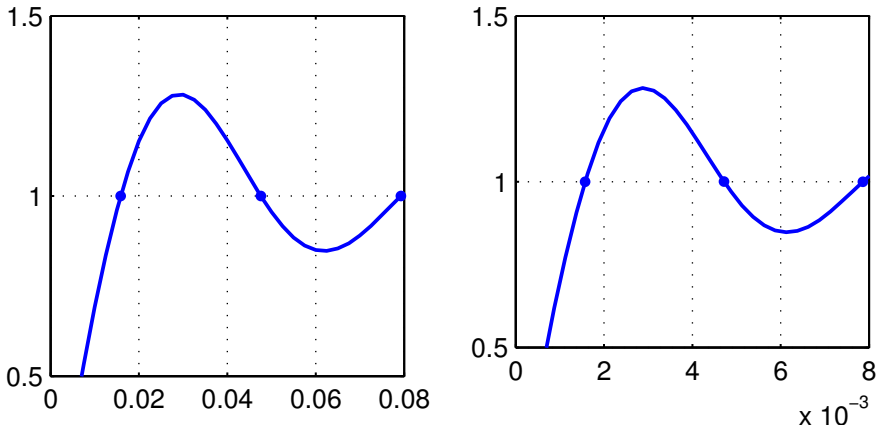
There is an overshoot problem here, known as the Gibbs phenomenon, that does not go away as $N \to \infty$. We can zoom in on the overshoot region by resetting the axes:

```
subplot(1,2,1), axis([0 .8 .5 1.5])
subplot(1,2,2), axis([0 .4 .5 1.5])
```



Here are analogous results with $N = 100$ and 1000.

```
f = chebfun('sign(x)',100);
subplot(1,2,1), plot(f,'.-'), grid on, axis([0 .08 .5 1.5])
f = chebfun('sign(x)',1000);
subplot(1,2,2), plot(f,'.-'), grid on, axis([0 .008 .5 1.5])
```



What is the amplitude of the Gibbs overshoot for Chebyshev interpolation of a step function? We can find out by using `max`:

```
for N = 2.^(1:8)
  gibbs = max(chebfun('sign(x)',N));
  fprintf('%5d  %13.8f\n', N, gibbs)
end
```

```
    2     1.00000000
    4     1.18807518
    8     1.26355125
   16     1.27816423
   32     1.28131717
   64     1.28204939
  128     1.28222585
  256     1.28226917
```

This gets a bit slow for larger N, but knowing that the maximum occurs around $x = 3/N$, we can speed it up by using Chebfun's { } notation to work on subintervals:

```
for N = 2.^(4:12)
  f = chebfun('sign(x)',N);
  fprintf('%5d  %13.8f\n', N, max(f{0,5/N}))
end
```

```
   16     1.27816423
   32     1.28131717
   64     1.28204939
  128     1.28222585
  256     1.28226917
  512     1.28227990
 1024     1.28228257
 2048     1.28228323
 4096     1.28228340
```

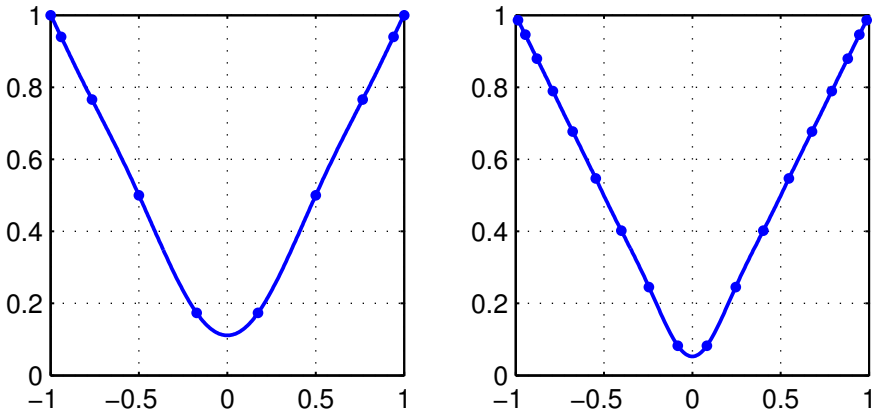
The overshoot converges to a number 1.282283455775... [Helmberg & Wagner 1997].

4.4 Smoothness and rate of convergence

The central dogma of approximation theory is this: the smoother the function, the faster the convergence as $N \to \infty$. What this means for Chebfun is that so long as a function is twice continuously differentiable, it can usually be approximated to machine precision for a workable value of N, even without subdivision of the interval.

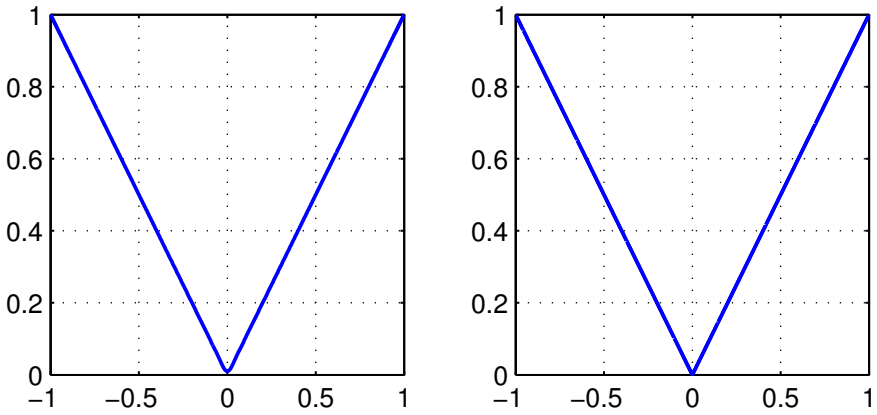
After the step function, a function with "one more derivative" of smoothness would be the absolute value. Here if we interpolate in N points, the errors decrease at the rate $O(N^{-1})$. For example:

```
clf
f10 = chebfun('abs(x)',10);
subplot(1,2,1), plot(f10,'.-'), ylim([0 1]), grid on
f20 = chebfun('abs(x)',20);
subplot(1,2,2), plot(f20,'.-'), ylim([0 1]), grid on
```



Chebfun has no difficulty computing interpolants of much higher order:

```
f100 = chebfun('abs(x)',100);
subplot(1,2,1), plot(f100), ylim([0 1]), grid on
f1000 = chebfun('abs(x)',1000);
subplot(1,2,2), plot(f1000), ylim([0 1]), grid on
```



Such plots look good to the eye, but they do not achieve machine precision. We can confirm this by using `splitting on` to compute a true absolute value and then measuring some norms.

```
fexact = chebfun('abs(x)','splitting','on');
err10 = norm(f10-fexact,inf)
err100 = norm(f100-fexact,inf)
err1000 = norm(f1000-fexact,inf)
```

```
err10 =
   0.111111111111111
err100 =
   0.010101010101010
err1000 =
   0.001001001001002
```

Notice the clean linear decrease of the error as N increases.

If f is a bit smoother, polynomial approximation to machine precision becomes practical:

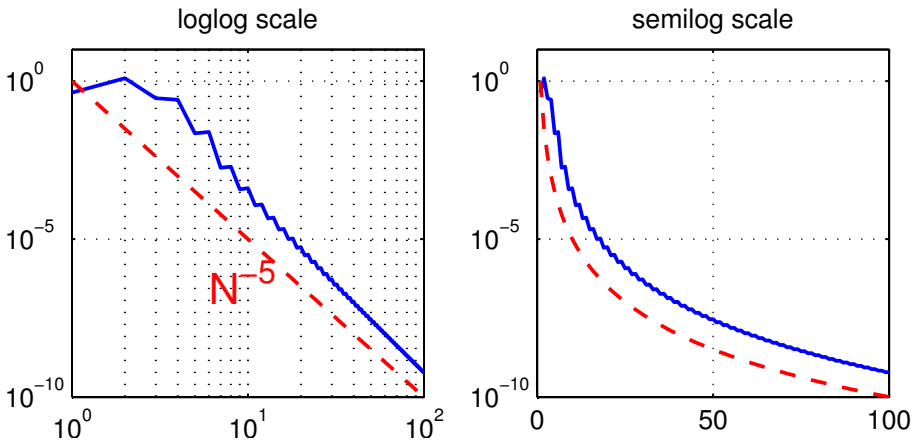
```
  length(chebfun('abs(x).*x'))
  length(chebfun('abs(x).*x.^2'))
  length(chebfun('abs(x).*x.^3'))
  length(chebfun('abs(x).*x.^4'))
```

```
ans =
       32092
ans =
        2935
ans =
   884
ans =
   419
```

Of course, these particular functions would be easily approximated by piecewise smooth chebfuns.

It is interesting to plot convergence as a function of N. Here is an example from [Battles & Trefethen 2004] involving the next function from the sequence above.

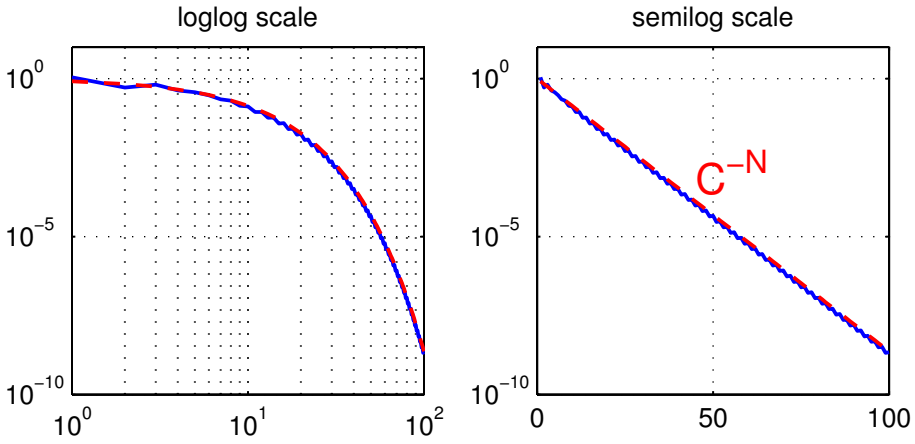
```
s = 'abs(x).^5';
exact = chebfun(s,'splitting','off');
NN = 1:100; e = [];
for N = NN
  e(N) = norm(chebfun(s,N)-exact);
end
clf
subplot(1,2,1)
loglog(e), ylim([1e-10 10]), title('loglog scale')
hold  on, loglog(NN.^(-5),'--r'), grid on
text(6,4e-7,'N^{-5}','color','r','fontsize',16)
subplot(1,2,2)
semilogy(e), ylim([1e-10 10]), grid on, title('semilog scale')
hold  on, semilogy(NN.^(-5),'--r'), grid on
```



The figure reveals very clean convergence at the rate N^{-5}. According to Theorem 2 of the next section, this happens because f has a fifth derivative of bounded variation.

Here is an example of a smoother function, one that is in fact analytic. According to Theorem 3 of the next section, if f is analytic, its Chebyshev interpolants converge geometrically. In this example we take f to be the Runge function, for which interpolants in equally spaced points would not converge at all (in fact they diverge exponentially – see Section 4.7).

```
s = '1./(1+25*x.^2)';
exact = chebfun(s);
for N = NN
  e(N) = norm(chebfun(s,N)-exact);
end
clf, subplot(1,2,1)
loglog(e), ylim([1e-10 10]), grid on, title('loglog scale')
c = 1/5 + sqrt(1+1/25);
hold  on, loglog(c.^(-NN),'--r'), grid on
subplot(1,2,2)
semilogy(e), ylim([1e-10 10]), title('semilog scale')
hold  on, semilogy(c.^(-NN),'--r'), grid on
text(45,1e-3,'C^{-N}','color','r','fontsize',16)
```



This time the convergence is equally clean but quite different in nature. Now the straight line appears on the semilogy axes rather than the loglog axes, revealing the geometric convergence.

4.5 Five theorems

The mathematics of Chebfun can be captured in five theorems about interpolants in Chebyshev points. The first three can be found in [Battles & Trefethen 2004], and all are discussed in [Trefethen 2013]. Let f be a continuous function on $[-1,1]$, and let p denote its interpolant in N Chebyshev points and p^* its best degree N approximation with respect to the maximum norm $\|\cdot\|$.

The first theorem asserts that Chebyshev interpolants are "near-best" [Ehlich & Zeller 1966].

THEOREM 1.

$$\|f-p\| \le (2+(2/\pi)\log(N))\|f-p^*\|.$$

This theorem implies that even if N is as large as 100,000, one can lose no more than one digit by using p instead of \\(p^*\\). Whereas Chebfun will readily compute such a p, it is unlikely that anybody has ever computed a nontrivial p^* for a value of N so large.

The next theorem asserts that if f is k times differentiable, roughly speaking, then the Chebyshev interpolants converge at the algebraic rate $1/N^k$ [Mastroianni & Szabados 1995].

THEOREM 2. Let $f, f', \ldots, f^{(k-1)}$ be absolutely continuous for some $k \ge 1$, and let $f^{(k)}$ be a function of bounded variation. Then $\|f-p\| = O(N^{-k})$ as $N \to \infty$.

Smoother than this would be a C^∞ function, i.e. infinitely differentiable, and smoother still would be a function analytic on $[-1,1]$, i.e., one whose Taylor series at each point of $[-1,1]$ converges at least in a small neighborhood of that point. For analytic functions the convergence is geometric. The essence of the following theorem is due to Bernstein in 1912, though it is not clear where an explicit statement first appeared in print.

THEOREM 3. If f is analytic and bounded in the "Bernstein ellipse" of foci 1 and -1 with semimajor and semiminor axis lengths summing to r, then $\|f-p\| = O(r^{-N})$ as $N \to \infty$.

More precisely, if $|f(z)| \le M$ in the ellipse, then the bound on the right can be taken as $4Mr^{-n}/(r-1)$.

The next theorem asserts that Chebyshev interpolants can be computed by the barycentric formula [Salzer 1972]. The sumation with a double prime denotes the sum from $k=0$ to $k=N$ with both terms $k=0$ and $k=N$ multiplied by $1/2$.

THEOREM 4.

$$p(x) = \sum_{k=0}^{N}{}'' \frac{(-1)^k f(x_k)}{x-x_k} \Bigg/ \sum_{k=0}^{N}{}'' \frac{(-1)^k}{x-x_k}.$$

See [Berrut & Trefethen 2005] and [Trefethen 2013] for information about barycentric interpolation.

The final theorem asserts that the barycentric formula has no difficulty with rounding errors. Our "theorem" is really just an advertisement; see [Higham 2004] for a precise statement and proof. Earlier work on this subject appeared in [Rack & Reimer 1982].

THEOREM 5. The barycentric formula of Theorem 4 is numerically stable.

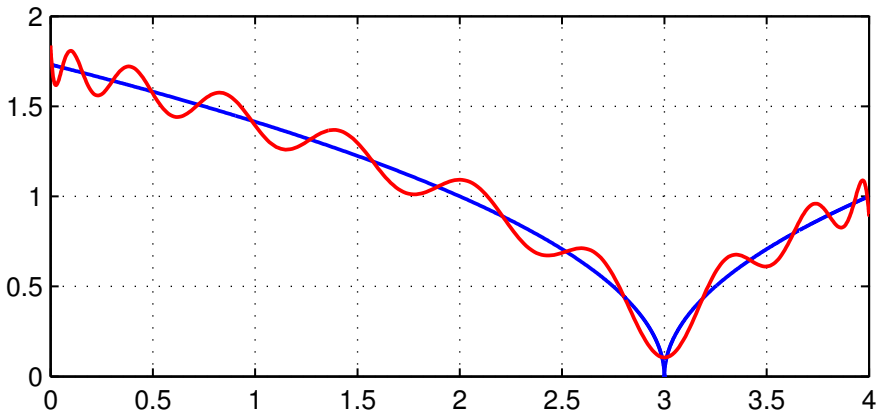
This stability result may seem surprising when one notes that for x close to x_k, the barycentric formula involves divisions by numbers that are nearly zero. Nevertheless it is provably stable. If x is exactly equal to some x_k, then one bypasses the formula and returns the exact value $p(x) = f(x_k)$.

4.6 Best approximations and the Remez algorithm

For practical computations, it is rarely worth the trouble to compute a best (minimax) approximation rather than simply a Chebyshev interpolant. Nevertheless best approximations are a beautiful and well-established idea, and it is certainly interesting to be able to compute them. Chebfun makes this possible with the command `remez`, named after Evgeny Remez, who devised the standard algorithm for computing these approximations in 1934. This capability is due to Ricardo Pachon; see [Pachon & Trefethen 2009].

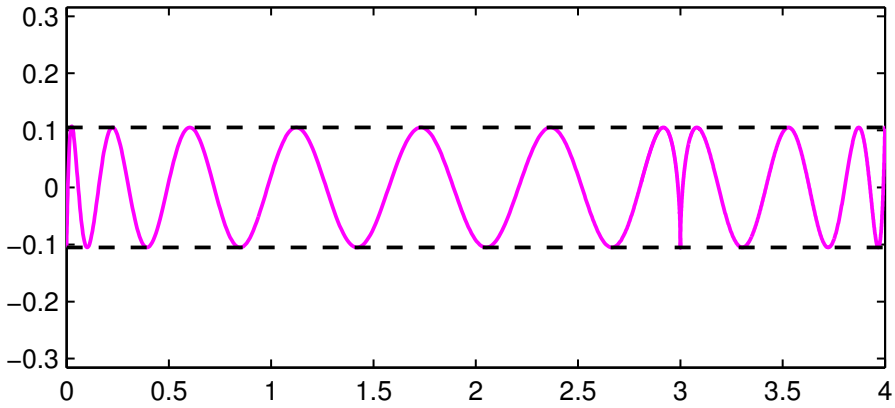
For example, here is a function on the interval $[0, 4]$ together with its best approximation by a polynomial of degree 20:

```
f = chebfun('sqrt(abs(x-3))',[0,4],'splitting','on');
p = remez(f,20);
clf, plot(f,'b',p,'r'), grid on
```



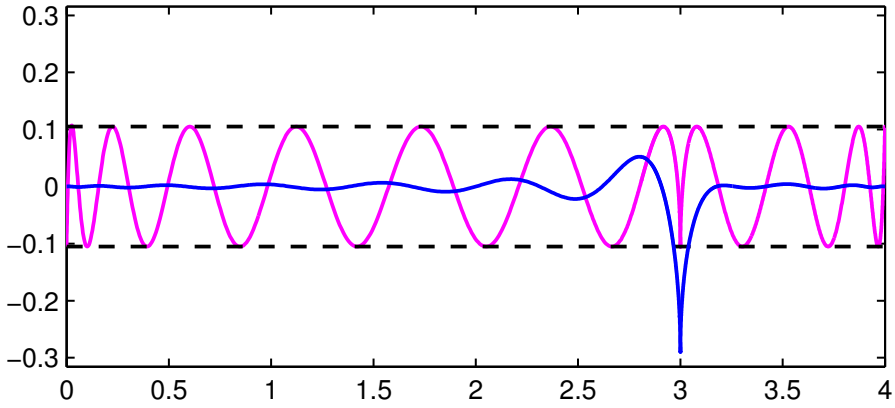
A plot of the error curve $(f - p)(x)$ shows that it equioscillates between $20 + 2 = 22$ alternating extreme values. Note that a second output argument from `remez` returns the error as well as the polynomial.

```
[p,err] = remez(f,20);
plot(f-p,'m'), hold on
plot([0 4],err*[1 1],'--k'), plot([0 4],-err*[1 1],'--k')
ylim(3*err*[-1,1])
```



Let's add the error curve for the degree 20 (i.e. 21-point) Chebyshev interpolant to the same plot:

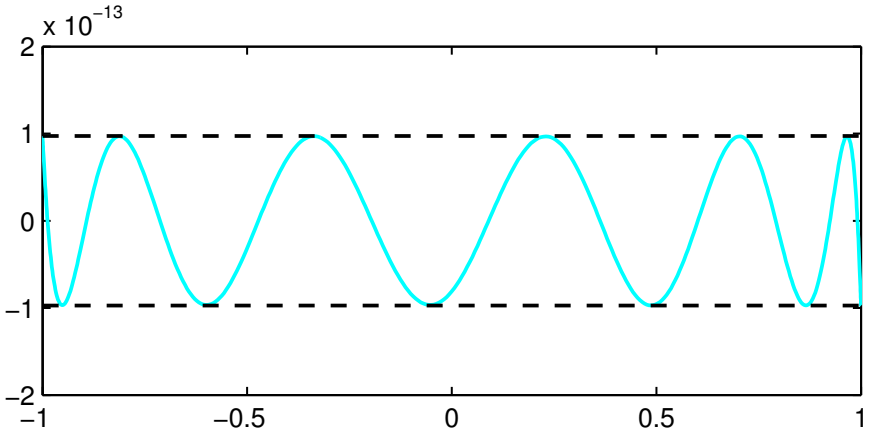
```
pinterp = chebfun(f,[0,4],21);
plot(f-pinterp,'b')
```



Notice that although the best approximation has a smaller maximum error, it is a worse approximation for most values of x.

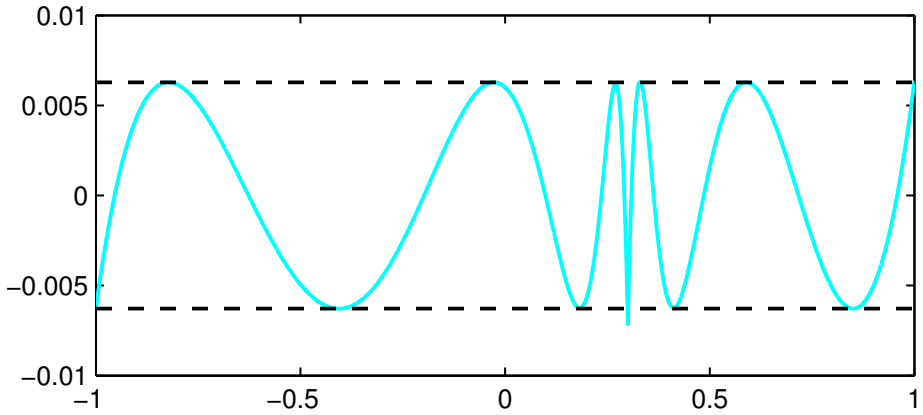
Chebfun's `remez` command can compute certain rational best approximants too, though it is somewhat fragile. If your function is smooth, a possibly more robust approach to computing best approximations is Caratheodory-Fejer approximation, implemented in the code `cf` due to Joris Van Deun [Van Deun & Trefethen 2011]. For example:

```
f = chebfun('exp(x)');
[p,q] = cf(f,5,5);
r = p./q;
err = norm(f-r,inf);
clf, plot(f-r,'c'), hold on
plot([-1 1],err*[1 1],'--k'), plot([-1 1],-err*[1 1],'--k')
ylim(2e-13*[-1 1])
```



CF approximation often comes close to optimal for non-smooth functions too, provided you specify a fourth argument to tell the system which Chebyshev grid to use:

```
f = abs(x-.3);
[p,q,r_handle,lam] = cf(f,5,5,300);
clf, plot(f-p./q,'c'), hold on
plot([-1 1],lam*[1 1],'--k'), plot([-1 1],-lam*[1 1],'--k')
```

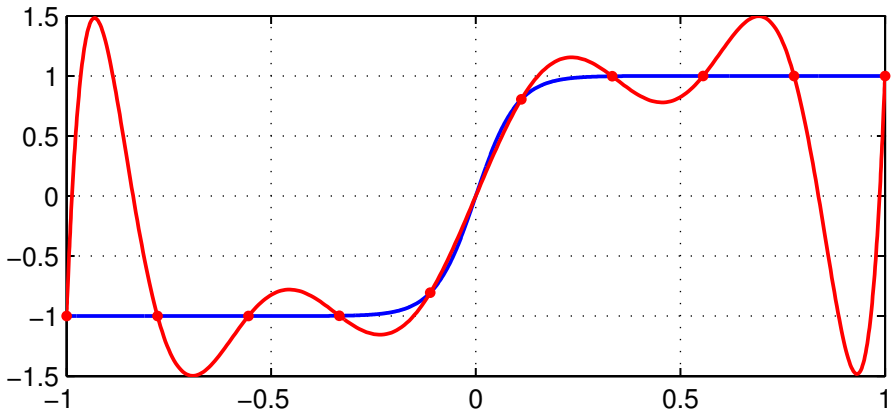


4.7 The Runge phenomenon

Chebfun is based on polynomial interpolants in Chebyshev points, not equispaced points. It has been known for over a century that the latter choice is disastrous, even for interpolation of smooth functions [Runge 1901]. One should never use equispaced polynomial interpolants for practical work (unless you will only need the result near the center of the interval of interpolation), but like best approximations, they are certainly interesting.

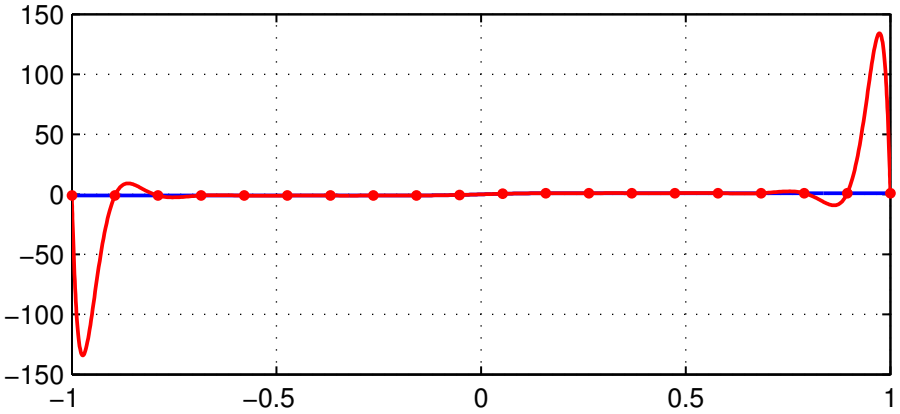
In Chebfun, we can compute them with the `interp1` command. For example, here is an analytic function and its equispaced interpolant of degree 9:

```
f = tanh(10*x);
s = linspace(-1,1,10);
p = chebfun.interp1(s,f(s)); hold off
plot(f), hold on, plot(p,'r'), grid on, plot(s,p(s),'.r')
```



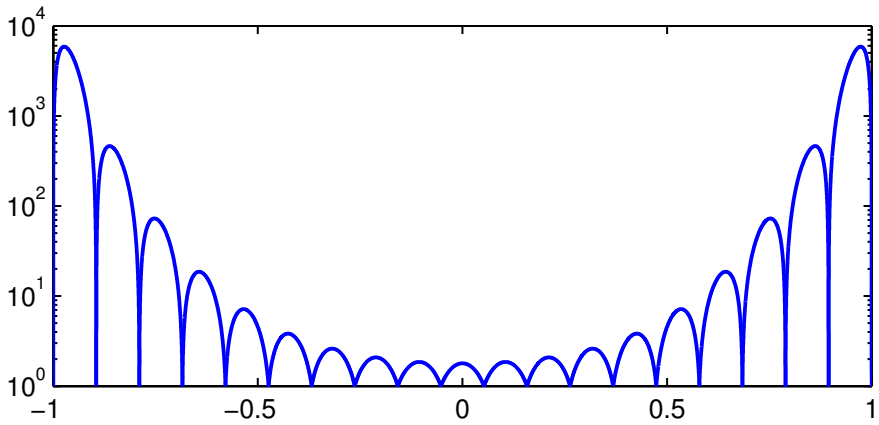
Perhaps this doesn't look too bad, but here is what happens for degree 19. Note the vertical scale.

```
s = linspace(-1,1,20);
p = chebfun.interp1(s,f(s)); hold off
plot(f), hold on, plot(p,'r'), grid on, plot(s,p(s),'.r')
```



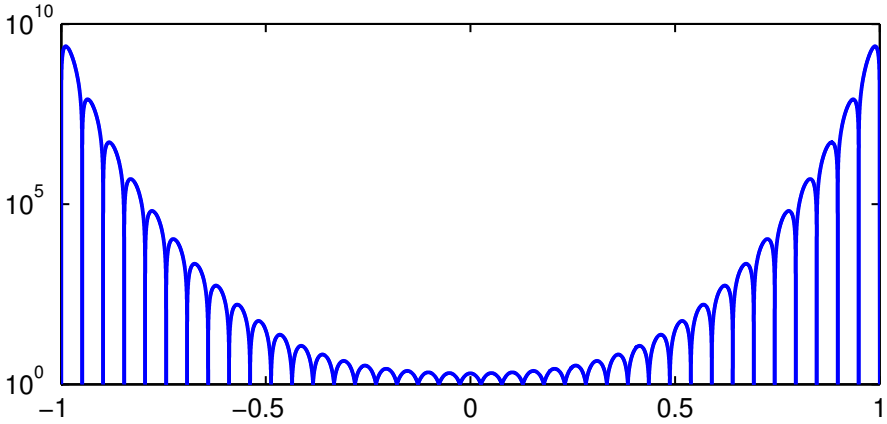
Approximation experts will know that one of the tools used in analyzing effects like this is known as the Lebesgue function associated with a given set of interpolation points. Chebfun has a command `lebesgue` for computing these functions. The problem with interpolation in 20 equispaced points is reflected in a Lebesgue function of size 10^4 – note the semilog scale:

```
clf, semilogy(lebesgue(s))
```



For 40 points it is much worse:

```
semilogy(lebesgue(linspace(-1,1,40)))
```



As the degree increases, polynomial interpolants in equispaced points diverge exponentially, and no other method of approximation based on equispaced data can completely get around this problem [Platte, Trefethen & Kuijlaars 2011].

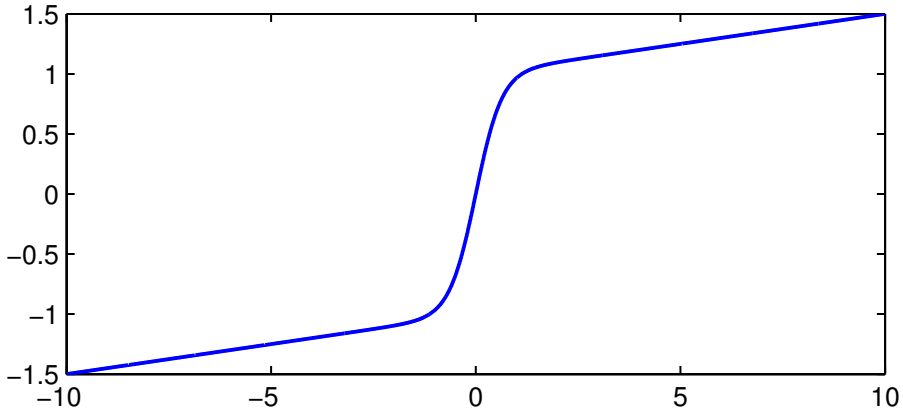
4.8 Rational approximations

Chebfun contains four different programs, at present, for computing rational approximants to a function f. We say that a rational function is of type (m, n) if it can be written as a quotient of one polynomial of degree at most m and another of degree at most n.

To illustrate the possibilities, consider the function

```
f = chebfun('tanh(pi*x/2) + x/20',[-10,10]);
length(f)
plot(f)
```

```
ans =
   350
```



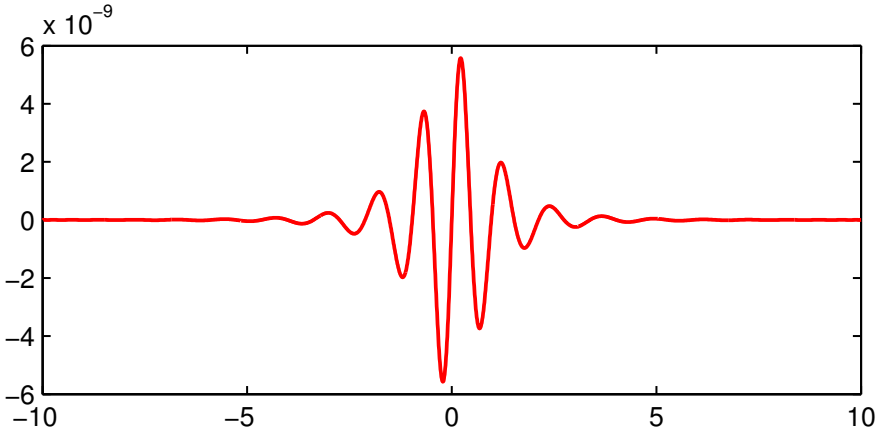
We can use the command `chebpade`, developed by Ricardo Pachon, to compute a Chebyshev-Pade approximant, defined by the condition that the Chebyshev series of p/q should match that of f as far as possible [Baker & Graves-Morris 1996]. (This is the so-called "Clenshaw-Lord" Chebyshev-Pade approximation; if the flag `maehly` is specified the code alternatively computes the linearized variation known as the "Maehly" approximation.) Chebyshev-Pade approximation is the analogue for functions defined on an interval of Pade approximation for functions defined in a neighborhood of a point.

```
[p,q] = chebpade(f,40,4);
r = p./q;
```

The functions f and r match to about 8 digits:

```
norm(f-r)
plot(f-r,'r')
```

```
ans =
     4.884639793650636e-09
```



Mathematically, f has poles in the complex plane at $\pm i$, $\pm 3i$, $\pm 5i$, and so on. We can obtain approximations to these values by looking at the roots of q:

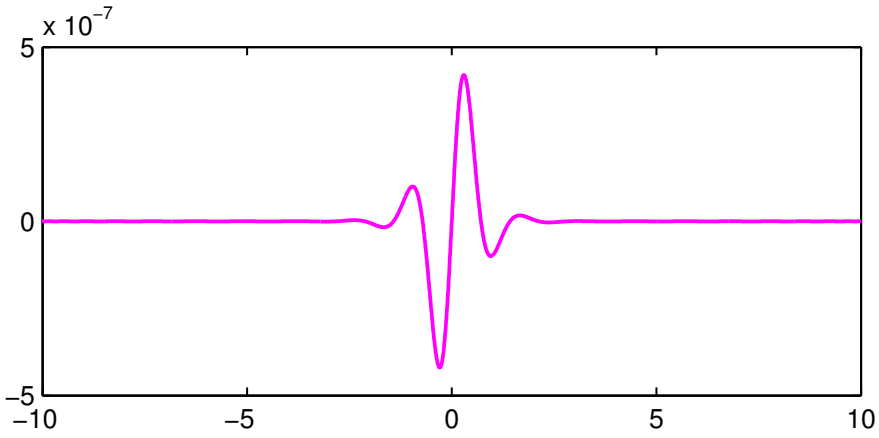
```
roots(q,'complex')
```

```
ans =
  0.000000000000000 - 1.000000750727884i
  0.000000000000000 + 1.000000750727884i
  0.000000000000000 - 3.004284960255872i
  0.000000000000000 + 3.004284960255872i
```

A similar but perhaps faster and more robust approach to rational interpolation is encoded in the command `ratinterp`, which computes a type (m, n) interpolant through $m+n+1$ Chebyshev points (or, optionally, a different set of points). This capability was developed by Ricardo Pachon, Pedro Gonnet and Joris Van Deun [Pachon, Gonnet & Van Deun 2012]. The results are similar:

```
[p,q] = ratinterp(f,40,4);
r = p./q;
norm(f-r)
plot(f-r,'m')
```

```
ans =
     3.501041670519031e-07
```



Again the poles are not bad:

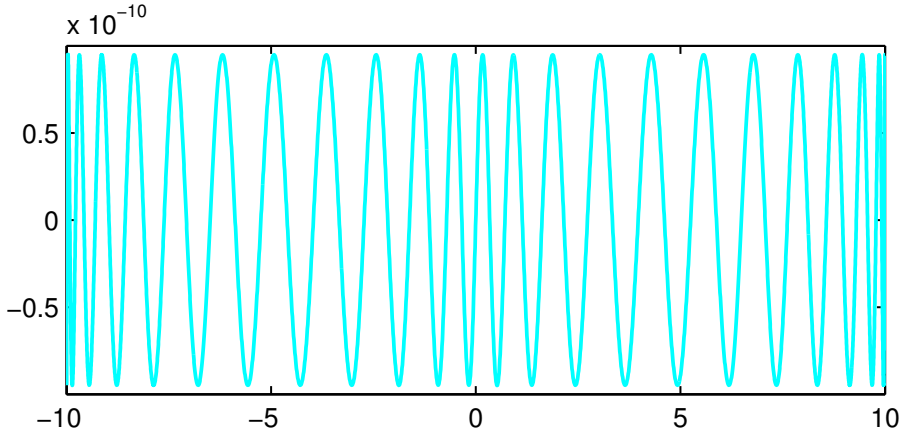
```
roots(q,'complex')
```

```
ans =
  0.000000000000000 - 1.000011080641719i
  0.000000000000000 + 1.000011080641719i
  0.000000000000000 - 3.010648453090407i
  0.000000000000000 + 3.010648453090407i
```

The third and fourth options for rational approximation, as mentioned in Section 4.6, are best approximants computed by `remez` and Caratheodory-Fejer approximants computed by `cf` [Trefethen & Gutknecht 1983, Van Deun & Trefethen 2011]. As mentioned in Section 4.6, CF approximants often agree with best approximations to machine precision if f is smooth. We explore the same function yet again, and this time obtain an equioscillating error curve:

```
[p,q] = cf(f,40,4);
r = p./q;
norm(f-r)
plot(f-r,'c')
```

```
ans =
     2.999172427815131e-10
```



And the poles:

```
roots(q,'complex')
```

```
ans =
  0.000000000000004 - 1.000000066684812i
  0.000000000000004 + 1.000000066684812i
  0.000000000002217 - 3.001936139233654i
  0.000000000002217 + 3.001936139233654i
```

It is tempting to vary parameters and functions to explore more poles and what accuracy can be obtained for them. But rational approximation and analytic continuation are very big subjects and we shall resist the temptation. See Chapter 28 of [Trefethen 2013] and [Webb, Trefethen & Gonnet 2012].

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[Van Deun & Trefethen 2011] J. Van Deun and L. N. Trefethen, A robust implementation of the Caratheodory-Fejer method for rational approximation, *BIT Numerical Mathematics*, 51 (2011), 1039-1050.

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5. Complex Chebfuns

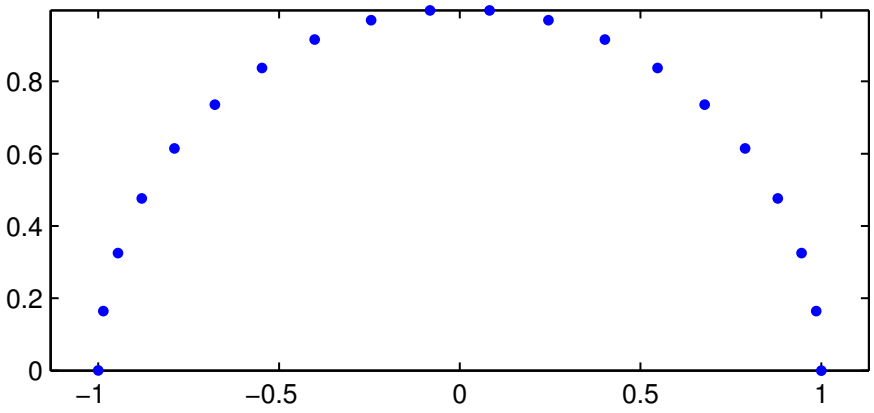
Lloyd N. Trefethen, November 2009, latest revision June 2014

Contents

5.1 Complex functions of a real variable

One of the attractive features of MATLAB is that it handles complex arithmetic well. For example, here are 20 points on the upper half of the unit circle in the complex plane:

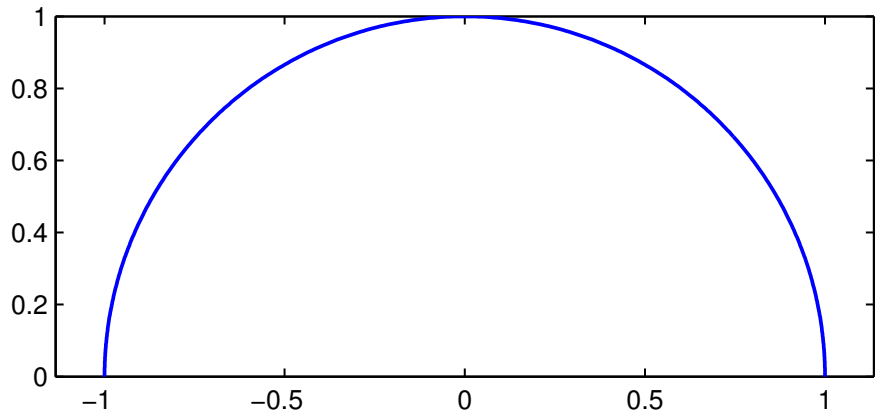
```
s = linspace(0,pi,20);
f = exp(1i*s);
plot(f,'.')
axis equal
```



In MATLAB, both the variables `i` and `j` are initialized as i, the square root of -1, but this code uses `1i` instead (just as one might write, for example, `3+2i` or `2.2-1.1i`). Writing the imaginary unit in this fashion is a common trick among MATLAB programmers, for it avoids the risk of surprises caused by `i` or `j` having been overwritten by other values. The `axis equal` command ensures that the real and imaginary axes are scaled equally.

Here is a Chebfun analogue.

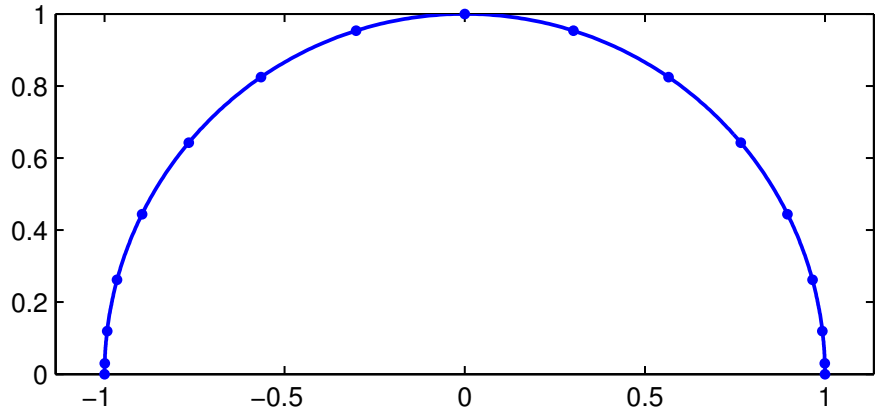
```
s = chebfun(@(s) s,[0 pi]);
f = exp(1i*s);
plot(f)
axis equal
```



The Chebfun semicircle is represented by a polynomial of low degree:

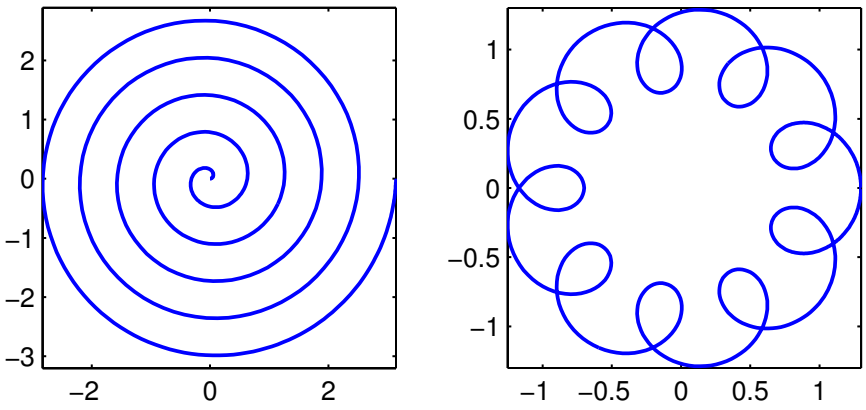
```
length(f)
plot(f,'.-')
axis equal
```

```
ans =
    17
```

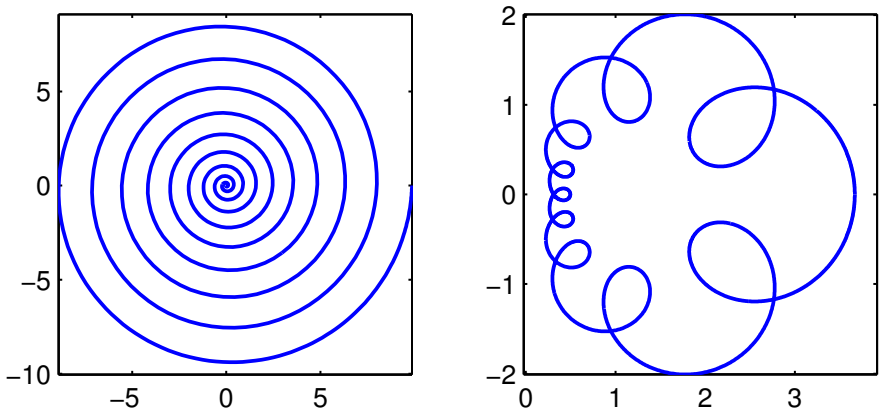


We can have fun with variations on the theme:

```
subplot(1,2,1), g = s.*exp(10i*s); plot(g), axis equal
subplot(1,2,2), h = exp(2i*s)+.3*exp(20i*s); plot(h), axis equal
```




```
subplot(1,2,1), plot(g.^2), axis equal
subplot(1,2,2), plot(exp(h)), axis equal
```



Such plots make pretty pictures, but as always with Chebfun, the underlying operations involve precise mathematics carried out to many digits of accuracy. For example, the integral of `g` is $-\pi i/10$,

```
sum(g)
```

```
ans =
  0.000000000000000 - 0.314159265358979i
```

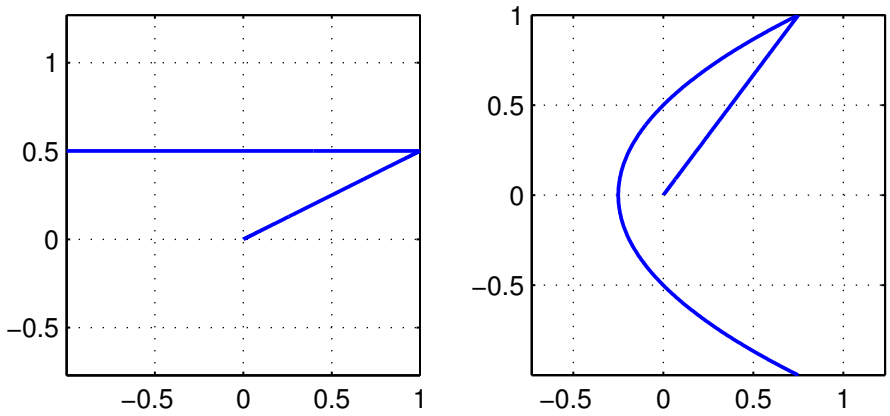
and the integral of `h` is zero:

```
sum(h)
```

```
ans =
     -2.430648284784403e-18 - 1.522474749680446e-16i
```

Piecewise smooth complex chebfuns are also possible. For example, the following starts from a chebfun `z` defined as $(1+0.5i)s$ for s on the interval $[0,1]$ and $1+0.5i-2(s-1)$ for s on the interval $[1,2]$.

```
z = chebfun({@(s) (1+.5i)*s, @(s) 1+.5i-2*(s-1)},[0 1 2]);
subplot(1,2,1), plot(z), axis equal, grid on
subplot(1,2,2), plot(z.^2), axis equal, grid on
```



Actually, this way of constructing a piecewise chebfun is rather clumsy. An easier method is to use the `join` command, in which a construction like `join(f,g,h)` constructs a single chebfun with the same values as `f`, `g`, and `h`, but on a domain concatenated together. Thus if the domains of `f`, `g`, `h` are $[a,b]$, $[c,d]$, and $[e,f]$, then `join(f,g,h)` has three pieces with domains $[a,b]$, $[b,b+(d-c)]$, $[b+(d-c),b+(d-c)+(f-e)]$. Using this trick, we can construct the chebfun `z` above in the following alternative manner:

```
s = chebfun(@(s) s,[0 1]);
zz = join((1+.5i)*s, 1+.5i-2*s);
norm(z-zz)
```

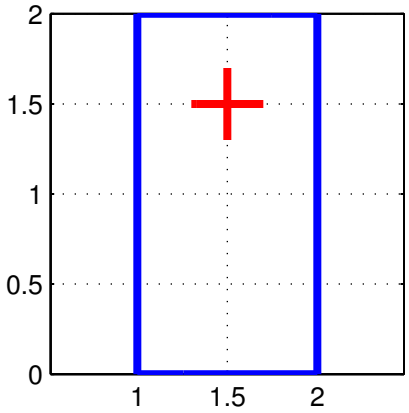
```
ans =
     1.281975124255710e-16
```

5.2 Analytic functions and conformal maps

A function is **analytic** if it is differentiable in the complex sense, or equivalently, if it has a convergent Taylor series near each point in its domain of definition. Analytic functions do interesting things in the complex plane. In particular, away from points where the derivative is zero, they are **conformal maps**, which means that though they may scale and rotate an infinitesimal region, they preserve angles between intersecting curves.

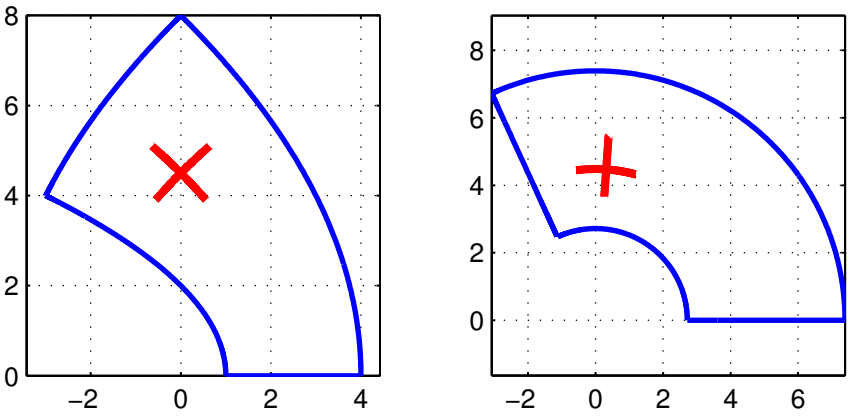
For example, suppose we define `R` to be a chebfun corresponding to the four sides of a rectangle and we define `X` to be another chebfun corresponding to a cross inside `R`.

```
s = chebfun('s',[0 1]);
R = join(1+s, 2+2i*s, 2+2i-s, 1+2i-2i*s);
LW = 'linewidth'; lw1 = 2; lw2 = 3;
clf, subplot(1,2,1), plot(R,LW,lw2), grid on, axis equal
X = join(1.3+1.5i+.4*s, 1.5+1.3i+.4i*s);
hold on, plot(X,'r',LW,lw2)
```



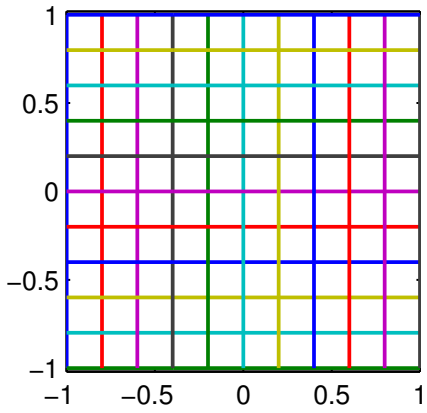
Here we see what happens to `R` and `X` under the maps z^2 and $\exp(z)$:

```
clf
subplot(1,2,1), plot(R.^2,LW,lw1), grid on, axis equal
hold on, plot(X.^2,'r',LW,lw2)
subplot(1,2,2), plot(exp(R),LW,lw1), grid on, axis equal
hold on, plot(exp(X),'r',LW,lw2)
```



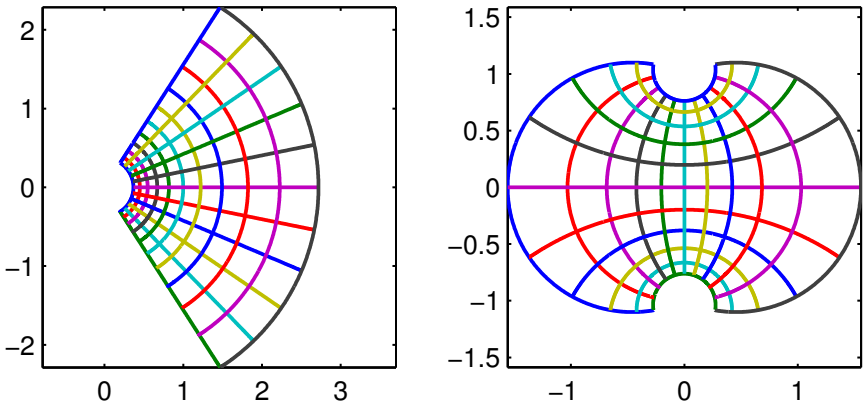
We can take the same idea further and construct a grid in the complex plane, each segment of which is a piece of a chebfun that happens to be linear. In this case we accumulate the various pieces as successive columns of a quasimatrix, i.e., a "matrix" whose columns are chebfuns.

```
x = chebfun(@(x) x);
S = chebfun;                    % make an empty chebfun
for d = -1:.2:1
  S = [S d+1i*x 1i*d+x];        % add 2 more lines to the collection
end
clf,
subplot(1,2,1), plot(S), axis equal
```



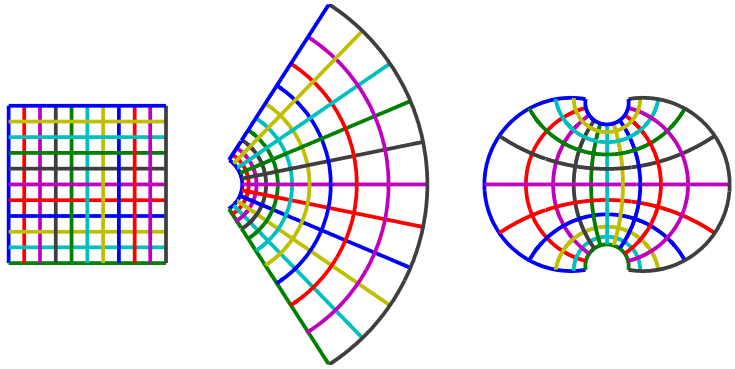
Here are the exponential and tangent of the grid:

```
subplot(1,2,1), plot(exp(S)), axis equal
subplot(1,2,2), plot(tan(S)), axis equal
```



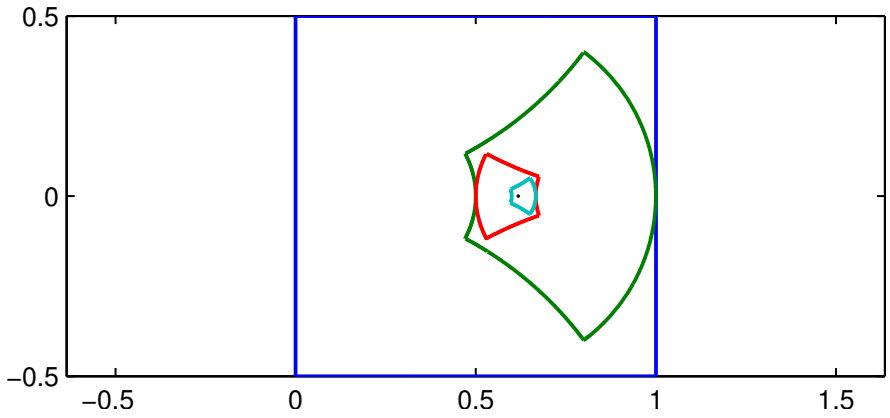
Here is a sequence that puts all three images together on a single scale:

```
clf
plot(S), hold on
plot(1.6+exp(S))
plot(6.6+tan(S))
axis equal, axis off
```



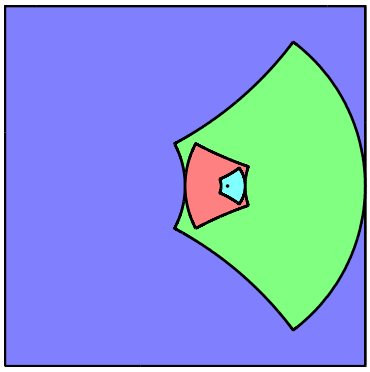
A particularly interesting set of conformal maps are the **Moebius transformations**, the rational functions of the form $(az + b)/(cz + d)$ for constants a, b, c, d. For example, here is a square and its image under the map $w = 1/(1 + z)$, and the image of the image under the same map, and the image of the image of the image. We also plot the limit point given by the equation $z = 1/(1 + z)$, i.e., $z = (\sqrt{5} - 1)/2$.

```
moebius = @(z) 1./(1+z);
s = chebfun(@(s) s,[0 1]);
S = join(-.5i+s, 1-.5i+1i*s, 1+.5i-s, .5i-1i*s);
clf
for j = 1:3
  S = [S moebius(S(:,j))];
end
plot(S)
hold on, axis equal
plot((sqrt(5)-1)/2,0,'.k','markersize',4)
```



Here's a prettier version of the same image using the Chebfun `fill` command.

```
S = join(-.5i+s, 1-.5i+1i*s, 1+.5i-s, .5i-1i*s);
clf
fill(real(S),imag(S),[.5 .5 1]), axis equal, hold on
S = moebius(S); fill(real(S),imag(S),[.5 1 .5])
S = moebius(S); fill(real(S),imag(S),[1 .5 .5])
S = moebius(S); fill(real(S),imag(S),[.5 1 1 ])
plot((sqrt(5)-1)/2,0,'.k','markersize',4)
axis off
```



5.3 Contour integrals

If s is a real parameter and $z(s)$ is a complex function of s, then we can define a contour integral in the complex plane like this:

$$\int f(z(s))z'(s)ds.$$

The contour in question is the curve described by $z(s)$ as s varies over its range.

For example, in the example at the end of Section 5.1 the contour consists of two straight segments that begin at 0 and end at $-1+.5i$. We can compute the integral of $\exp(-z^2)$ over the contour like this:

```
f = exp(-z.^2);
I = sum(f.*diff(z))
```

```
I =
 -0.842544559526136 + 0.166587147924074i
```

Notice how easily the contour integral is realized in Chebfun, even over a contour consisting of several pieces. This particular integral is related to the complex error function [Weideman 1994].

According to **Cauchy's theorem**, the integral of an analytic function around a closed curve is zero, or equivalently, the integral between two points z_1 and z_2 is path-independent. To verify this, we can compute the same integral over the straight segment going directly from 0 to $-1+0.5i$:

```
w = chebfun('(-1+.5i)*s',[0 1]);
f = exp(-w.^2);
I2 = sum(f.*diff(w))
```

```
I2 =
 -0.842544559526136 + 0.166587147924074i
```

A **meromorphic function** is a function that is analytic in a region of interest in the complex plane apart from possible poles. According to the **Cauchy integral formula**, $1/2\pi i$ times the integral of a meromorphic function f around a closed contour is equal to the sum of the residues of f associated with any poles it may have in the enclosed region. The **residue** of f at a point z_0 is the coefficient of the degree -1 term in its Laurent expansion at z_0. For example, the function $\exp(z)/z^3$ has Laurent series $z^{-3} + z^{-2} + (1/2)z^{-1} + (1/6)z^0 + \dots$ at the origin, and so its residue there is $1/2$. We can confirm this by computing the contour integral around a circle:

```
z = chebfun('exp(1i*s)',[0 2*pi]);
f = exp(z)./z.^3;
I = sum(f.*diff(z))/(2i*pi)
```

```
I =
  0.500000000000000 + 0.000000000000000i
```

Notice that we have just computed the degree 2 Taylor coefficient of $\exp(z)$.

When Chebfun integrates around a circular contour like this, it does not automatically take advantage of the fact that the integrand is periodic. That would be Fourier analysis as opposed to Chebyshev analysis, and beginning with Version 5, a "Fourfun" approach to such problems has been available, at least when the arguments are smooth (compare [Davis 1959]). For example, we could repeat the above calculation in Fourier mode like this:

```
z = chebfun('exp(1i*s)',[0 2*pi],'periodic');
f = exp(z)./z.^3;
I = sum(f.*diff(z))/(2i*pi)
```

```
I =
  0.500000000000002 + 0.000000000000000i
```

Chebyshev methods are more flexible, as a rule, but Fourier methods have advantages sometimes of efficiency (up to a factor of $\pi/2$ per dimension) and accuracy. For techniques that recover some of that factor of $\pi/2$ even for nonperiodic problems, see [Hale & Trefethen 2008].

The contour does not have to have radius 1, or be centered at the origin:

```
z = chebfun('1+2*exp(1i*s)',[0 2*pi],'periodic');
f = exp(z)./z.^3;
I2 = sum(f.*diff(z))/(2i*pi)
```

```
I2 =
  0.500000000000000 + 0.000000000000000i
```

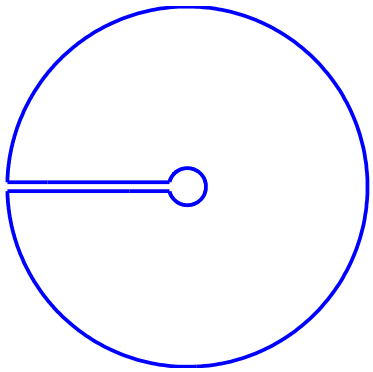
Nor does the contour have to be smooth. Here let us compute the same result by integration over a square (reverting to Chebyshev rather than Fourier technology).

```
s = chebfun('s',[-1 1]);
z = join(1+1i*s, 1i-s, -1-1i*s, -1i+s);
f = exp(z)./z.^3;
I3 = sum(f.*diff(z))/(2i*pi)
```

```
I3 =
  0.500000000000000 + 0.000000000000000i
```

In Chebfun one can also construct more interesting contours of the kind that appear in complex variables texts. Here is an example involving a "keyhole" contour:

```
c = [-2+.05i, -.2+.05i, -.2-.05i, -2-.05i];    % 4 corners
s = chebfun('s',[0 1]);
z = join(c(1)+s*(c(2)-c(1)), c(2)*c(3).^s./c(2).^s, ...
     c(3)+s*(c(4)-c(3)), c(4)*c(1).^s./c(4).^s);
clf, plot(z), axis equal, axis off
```



The integral of $f(z) = \log(z)\tanh(z)$ around this contour will be equal to $2\pi i$ times the sum of the residues at the poles of f at $\pm\pi i/2$.

```
f = log(z).*tanh(z);
I = sum(f.*diff(z))
Iexact = 4i*pi*log(pi/2)
```

```
I =
  0.000000000000005 + 5.674755637702216i
Iexact =
  0.000000000000000 + 5.674755637702224i
```

5.4 Cauchy integrals and locating zeros and poles

Here are some further examples of computations with Cauchy integrals. The Bernoulli number B_k is $k!$ times the kth Taylor coefficient of $z/((\exp(z)-1)$. Here is B_{10} compared with its exact value $5/66$.

```
k = 10;
z = chebfun('5*exp(1i*s)',[0 2*pi]);
f = z./((exp(z)-1));
B10 = factorial(k)*sum((f./z.^(k+1)).*diff(z))/(2i*pi)
exact = 5/66
```

```
B10 =
  0.075757575757581 - 0.000000000000004i
exact =
   0.075757575757576
```

Notice that we have taken `z` to be a circle of radius 5. If the radius is 1, the accuracy is a good deal lower:

```
z = chebfun('exp(1i*s)',[0 2*pi]);
f = z./((exp(z)-1));
B10 = factorial(k)*sum((f./z.^(k+1)).*diff(z))/(2i*pi)
```

```
B10 =
  0.075757575477819 - 0.000000000113937i
```

This problem of numerical instability would arise no matter how one calculated the integral over the unit circle; it is not the fault of Chebfun. For a study of how to pick the optimal radius, see [Bornemann 2009].

Another use of Cauchy integrals is to count zeros or poles of functions in specified regions. According to the **principle of the argument**, the number of zeros minus the number of poles of f in a region is

$$N = \frac{1}{2\pi i} \int \frac{f'(z)}{f(z)} dz,$$

where the integral is taken over the boundary. Since $f' = df/dz = (df/ds)(ds/dz)$, we can rewrite this as

$$N = \frac{1}{2\pi i} \int \frac{1}{f} \frac{df}{ds} ds.$$

For example, the function $f(z) = \sin(z)^3 + \cos(z)^3$ clearly has no poles; how many zeros does it have in the disk about 0 of radius 2? The following calculation shows that the answer is 3:

```
z = chebfun('2*exp(1i*s)',[0 2*pi]);
f = sin(z).^3 + cos(z).^3;
N = sum((diff(f)./f))/(2i*pi)
```

```
N =
  2.999999999999998 + 0.000000000000000i
```

What is really going on here is a calculation of the change of the argument of f as the boundary is traversed. Another way to find that number is with the Chebfun overloads of the MATLAB commands `angle` and `unwrap`:

```
anglef = unwrap(angle(f));
N = (anglef(end)-anglef(0))/(2*pi)
```

```
N =
   2.999999999999999
```

Variations on this idea enable one to locate zeros and poles as well as count them. For example, we can locate a single zero with the formula

$$r = \frac{1}{2\pi i} \int z(df/ds)/f ds$$

[McCune 1966]. Here is the zero of the function above in the unit disk:

```
z = chebfun('exp(1i*s)',[0 2*pi],'periodic');
f = sin(z).^3 + cos(z).^3;
r = sum(z.*(diff(f)./f))/(2i*pi)
```

```
r =
 -0.785398163397449 - 0.000000000000000i
```

We can check the result by a more ordinary Chebfun calculation:

```
x = chebfun('x');
f = sin(x).^3 + cos(x).^3;
r = roots(f)
```

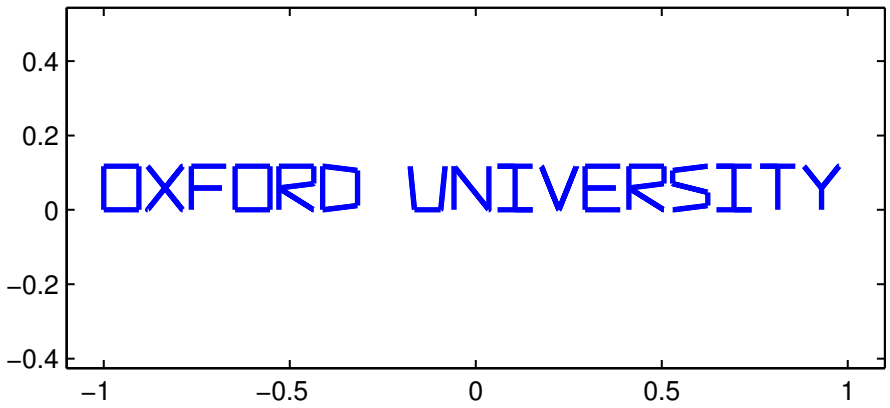
```
r =
  -0.785398163397449
```

To find multiple zeros via Cauchy integrals, and for many other generalizations of the ideas in this chapter, see [Austin, Kravanja & Trefethen 2013].

5.5 Alphabet soup

The Chebfun command `scribble` returns a piecewise linear complex chebfun corresponding to a word spelled out in capital letters. For example:

```
f = scribble('Oxford University');
LW = 'linewidth'; lw = 2;
plot(f,LW,lw), xlim(1.1*[-1 1]), axis equal
```



This chebfun happens to have 67 pieces:

```
length(domain(f))-1
```

```
ans =
    67
```

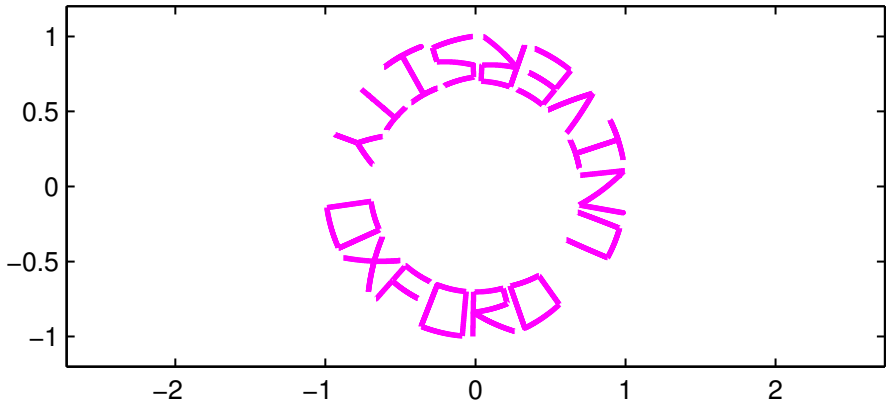
Though its applications are unlikely to be mathematical, f is a precisely defined mathematical object just like any other chebfun. If we wish, we can compute with it:

```
f(0), norm(f)
```

```
ans =
   0.129411764705882
ans =
   0.847576500999202
```

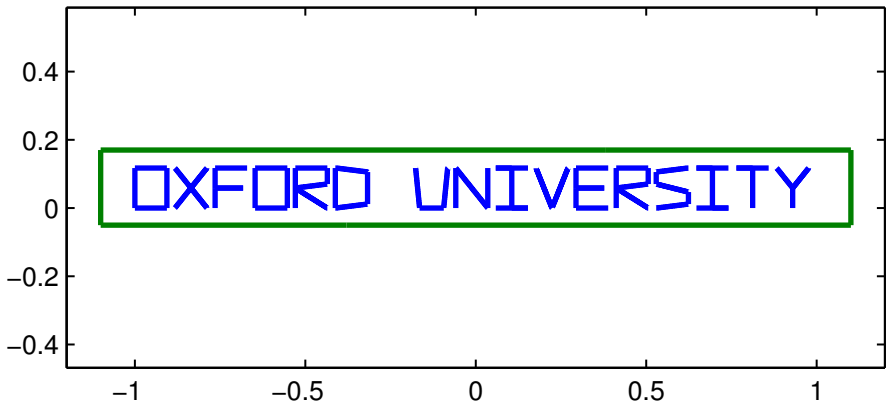
Perhaps more interesting is that we can apply functions to it and learn something in the process:

```
plot(exp(3i*f),'m',LW,lw), ylim(1.2*[-1 1]), axis equal
```

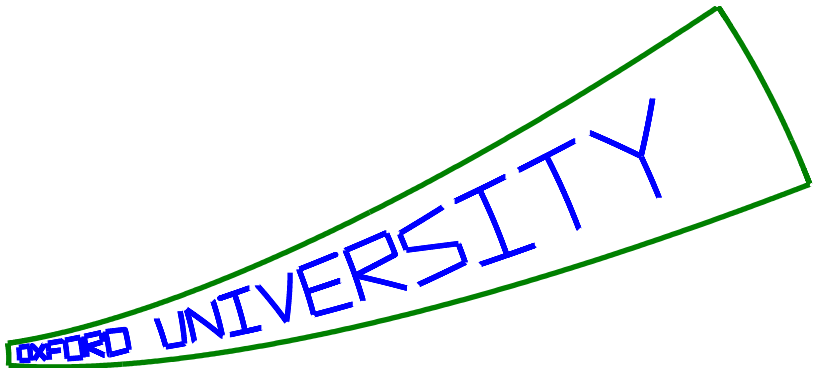


Does putting a box around enhance the image? (We do this by adding a second column of a Chebfun quasimatrix – see Chapter 6.)

```
L = f.ends(end);
s = chebfun(@(x) 2*x+2,[-1 -0.5]);
box = join(-1.1-.05i+2.2*s,1.1-.05i+.22i*s,1.1+.17i-2.2*s,-1.1+.17i-.22i*s);
f = [f box];
plot(f,LW,lw), xlim(1.2*[-1 1]), axis equal
```




```
clf, plot(exp((1+.2i)*f),LW,lw), axis equal, axis off
```




```
plot(tan(f),LW,lw), axis equal, axis off
```

Next May 16, you might wish to write a greeting card for Pafnuty Lvovich Chebyshev, accurate as always to 15 digits:

```
f = scribble('Happy Birthday Pafnuty!');
L = f.ends(end);
g = @(z) exp(-2.2i+(2.5i+.4)*z);
clf, plot(g(f),'r',LW,lw), axis equal, axis off
circle = 1.12*chebfun(@(x) exp(2i*pi*x/L),[0 L]);
```

```
ellipse = 1.2*(circle + 1./circle)/2 + 1i*mean(imag(f));
hold on, plot(g(ellipse),'b',LW,lw)
axis auto equal off
```

You can find an example "Birthday cards and analytic function" in the Complex Variables section of the Chebfun Examples collection, and further related explorations in the Geometry section.

5.6 References

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6. Quasimatrices and Least-Squares

Lloyd N. Trefethen, November 2009, latest revision June 2014

Contents

6.1 Quasimatrices and `spy`

A chebfun can have more than one column, or if it is transposed, it can have more than one row. In these cases we get a **quasimatrix**, a "matrix" in which one of the dimensions is discrete as usual but the other is continuous. Our default choice will be that of an " $\infty \times n$ " quasimatrix consisting of n columns, each of which is a chebfun. When it is important to specify the orientation we use the term **column quasimatrix** or **row quasimatrix**.

Here for example is the quasimatrix consisting of the first six powers of x on the interval $[-1,1]$. The command `size` can be used to identify the continuous dimension, and to find the numbers of rows or columns:

```
x = chebfun('x');
A = [1 x x.^2 x.^3 x.^4 x.^5];
size(A)
size(A,2)
```

```
ans =
   Inf     6
ans =
     6
```

Here is the third column of A evaluated at the point $x = 0.5$:

```
A(0.5,3)
```

```
ans =
   0.250000000000000
```

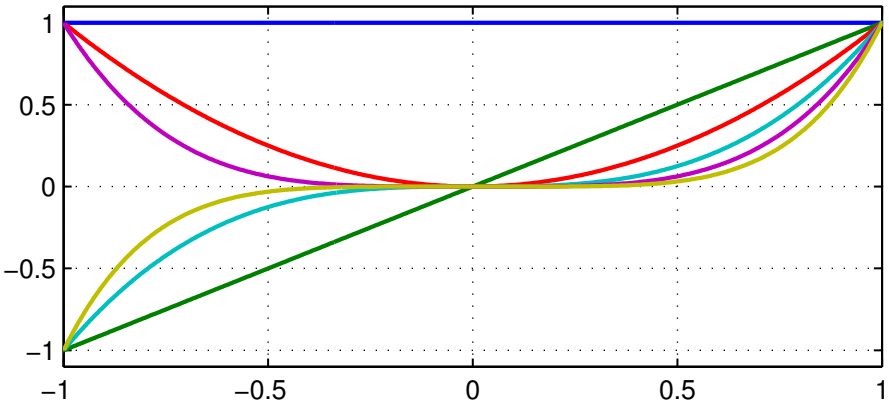
Here are the column sums, i.e., the integrals of $1, x, \ldots, x^5$ from -1 to 1:

```
format short, sum(A), format long
```

```
ans =
    2.0000         0    0.6667         0    0.4000         0
```

And here is a plot of the columns:

```
LW = 'linewidth';
plot(A,LW,1.6), grid on, ylim([-1.1 1.1])
```



The term quasimatrix comes from [Stewart 1998], and the same idea appears with different terminology in [de Boor 1991] and [Trefethen & Bau 1997, pp. 52-54]. The idea is a natural one, since so much of applied linear algebra deals with discrete approximations to the continuous, but it seems not to have been discussed explicitly very much until the appearance of Chebfun [Battles & Trefethen 2004, Battles 2006].

If `f` and `g` are column chebfuns, then `f'*g` is a scalar, their inner product. For example, here is the inner product of x^2 and x^4 over $[-1, 1]$ (equal to $2/7$):

```
A(:,3)'*A(:,5)
```

```
ans =
   0.285714285714286
```

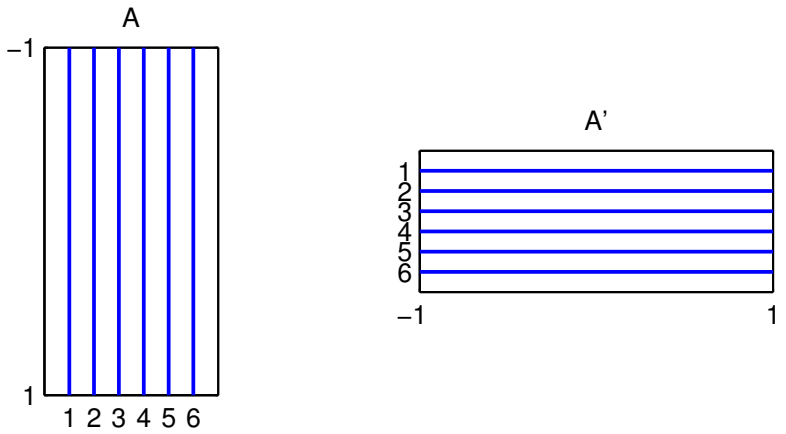
More generally, if `A` and `B` are column quasimatrices with m and n columns, respectively, then `A'*B` is the $m \times n$ matrix of inner products of those columns. Here is the 6×6 example corresponding to $B = A$:

```
format short, A'*A, format long
```

```
ans =
    2.0000    0.0000    0.6667         0    0.4000    0.0000
    0.0000    0.6667         0    0.4000    0.0000    0.2857
    0.6667         0    0.4000         0    0.2857         0
         0    0.4000         0    0.2857         0    0.2222
    0.4000    0.0000    0.2857         0    0.2222         0
    0.0000    0.2857         0    0.2222         0    0.1818
```

You can get an idea of the shape of a quasimatrix with the overloaded `spy` command

```
subplot(1,2,1), spy(A), title A
subplot(1,2,2), spy(A'), title('A''')
```



6.2 Backslash and least-squares

In MATLAB, the command `c = A\b` computes the solution to the system of equations $Ac = b$ if A is a square matrix, whereas if A is rectangular, with more rows than columns, it computes the least squares solution, the vector c that minimizes $\|Ac - b\|$. A quasimatrix is always rectangular, and \ has accordingly been overloaded to carry out the appropriate continuous least-squares computation. (The actual MATLAB command that handles backslash is `mldivide`.)

For example, continuing with the same chebfun `x` and quasimatrix `A` as above, consider the following sequence:

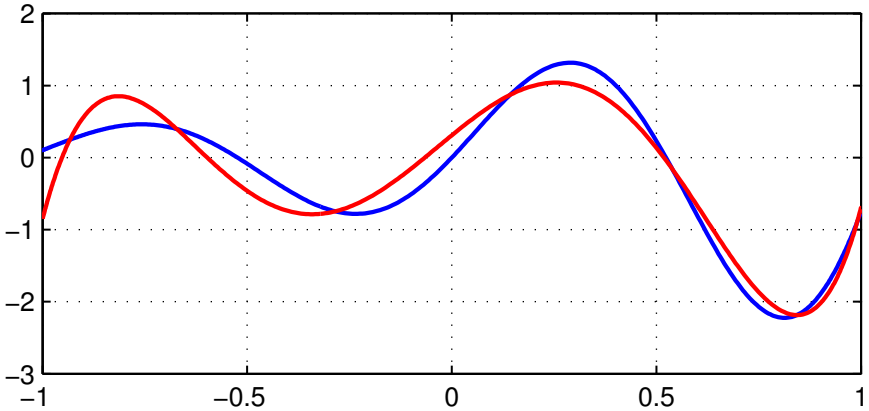
```
f = exp(x).*sin(6*x);
c = A\f
```

```
c =
   0.309654988398407
   4.640757102742464
  -2.157249816336408
 -20.041645425109152
   1.073963006923383
  15.477982292827999
```

The vector c can be interpreted as the vector of coefficients of the least-squares fit to f by a linear combination of the functions $1, x, \ldots, x^5$. Here is a plot of f (in blue) and the least-squares approximation (in red), which we label `ffit`.

```
ffit = A*c;
clf, plot(f,'b',ffit,'r',LW,1.6), grid on
error = norm(f-ffit)
```

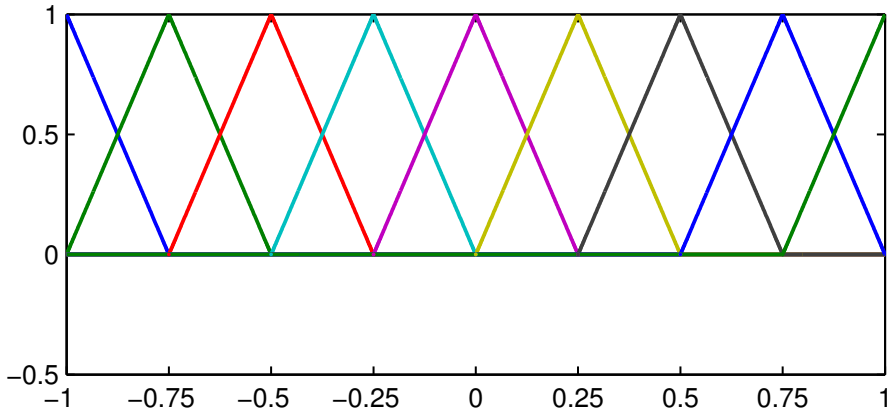
```
error =
   0.356073976001434
```



It is a general result that the least-squares approximation by a polynomial of degree n to a continuous function f must intersect f at least $n+1$ times in the interval of approximation.

Here is quite a different quasimatrix whose columns can be used to fit f. The columns correspond to hat functions located at points equally spaced from -1 to 1, and they are realized as piecewise smooth chebfuns.

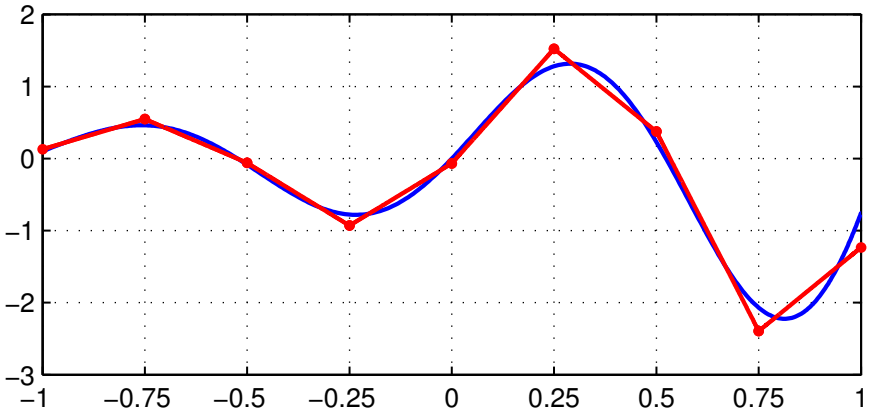
```
A2 = [];
for j = 0:8
  xj = -1 + j/4;
  A2 = [A2 max(0,1-4*abs(x-xj))];
end
plot(A2)
set(gca,'xtick',-1:.25:1)
```



A linear combination of these columns is a piecewise linear function with breakpoints at $-0.75, -0.50, \ldots, 0.75$. Here is the least-squares fit by such functions to f. Remember that although we happen to be fitting here by a function with a discrete flavor, all the operations are continuous ones involving integrals, not point evaluations.

```
c = A2\f;
ffit = A2*c;
plot(f,'b',ffit,'.-r',LW,1.6), grid on
set(gca,'xtick',-1:.25:1)
error = norm(f-ffit)
```

```
error =
   0.148137345378415
```



6.3 QR factorization

Matrix least-squares problems are ordinarily solved by QR factorization, and in the quasimatrix case, they are solved by quasimatrix QR factorization. This is the technology underlying the backslash operator described in the last section.

A quasimatrix QR factorization takes this form:

$$A = QR,$$

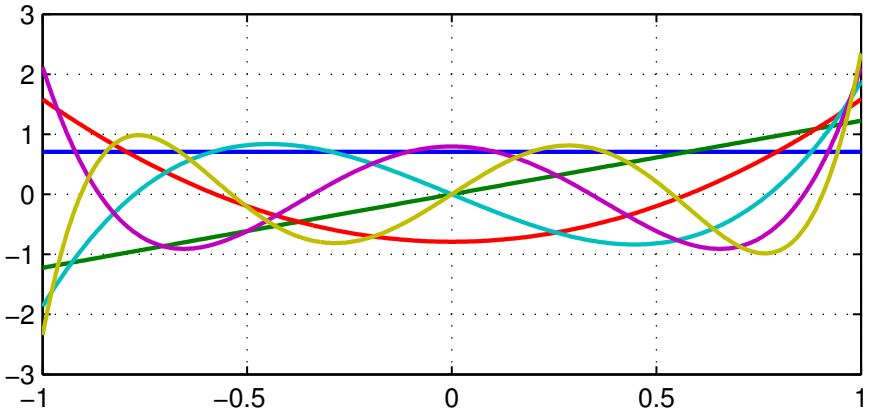
with

$$A: \ \infty \times n, \quad Q: \ \infty \times n, \quad R: \ n \times n.$$

The columns of A are arbitrary, the columns of Q are orthonormal, and R is an $n \times n$ upper-triangular matrix. This factorization corresponds to what is known in various texts as the "reduced", "economy size", "skinny", "abbreviated", or "condensed" QR factorization, since Q is rectangular rather than square and R is square rather than rectangular. In MATLAB the syntax for computing such things

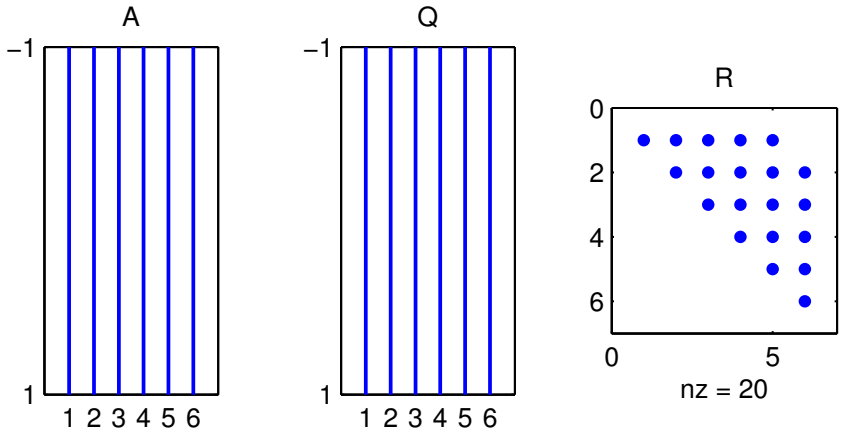
is `[Q,R] = qr(A,0)`, and the same command has been overloaded for chebfuns. The computation makes use of a quasimatrix analogue of Householder triangularization [Trefethen 2010]. Alternatively one can simply write `[Q,R] = qr(A)`:

```
[Q,R] = qr(A);
plot(Q,LW,1.6), grid on
```



The `spy` command confirms the shape of these various matrices. In principle half the dots in the upper-triangle should be zero because of the fact that the columns of A alternate even and odd functions, but rounding errors introduce nonzeros.

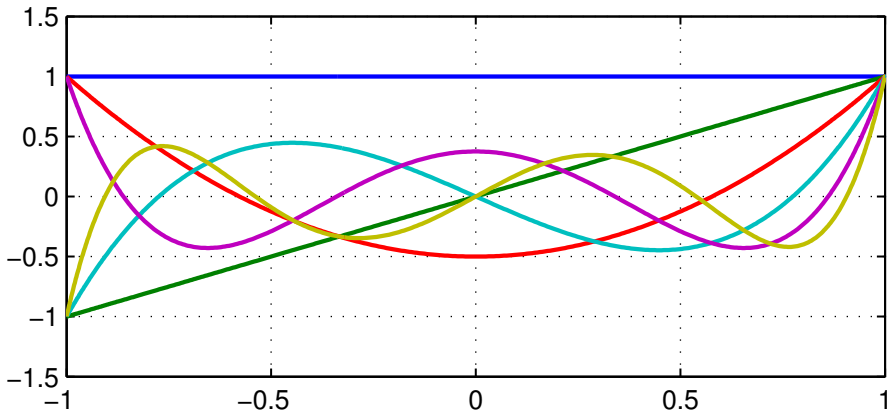
```
subplot(1,3,1), spy(A), title A
subplot(1,3,2), spy(Q), title Q
subplot(1,3,3), spy(R), title R
```



The plot shows **orthogonal polynomials**, namely the orthogonalizations of the monomials $1, x, \ldots, x^5$ over $[-1, 1]$. These are the famous Legendre polynomials P_k [Abramowitz & Stegun 1972], except that the latter are conventionally normalized by the condition $P(1) = 1$ rather than by having norm 1. We can renormalize to impose this condition as follows:

```
for j = 1:size(A,2)
  R(j,:) = R(j,:)*Q(1,j);
  Q(:,j) = Q(:,j)/Q(1,j);
```

```
end
clf, plot(Q,LW,1.6), grid on
```



(A slicker way to produce this plot in Chebfun would be to execute `plot(legpoly(0:5))`.)

If $A = QR$, then $AR^{-1} = Q$, and here is R^{-1}:

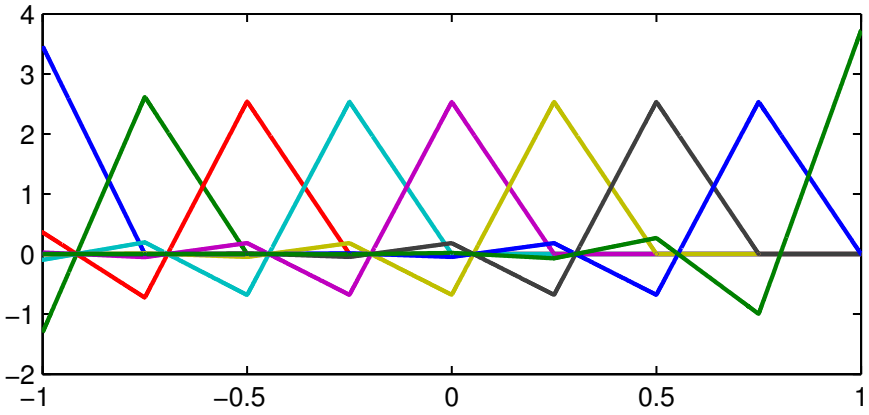
```
format short, inv(R), format long
```

```
ans =
    1.0000   -0.0000   -0.5000    0.0000    0.3750   -0.0000
         0    1.0000    0.0000   -1.5000   -0.0000    1.8750
         0         0    1.5000   -0.0000   -3.7500    0.0000
         0         0         0    2.5000    0.0000   -8.7500
         0         0         0         0    4.3750   -0.0000
         0         0         0         0         0    7.8750
```

Column k of R^{-1} is the vector of coefficients of the expansion of column k of Q as a linear combination of the columns of A, that is, the monomials $1, x, x^2, \ldots$. In other words, column k of R^{-1} is the vector of coefficients of the degree k Legendre polynomial. For example, we see from the matrix that $P_3(x) = 2.5x^3 - 1.5x$.

Here is what the hat functions look like after orthonormalization:

```
[Q2,R2] = qr(A2);
plot(Q2,LW,1.6)
```



6.4 `svd`, `norm`, `cond`

An $m \times n$ matrix A defines a map from R^n to R^m, and in particular, A maps the unit ball in R^n to a hyperellipsoid of dimension $\le n$ in R^m. The (reduced, skinny, condensed,...) **SVD** or **singular value decomposition** exhibits this map by providing a factorization $AV = US$ or equivalently $A = USV^*$, where U is $m \times n$ with orthonormal columns, S is diagonal with nonincreasing nonnegative diagonal entries known as the **singular values**, and V is $n \times n$ and orthogonal. A maps v_j, the j th column of V or the j th **right singular vector**, to s_j times u_j, the j th column of U or the j th **left singular vector**, which is the vector defining the j th largest semiaxis of the hyperellipsoid. See Chapters 4 and 5 of [Trefethen & Bau 1997].

If A is an $\infty \times n$ quasimatrix, everything is analogous:

$$A = USV', \qquad A : \infty \times n, \quad U : \infty \times n, \quad S : n \times n, \quad V : n \times n.$$

The image of the unit ball in R^n under A is still a hyperellipsoid of dimension $\le n$, which now lies within an infinite-dimensional function space. The columns of Q are orthonormal functions and S and V have the same properties as in the matrix case.

For example, here are the singular values of the matrix A defined earlier with columns $1, x, \ldots, x^5$:

```
s = svd(A,0)
```

```
s =
   1.532062889375340
   1.032551897396699
   0.518125864967969
   0.258419769500035
   0.080938947808205
   0.035425077461572
```

The largest singular value is equal to the norm of the quasimatrix, which is defined by $\|A\| = \max_x \|Ax\|/\|x\|$.

```
norm(A,2)
```

```
ans =
   1.532062889375340
```

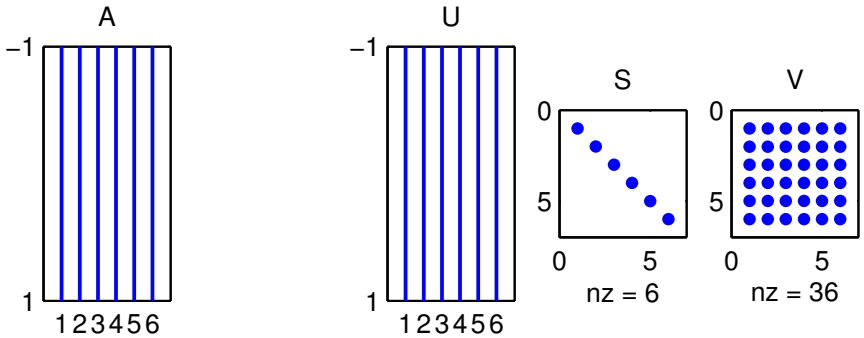
(Note that we must include the argument `2` here: for reasons of speed, the default for quasimatrices, unlike the usual MATLAB matrices, is the Frobenius norm rather than the 2-norm.) The SVD enables us to identify exactly what vectors are involved in achieving this maximum ratio. The optimal vector x is v_1, the first right singular vector of A,

```
[U,S,V] = svd(A);
v1 = V(:,1)
```

```
v1 =
   0.913034433780914
   0.000000000000000
   0.344611116356111
   0.000000000000000
   0.218200140270718
   0.000000000000000
```

We can use spy to confirm the shapes of the matrices. As with `spy( R)` earlier, here `spy(V)` should in principle show a checkerboard, but nonzeros are introduced by rounding errors.

```
subplot(1,5,1), spy(A), title A
subplot(1,5,3), spy(U), title U
subplot(1,5,4), spy(S), title S
subplot(1,5,5), spy(V), title V
```



We confirm that the norm of v_1 is 1:

```
norm(v1)
```

```
ans =
   1.000000000000000
```

This vector is mapped by A to the chebfun $s_1 u_1$:

```
  u1 = U(:,1);
  norm(u1)
```

```
ans =
     1
```

```
  s1 = S(1,1)
```

```
s1 =
   1.532062889375340
```

```
  norm(A*v1)
```

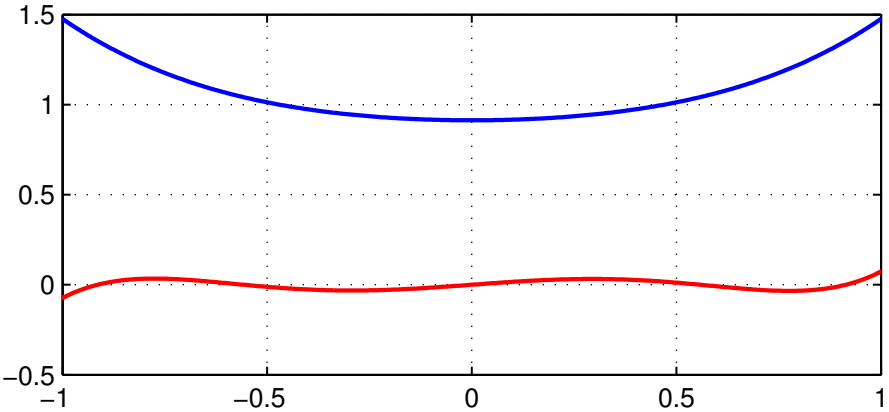
```
ans =
   1.532062889375341
```

```
  norm(A*v1-s1*u1)
```

```
ans =
     4.142554461132179e-16
```

Similarly, the minimal singular value and corresponding singular vectors describe the minimum amount that A can enlarge an input. The following commands plot the extreme functions Av_1 (blue) and Av_n (red). We can interpret these as the largest and smallest degree 5 polynomials, as measured in the 2-norm over $[-1, 1]$, whose coefficient vectors have 2-norm equal to 1.

```
  clf, plot(A*v1,LW,1.6), grid on, hold on
  vn = V(:,end);
  plot(A*vn,'r',LW,1.6), hold off
```



The ratio of the largest and smallest singular values – the eccentricity of the hyperellipsoid – is the condition number of A:

```
  max(s)/min(s)
```

```
ans =
  43.247975704139762

  cond(A)

ans =
  43.247975704139762
```

The fact that `cond(A)` is a good deal greater than 1 reflects the ill-conditioning of the monomials $1, x, \ldots, x^5$ as a basis for degree 5 polynomials in $[-1, 1]$. The effect becomes rapidly stronger as we take more terms in the sequence:

```
  cond([A x.^6 x.^7 x.^8 x.^9 x.^10 x.^11 x.^12 x.^13 x.^14 x.^15])

ans =
     2.298938277191532e+05
```

By contrast a quasimatrix formed of suitably normalized Legendre polynomials has condition number 1, since they are orthonormal:

```
cond(legpoly(0:15,'norm'))

ans =
   1.000000000000002
```

A quasimatrix of Chebyshev polynomials doesn't quite achieve condition number 1, but it comes close:

```
cond(chebpoly(0:15))

ans =
   4.597747107616717
```

Chebyshev polynomials form an excellent basis for expansions on $[-1, 1]$, a fact that is at the heart of Chebfun.

6.5 Other norms

The definition $\|A\| = \max_x \|Ax\|/\|x\|$ makes sense in other norms besides the 2-norm, and the particularly important alternatives are the 1-norm and the ∞-norm. The 1-norm of a column quasimatrix is the "maximum column sum", i.e., the maximum of the 1-norms of its columns. In the case of our quasimatrix A, the maximum is attained by the first column, which has norm 2:

```
  norm(A,1)

ans =
     2
```

The ∞-norm is the "maximum row sum", which for a column quasimatrix corresponds to the maximum of the chebfun obtained by adding the absolute values of the columns. In the case of A, the sum is $1 + |x| + \cdots + |x|^5$, which attains its maximum value 6 at $x = -1$ and 1:

```
norm(A,inf)
```

```
ans =
   6.000000000000002
```

The norms of row quasimatrices are analogous, with `norm(A',inf) = norm(A,1)` and `norm(A',1) = norm(A,inf)`. Like MATLAB itself applied to a rectangular matrix, Chebfun does not define `cond(A,1)` or `cond(A,inf)` if `A` is a quasimatrix.

The Frobenius or Hilbert-Schmidt norm is equal to the square root of the sum of the squares of the singular values:

```
norm(A,'fro')
```

```
ans =
   1.938148951041007
```

6.6 `rank`, `null`, `orth`, `pinv`

Chebfun also contains overloads for some further MATLAB operations related to orthogonal matrix factorizations. Perhaps the most useful of these is `rank(A)`, which computes the singular values of A and makes a judgement as to how many of them are significantly different from zero. For example, with x still defined as before, here is an example showing that the functions 1, $\sin(x)^2$, and $\cos(x)^2$ are linearly dependent:

```
B = [1 sin(x).^2 cos(x).^2];
rank(B)
```

```
ans =
     2
```

Since B is rank-deficient, is has a nontrivial nullspace, and the command `null(B)` will find an orthonormal basis for it:

```
null(B)
```

```
ans =
  -0.577350269189626
   0.577350269189626
   0.577350269189626
```

Similarly the command `orth(B)` finds an orthonormal basis for the range of B, which in this case has dimension 2:

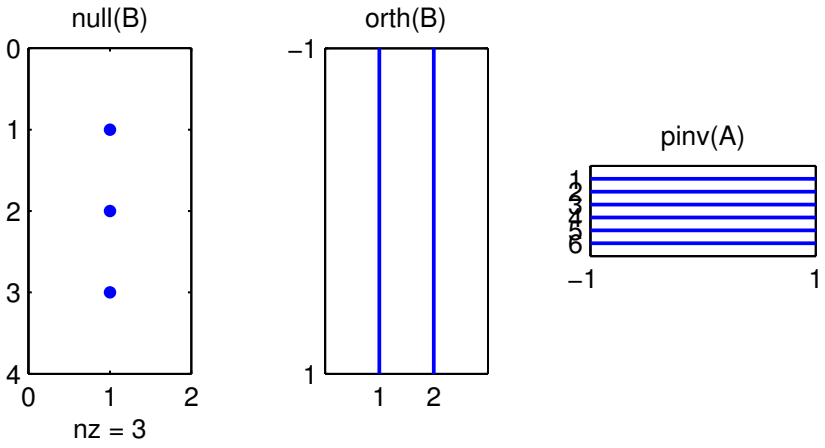
```
  orth(B)

ans =
   chebfun column1 (1 smooth piece)
       interval       length   endpoint values
[      -1,       1]       17      0.61     0.61
Epslevel = 2.340672e-16.  Vscale = 7.634146e-01.
   chebfun column2 (1 smooth piece)
       interval       length   endpoint values
[      -1,       1]       17      -1.4     -1.4
Epslevel = 1.555489e-15.  Vscale = 1.425588e+00.
```

If `A` is an $\infty \times n$ column quasimatrix, the command `pinv(A)` computes the pseudoinverse of `A`, an $n \times \infty$ row quasimatrix such that `pinv(A)*c = A\c`.

Here is a summary of the dimensions of these objects:

```
  subplot(1,3,1), spy(null(B)), title null(B)
  subplot(1,3,2), spy(orth(B)), title orth(B)
  subplot(1,3,3), spy(pinv(A)), title pinv(A)
```



6.7 Array-valued chebfuns vs. arrays of chebfuns

In Chebfun, quasimatrices are actually implemented in two different ways. When its columns are smooth functions, a quasimatrix is normally represented as an **array-valued chebfun**. If a quasimatrix has singularities, or breakpoints that differ from one column to another, it is represented in a different fashion as an **array of chebfuns**. This representation is more flexible, though slower for some operations. In principle, users should never see the difference.

6.8 References

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7. Linear Differential Operators and Equations

Tobin A. Driscoll, November 2009, latest revision June 2014

Contents

7.1 Introduction

Chebfun has powerful capabilities for solving ordinary differential equations as well as partial differential equations involving one space and one time variable. The present chapter is devoted to chebops, the fundamental Chebfun tools for solving differential (or integral) equations. In particular we focus here on the linear case. We shall see that one can solve a linear two-point boundary value problem to high accuracy by a single backslash command. Nonlinear extensions are described in Section 7.9 and in Chapter 10, and for PDEs, try `help pde15s`.

7.2 About linear chebops

A chebop represents a differential or integral operator that acts on chebfuns. This chapter focusses on the linear case, though from a user's point of view, linear and nonlinear problems are quite similar. One thing that makes linear operators special is that `eigs` and `expm` can be applied to them, as we shall describe in Sections 7.5 and 7.6.

Like chebfuns, chebops start from premise of approximation by piecewise polynomial interpolants; in the context of differential equations, such techniques are called spectral collocation methods. As with chebfuns, the discretizations are chosen automatically to achieve the maximum possible accuracy available from double precision arithmetic. In fact, beginning with version 5, Chebfun actually offers two different methods for solving these problems, which go by the names of rectangular collocation

(or Driscoll-Hale) spectral methods and ultraspherical (or Olver-Townsend) spectral methods. See Sections 7.7 and 8.10.

The linear part of the chebop package was conceived at Oxford by Bornemann, Driscoll, and Trefethen [Driscoll, Bornemann & Trefethen 2008], and the implementation is due to Driscoll, Hale, and Birkisson [Birkisson & Driscoll 2011, Driscoll & Hale 2014] . Much of the functionality of linear chebops is actually implemented in various classes built around the idea of what we call a "linop", but users generally do not deal with linops and related structures directly.

7.3 Chebop syntax

A chebop has a domain, an operator, and sometimes boundary conditions. For example, here is the chebop corresponding to the second-derivative operator on $[-1, 1]$:

```
L = chebop(-1, 1);
L.op = @(x,u) diff(u,2);
```

(For scalar operators like this, one may dispense with the `x` and just write `L.op = @(u) diff(u,2)`.) This operator can now be applied to chebfuns defined on $[-1, 1]$. For example, taking two derivatives of $\sin(3x)$ multiplies its amplitude by 9:

```
u = chebfun('sin(3*x)');
norm(L(u), inf)
```

```
ans =
   9.000000000000084
```

Both the notations `L*u` and `L(u)` are allowed, with the same meaning.

```
min(L*u)
```

```
ans =
  -9.000000000000084
```

Mathematicians generally prefer Lu if L is linear and $L(u)$ if it is nonlinear.

A chebop can also have left and/or right boundary conditions. For a Dirichlet boundary condition it's enough to specify a number:

```
L.lbc = 0;
L.rbc = 1;
```

More complicated boundary conditions can be specified with anonymous functions, which are then forced to take zero values at the boundary. For example, the following sequence imposes the conditions $u = 0$ at the left boundary and $u' = 1$ at the right:

```
L.lbc = @(u) u;
L.rbc = @(u) diff(u) - 1;
```

We can see a summary of `L` by typing the name without a semicolon:

```
L
```

```
L =
   Linear operator:
      diff(u,2) = 0
   operating on chebfun objects defined on:
      [-1 1]
   with
    left boundary conditions:
      u = 0
    right boundary conditions:
      @(u)diff(u)-1 = 0
```

Boundary conditions are needed for solving differential equations, but they have no effect when a chebop is simply applied to a chebfun. Thus, despite the boundary conditions just specified, `L*u` gives the same answer as before:

```
norm(L*u, inf)
```

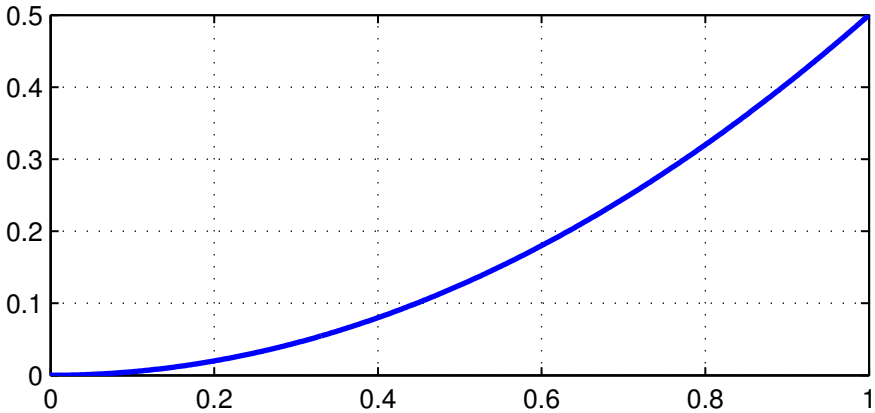
```
ans =
   9.000000000000084
```

Here is an example of an integral operator, the operator that maps u defined on $[0, 1]$ to its indefinite integral:

```
L = chebop(0, 1);
L.op = @(x,u) cumsum(u);
```

For example, the indefinite integral of x is $x^2/2$:

```
x = chebfun('x', [0, 1]);
LW = 'linewidth';
hold off, plot(L*x, LW, 2), grid on
```



Chebops can be specified in various ways, including all in a single line. For example we could write

```
L = chebop(@(x,u) diff(u) + diff(u,2), [-1, 1])
```

```
L =
   Linear operator:
      diff(u)+diff(u,2) = 0
   operating on chebfun objects defined on:
      [-1 1]
```

Or we could include boundary conditions:

```
L = chebop(@(x,u) diff(u) + diff(u,2), [-1, 1], 0, @(u) diff(u))
```

```
L =
   Linear operator:
      diff(u)+diff(u,2) = 0
   operating on chebfun objects defined on:
      [-1 1]
   with
    left boundary conditions:
      u-BC = 0
    right boundary conditions:
      @(u)diff(u) = 0
```

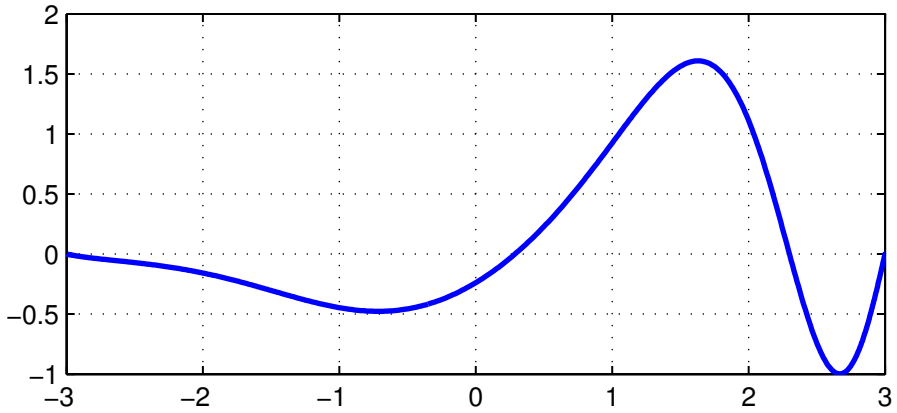
For operators applying to more than one variable (needed for solving systems of differential equations), see Section 7.8.

7.4 Solving differential and integral equations

In MATLAB, if `A` is a square matrix and `b` is a vector, then the command `x=A\b` solves the linear system of equations $Ax = b$. Similarly in Chebfun, if `L` is a differential operator with appropriate boundary conditions and `f` is a Chebfun, then `u=L\f` solves the differential equation $L(u) = f$. More generally `L` might be an integral or integro-differential operator. (Of course, just as you can solve $Ax = b$ only if A is nonsingular, you can solve $L(u) = f$ only if the problem is well-posed.)

For example, suppose we want to solve the differential equation $u'' + x^3 u = 1$ on the interval $[-3, 3]$ with Dirichlet boundary conditions. Here is a Chebfun solution:

```
L = chebop(-3, 3);
L.op = @(x,u) diff(u,2) + x.^3.*u;
L.lbc = 0; L.rbc = 0;
u = L\1; plot(u, LW, 2), grid on
```



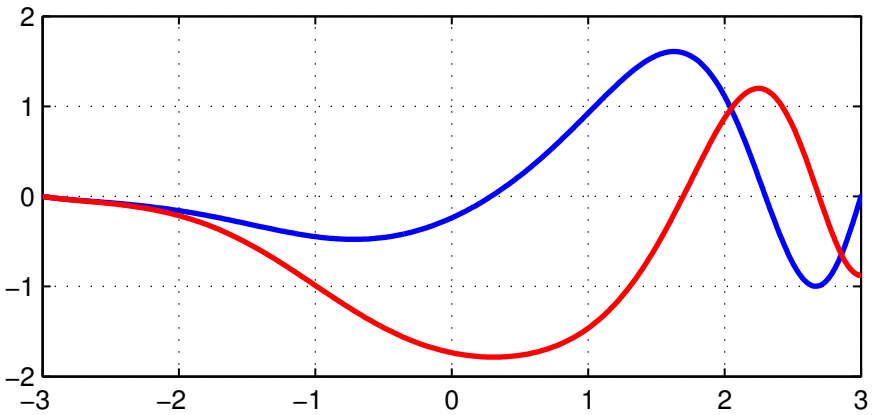
We confirm that the computed u satisfies the differential equation to high accuracy:

```
norm(L(u) - 1)
```

```
ans =
     1.812483165209799e-11
```

Let's change the right-hand boundary condition to $u' = 0$ and see how this changes the solution:

```
L.rbc = @(u) diff(u);
u = L\1;
hold on, plot(u, 'r', LW, 2)
```



An equivalent to backslash is the `solvebvp` command.

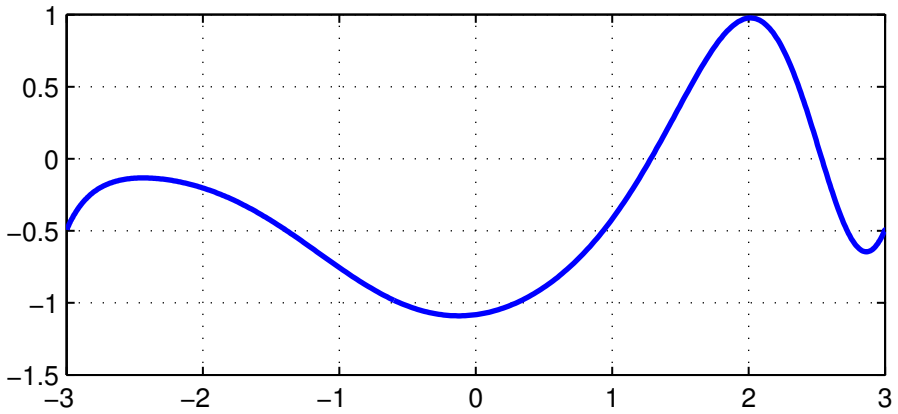
```
v = solvebvp(L, 1);
norm(u - v)
```

```
ans =
     0
```

Periodic boundary conditions can be imposed with the special boundary condition string `L.bc='periodic'`, which will find a periodic solution, provided that the right-side function is also

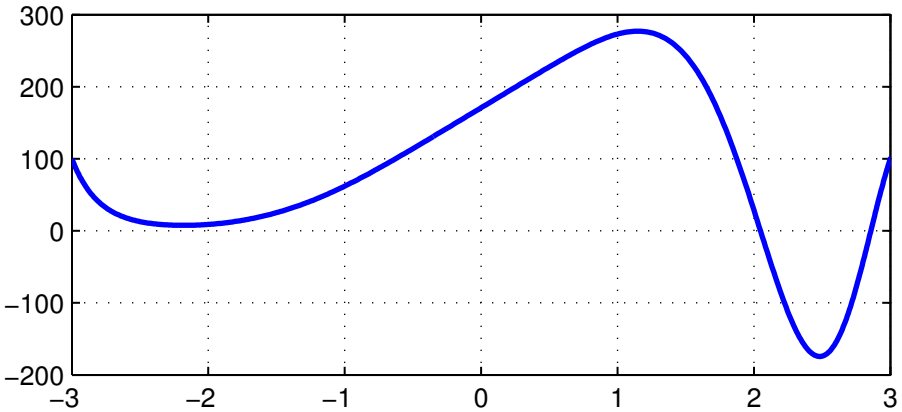
periodic. At the moment Chebfun does not include truly periodic spectral discretizations based on Fourier series, though we hope to introduce this feature before long.

```
L.bc = 'periodic';
u = L\1;
hold off, plot(u, LW, 2), grid on
```



A command like `L.bc=100` imposes the corresponding numerical Dirichlet condition at both ends of the domain:

```
L.bc = 100;
plot(L\1, LW, 2), grid on
```

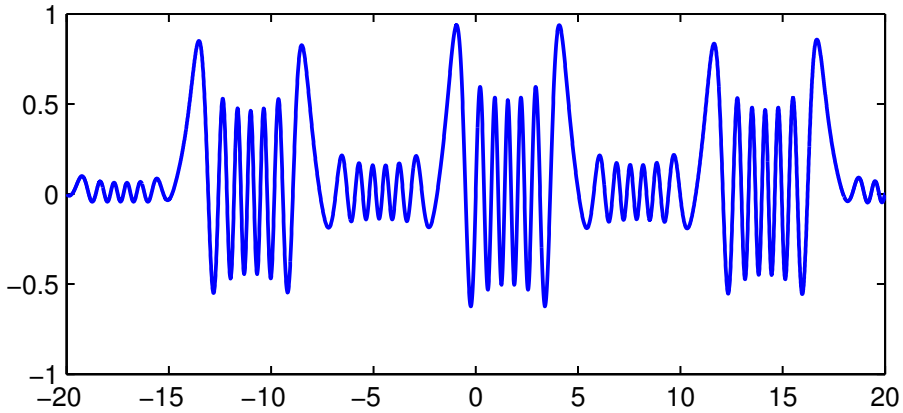


Boundary conditions can also be specified in a single line, as noted above:

```
L = chebop( @(x,u) diff(u,2) + 10000*u, [-1,1], 0, @(u) diff(u) );
```

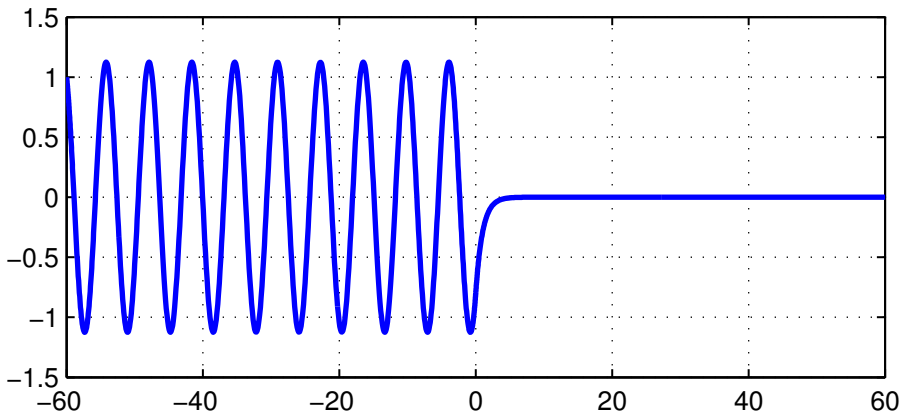
Thus it is possible to set up and solve a differential equation and plot the solution with a single line of Chebfun:

```
plot( chebop(@(x,u) diff(u,2) + 50*(1 + sin(x)).*u, [-20,20], 0, 0) \ 1 )
```



When Chebfun solves differential or integral equations, the coefficients may be piecewise smooth rather than globally smooth. For example, here is a 2nd order problem involving a coefficient that jumps from $+1$ (oscillation) for $x < 0$ to -1 (growth/decay) for $x > 0$:

```
L = chebop(-60, 60);
L.op = @(x,u) diff(u,2) - sign(x).*u;
L.lbc = 1; L.rbc = 0;
u = L\0;
plot(u, LW, 2), grid on
```



Further examples of Chebfun solutions of differential equations with discontinuous coefficients can be found in the Demos menu of chebgui.

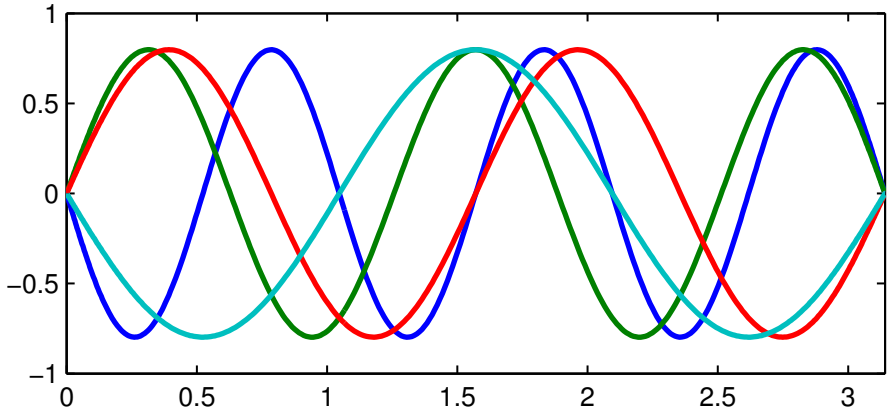
7.5 Eigenvalue problems: `eigs`

In MATLAB, `eig` finds all the eigenvalues of a matrix whereas `eigs` finds some of them. A differential or integral operator normally has infinitely many eigenvalues, so one could not expect an analog of `eig` for chebops. `eigs`, however, has been overloaded. Just like MATLAB `eigs`, Chebfun `eigs` finds six eigenvalues by default, together with eigenfunctions if requested. (For details see [Driscoll, Bornemann & Trefethen 2008].) Here is an example involving sine waves.

```
L = chebop( @(x,u) diff(u,2), [0, pi] );
L.bc = 0;
```

```
[V, D] = eigs(L);
diag(D)
clf, plot(V(:,1:4), LW, 2), ylim([-1 1])
```

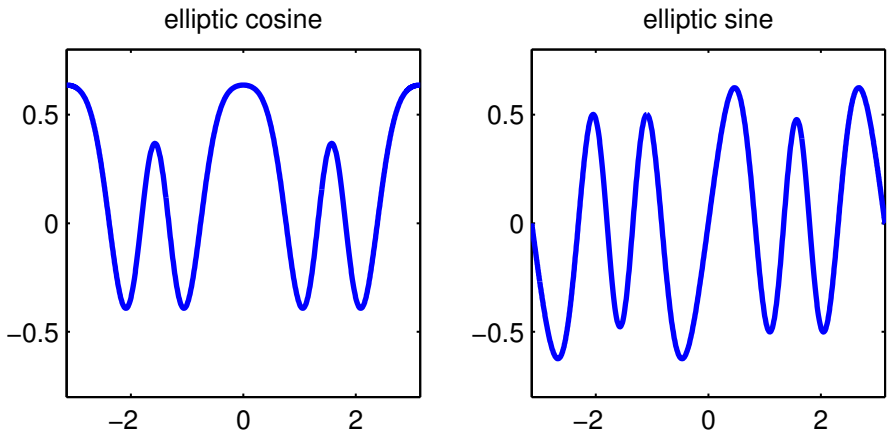
```
ans =
 -35.999999999996355
 -25.000000000000519
 -16.000000000003798
  -9.000000000001164
  -4.000000000000875
  -1.000000000000230
```



By default, `eigs` tries to find the six eigenvalues whose eigenmodes are "most readily converged to", which approximately means the smoothest ones. You can change the number sought and tell `eigs` where to look for them. Note, however, that you can easily confuse `eigs` if you ask for something unreasonable, like the largest eigenvalues of a differential operator.

Here we compute 10 eigenvalues of the Mathieu equation and plot the 9th and 10th corresponding eigenfunctions, known as an elliptic cosine and sine. Note the imposition of periodic boundary conditions.

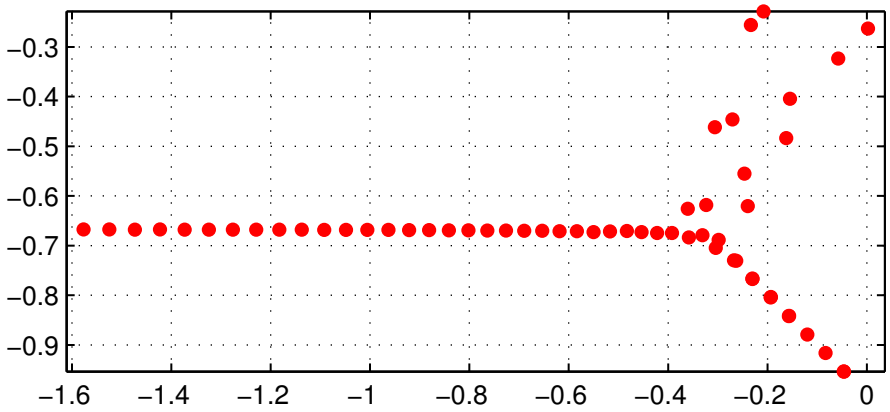
```
q = 10;
A = chebop(-pi, pi);
A.op = @(x,u) diff(u,2) - 2*q*cos(2*x).*u;
A.bc = 'periodic';
[V, D] = eigs(A, 16, 'LR');    % eigenvalues with largest real part
d = diag(D); [d, ii] = sort(d, 'descend'); V = V(:, ii');
subplot(1,2,1), plot(V(:, 9), LW, 2), ylim([-.8 .8]), title('elliptic cosine')
subplot(1,2,2), plot(V(:,10), LW, 2), ylim([-.8 .8]), title('elliptic sine')
```



`eigs` can also solve generalized eigenproblems, that is, problems of the form $Au = \lambda Bu$. For these one must specify two linear chebops `A` and `B`, with the boundary conditions all attached to `A`. Here is an example of eigenvalues of the Orr-Sommerfeld equation of hydrodynamic linear stability theory at a Reynolds number close to the critical value for eigenvalue instability [Schmid & Henningson 2001]. This is a fourth-order generalized eigenvalue problem, requiring two conditions at each boundary.

```
Re = 5772;
B = chebop(-1, 1);
B.op = @(x,u) diff(u,2) - u;
A = chebop(-1, 1);
A.op = @(x,u) (diff(u,4) - 2*diff(u, 2) + u)/Re - ...
    1i*(2*u + (1 - x.^2).*(diff(u, 2) - u));
A.lbc = @(u) [u; diff(u)];
A.rbc = @(u) [u; diff(u)];
lam = eigs(A, B, 60, 'LR');
MS = 'markersize';
clf, plot(lam, 'r.', MS, 16), grid on, axis equal
spectral_abscissa = max(real(lam))
```

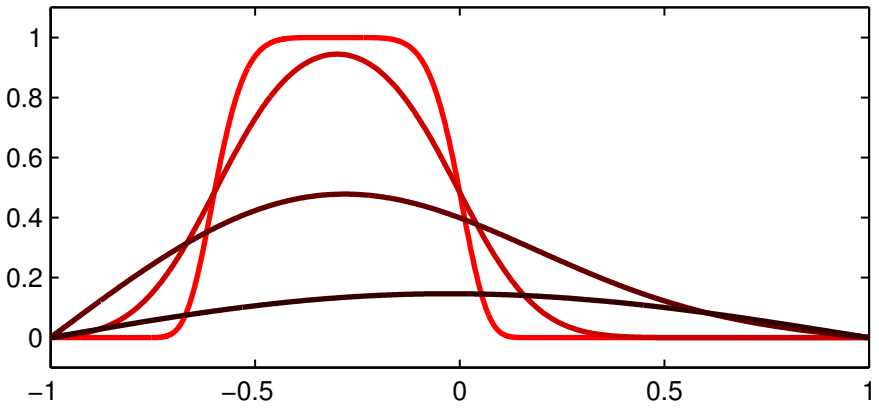
```
spectral_abscissa =
   0.002462425127391
```



7.6 Exponential of a linear operator: `expm`

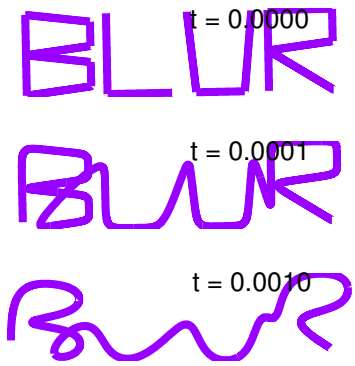
In MATLAB, `expm` computes the exponential of a matrix, and this command has been overloaded in Chebfun to compute the exponential of a linear operator. If L is a linear operator and $E(t) = \exp(tL)$, then the partial differential equation $u_t = Lu$ has solution $u(t) = E(t)u(0)$. Thus by taking L to be the 2nd derivative operator, for example, we can use `expm` to solve the heat equation $u_t = u_{xx}$:

```
A = chebop(@(x,u) diff(u,2), [-1, 1], 0);
f = chebfun('exp(-1000*(x+0.3).^6)');
clf, plot(f, 'r', LW, 2), hold on, c = [0.8 0 0];
for t = [0.01 0.1 0.5]
  u = expm(A, t, f);
  plot(u,'color', c, LW, 2), c = 0.5*c;
  ylim([-.1 1.1])
end
```



Here is a more fanciful analogous computation with a complex initial function obtained from the `scribble` command introduced in Chapter 5.

```
f = exp(.02i)*scribble('BLUR'); clf
D = chebop(@(x,u) diff(u,2), [-1 1]);
D.bc = 'neumann';
k = 0;
for t = [0 .0001 .001]
  k = k + 1; subplot(3,1,k)
  if t==0, u = f; else u = expm(D, t, f); end
  plot(u, 'linewidth', 3, 'color', [.6 0 1])
  xlim(1.05*[-1 1]), axis equal
  text(0.01, .46, sprintf('t = %6.4f', t), 'fontsize', 10), axis off
end
```



7.7 Algorithms: rectangular collocation vs. ultraspherical

Let us say a word about how Chebfun carries out these computations. Until Chebfun version 5, the methods involved were all Chebyshev spectral methods on automatically chosen grids. The general ideas are presented in [Trefethen 2000], [Driscoll, Bornemann & Trefethen 2008], and [Driscoll 2010], but Chebfun actually uses modifications of these methods described in [Driscoll & Hale 2014] involving a novel mix of Chebyshev grids of the first and second kinds. These **rectangular collocation** or **Driscoll-Hale** spectral discretizations start from the idea that a differential operator is discretized as a rectangular matrix that maps from one grid to another with fewer points. The matrix is then made square again by the incorporation of boundary conditions. When a differential equation is solved in Chebfun, the problem is solved on a sequence of grids of size $33, 65, \ldots$ until convergence is achieved in the usual Chebfun sense defined by decay of Chebyshev expansion coefficients.

One matter you might not guess was challenging is the determination of whether or not an operator is linear! This is important since if an operator is linear, special actions are possible possible such as application of `eigs` and `expm` and solution of differential equations in a single step without iteration. Chebfun includes special devices to determine whether a chebop is linear so that these effects can be realized [Birkisson 2014].

As mentioned, the discretization length of a Chebfun solution is chosen automatically according to the intrinsic resolution requirements. However, the matrices that arise in Chebyshev spectral methods are notoriously ill-conditioned. Thus the final accuracy in solving differential equations in Chebfun's default mode is rarely close to machine precision. Typically one loses two or three digits for second-order differential equations and five or six for fourth-order problems.

This problem of ill-conditioning was one of the motivations for the development of the other discretization method used by Chebfun, known as **ultraspherical** or **Olver-Townsend** spectral discretizations [Olver & Townsend 2013]. Here, rather than a single Chebyshev basis, several different bases of ultraspherical polynomials are used, depending on the order of the differential operator. This leads to better conditioned matrices that are also sparser, especially for linear problems with constant or smooth coefficients. By default, Chebfun uses rectangular collocation discretizations, but most problems can equally be solved in ultraspherical mode, and the results may be more accurate. For example, here we solve a problem whose exact solution is $\cos(x)$ in the rectangular fashion and check the error at $x = 5$:

```
tic
u = chebop(@(x, u) diff(u, 2) + u, [-10,10], cos(10), cos(10))\0;
toc
```

```
error = u(5) - cos(5)
```

```
Elapsed time is 0.099327 seconds.
error =
    -2.159383782895930e-14
```

We can switch to ultraspherical mode and run the same experiment again like this:

```
cheboppref.setDefaults('discretization', @ultraS)
tic
u = chebop(@(x,u) diff(u,2) + u, [-10,10], cos(10), cos(10))\0;
toc
error = u(5) - cos(5)
cheboppref.setDefaults('factory')   % reset to standard mode
```

```
Elapsed time is 0.227108 seconds.
error =
    -8.326672684688674e-16
```

7.8 Block operators and systems of equations

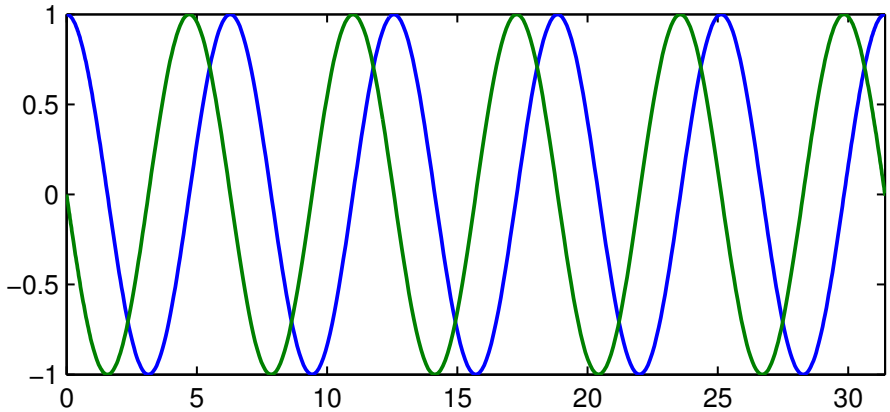
Some problems involve several variables coupled together. In Chebfun, these are treated with the use of quasimatrices, that is, chebfuns with several columns, as described in Chapter 6.

For example, suppose we want to solve the coupled system $u' = v$, $v' = -u$ with initial data $u = 1$ and $v = 0$ on the interval $[0, 10\pi]$. (This comes from writing the equation $u'' = -u$ in first-order form, with $v = u'$.) We can solve the problem like this:

```
L = chebop(0, 10*pi);
L.op = @(x, u, v) [diff(u) - v; diff(v) + u];
L.lbc = @(u, v) [u-1; v];
rhs = [0; 0];
U = L\rhs;
```

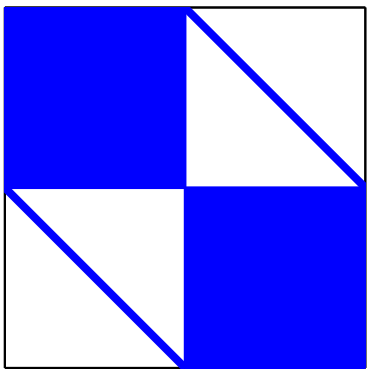
The solution `U` is an $\infty \times 2$ Chebfun quasimatrix with columns `u=U(:,1)` and `v=U(:,2)`. Here is a plot:

```
clf, plot(U)
```



The overloaded `spy` command helps clarify the structure of this operator we just made use of:

```
spy(L)
```



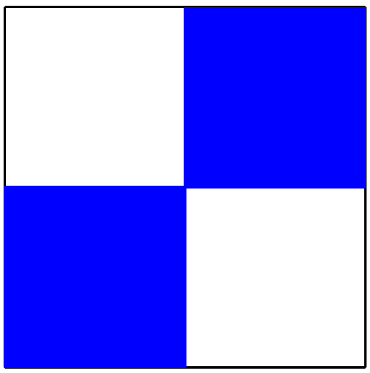
This image shows that L maps a pair of functions $[u; v]$ to a pair of functions $[w; y]$, where the dependences of w on u and y on v are global (because of the derivative) whereas the dependences of w on v and y on u are local (diagonal).

To illustrate the solution of an eigenvalue problem involving a block operator, we can take much the same idea. The eigenvalue problem $u'' = c^2 u$ with $u = 0$ at the boundaries can be written in first order form as $u' = cv$, $v' = cu$.

```
L = chebop(0, 10*pi);
L.op = @(x, u, v) [diff(v); diff(u)];
L.lbc = @(u,v) u;
L.rbc = @(u,v) u;
```

The operator in this eigenvalue problem has a simpler structure than before:

```
clf, spy(L)
```



Here are the first 7 eigenvalues:

```
[eigenfunctions,D] = eigs(L, 7);
eigenvalues = diag(D)
```

```
eigenvalues =
 -0.000000000000001 + 0.000000000000000i
 -0.000000000000001 - 0.100000000000001i
 -0.000000000000001 + 0.100000000000001i
 -0.000000000000000 - 0.200000000000000i
 -0.000000000000000 + 0.200000000000000i
 -0.000000000000002 - 0.300000000000000i
 -0.000000000000002 + 0.300000000000000i
```

The `eigenfunctions` result has the first seven eigenfunctions for each of the two variables, u and v:

```
eigenfunctions
```

```
eigenfunctions =
   2 x 7 chebmatrix of block types:
    '<a href="matlab: disp(eigenfunctions{1,1})">chebfun</a>'    '<a href="matlab: disp(eigenfunctio
    '<a href="matlab: disp(eigenfunctions{2,1})">chebfun</a>'    '<a href="matlab: disp(eigenfunctio
```

It's often convenient to convert a chebmatrix result to a chebfun. In this case, we want to extract the u and v variables separately:

```
U = chebfun( eigenfunctions(1,:) );
V = chebfun( eigenfunctions(2,:) );
size(V)
```

```
ans =
   Inf     7
```

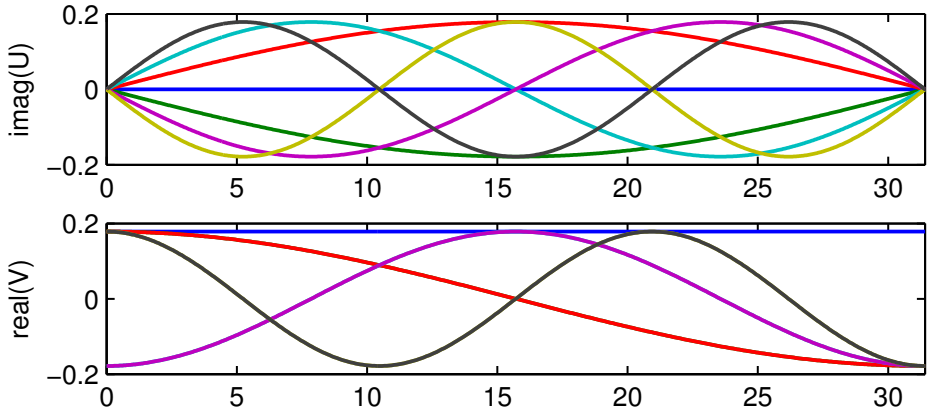
Both `U` and `V` are complex, but only due to roundoff:

```
normRealU = norm(real(U))
normImagV = norm(imag(V))
```

```
normRealU =
     3.860309020791207e-11
normImagV =
     3.860165926729633e-11
```

This fact makes it easy to plot them.

```
subplot(2,1,1)
plot(imag(U)), ylabel('imag(U)')
subplot(2,1,2)
plot(real(V)), ylabel('real(V)')
```



7.9 Nonlinear equations by Newton iteration

As mentioned at the beginning of this chapter, nonlinear differential equations are discussed in Chapter 10. As an indication of some of the possibilities, however, we now illustrate how a sequence of linear problems may be useful in solving nonlinear problems. For example, the nonlinear BVP

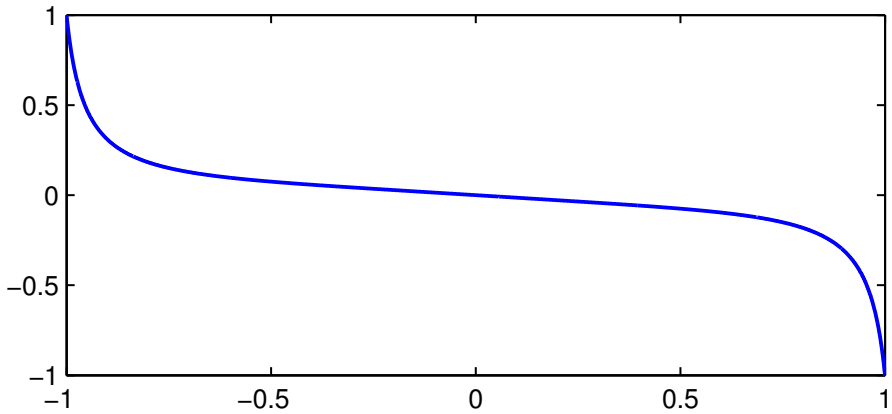
$$0.001u'' - u^3 = 0, \qquad u(-1) = 1, \quad u(1) = -1$$

could be solved by Newton iteration as follows.

```
L = chebop(-1, 1);
L.op = @(x,u) 0.001*diff(u, 2);
J = chebop(-1, 1);
x = chebfun('x');
u = -x;  nrmdu = Inf;
while nrmdu > 1e-10
  r = L*u - u.^3;
  J.op = @(du) .001*diff(du, 2) - 3*u.^2.*du;
  J.bc = 0;
  du = -(J\r);
  u = u + du;  nrmdu = norm(du)
end
clf, plot(u)
```

```
nrmdu =
   0.260668532007021
nrmdu =
   0.164126069559937
nrmdu =
   0.098900892365439
nrmdu =
   0.053787171683932
nrmdu =
   0.021518152858429
```

```
nrmdu =
   0.003586696693250
nrmdu =
     8.951602488918563e-05
nrmdu =
     5.357404593788955e-08
nrmdu =
     2.063725137129868e-14
```



Note the beautifully fast convergence, as one expects with Newton's method. The chebop `J` defined in the ***while*** loop is a Jacobian operator (=Frechet derivative), which we have constructed explicitly by differentiating the nonlinear operator defining the ODE. In Section 10.4 we shall see that this whole Newton iteration can be automated by use of Chebfun's "nonlinear backslash" capability, which utilizes automatic differentiation to construct the Frechet derivative automatically. In fact, all you need to type is

```
N = chebop(-1, 1);
N.op = @(x,u) 0.001*diff(u, 2) - u.^3;
N.lbc = 1; N.rbc = -1;
v = N\0;
```

The result is the same as before to many digits of accuracy:

```
norm(u - v)
```

```
ans =
     3.141073177047297e-13
```

7.10 BVP systems with unknown parameters

Sometimes ODEs or systems of ODEs contain unknown parameter values that must be computed for as part of the solution. An example of this is MATLAB's built-in `mat4bvp` example. These parameters can always be included in system as unknowns with zero derivatives, but this can be computationally inefficient. Chebfun allows the option of explicit treatment of the parameters. Often the dependence of the solution on these parameters is nonlinear (as in the case below), and this discussion might better have been left to Chapter 10, but since, from the user perspective, there is little difference in this case, we include it here.

Below is an example of such a parameterised problem, which is represents a linear pendulum with a forcing sine-wave term of an unknown frequency T. The task is to compute the solution for which

$$u(-pi) = u(pi) = u'(pi) = 1$$

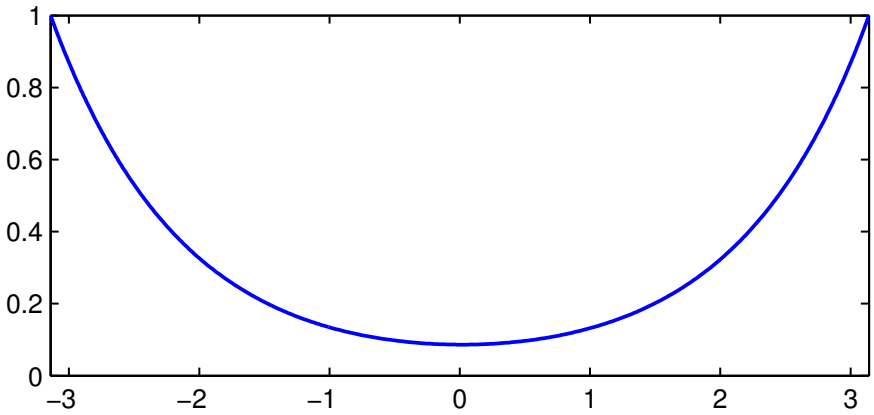
.

```
N = chebop(@(x, u , T) diff(u,2) - u - sin(T.*x/pi), [-pi pi]);
N.lbc = @(u,T) u - 1;
N.rbc = @(u,T) [u - 1; diff(u) - 1];
uT = N\0;
```

Here, the output `uT` is a chebmatrix – an object that is amongst other features able to vertically concatenate chebfuns and scalar. The first entry corresponds to the chebfun `u` while the second is the scalar `T`. We can access the elements of `uT` via curly-brackets syntax as follows:

```
u = uT{1}; T = uT{2}
plot(u)
```

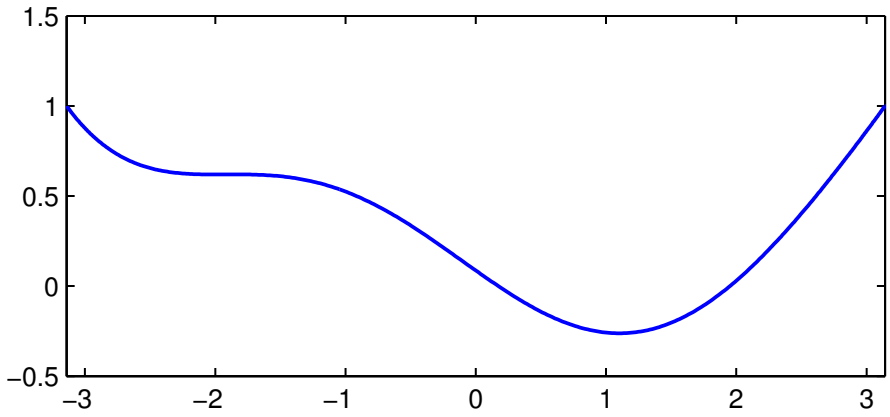
```
T =
   0.005438812795289
```



As the system is nonlinear in T, we can expect that there will be more than one solution. Indeed, if we choose a different initial guess for T, we can converge to one of these.

```
N.init = [chebfun(1, [-pi pi]); 4];
uT = N\0;
u = uT{1}; T = uT{2}
plot(u)
```

```
T =
   4.044049959218974
```



7.11 References

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8. Chebfun Preferences

Lloyd N. Trefethen, November 2009, last revised June 2014

Contents

8.1 Introduction

Like any software package, Chebfun is based on certain design decisions. Some of these can be adjusted by the user, like the maximum number of points at which a function will be sampled before Chebfun gives up trying to resolve it. Extensive information about these possibilities can be found by executing `help chebfunpref`, or for chebops, as used to solve integral and differential equations, `help cheboppref`. To see the list of preferences and their current values, execute `chebfunpref` or `cheboppref`:

```
chebfunpref
```

```
chebfunpref object with the following preferences:
    domain:                     [-1, 1]
    splitting:                  0
    splitPrefs
        splitLength:            160
        splitMaxLength:         6000
    blowup:                     0
    blowupPrefs
        exponentTol:            1.100000e-11
        maxPoleOrder:           20
        defaultSingType:        'sing'
```

```
    enableDeltaFunctions:         1
    deltaPrefs
        deltaTol:                 1.000000e-09
        proximityTol:             1.000000e-11
    cheb2Prefs
        maxRank:                  513
        sampleTest:               1
    tech:                         @chebtech2
    <a href="matlab: help chebtech2/techPref">techPrefs</a>
        eps:                      2.220446049250313e-16
        minSamples:               17
        maxLength:                65537
        fixedLength:              NaN
        extrapolate:              0
        sampleTest:               1
        refinementFunction:       'nested'
        happinessCheck:           'classic'
```

More detailed information from further down in the preference structure will come from, for example, `help chebtech/techpref`.

To ensure that all preferences are set to their factory values, execute

```
chebfunpref.setDefaults('factory')
```

In this chapter we explore some of these adjustable preferences, showing how special effects can be achieved by modifying them. Besides showing off some useful techniques, this review will also serve to deepen the user's understanding of Chebfun by poking about a bit at its edges.

A general point to be emphasized is the distinction between creating a chebfun directly from the constructor and creating one by operating on previous chebfuns. In the former case we can include preferences directly in the constructor command, and we recommend this as good practice:

```
f = chebfun('x.^x',[0,1],'splitting','on');
```

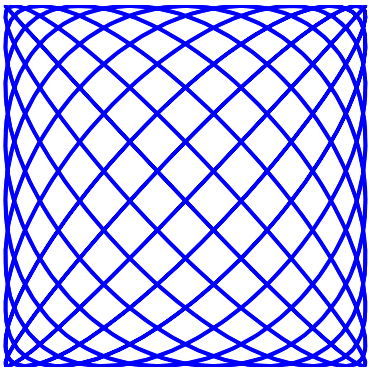
In the latter case, however, one can turn the preference on and off again.

```
x = chebfun('x',[0,1]);
chebfunpref.setDefaults('splitting',true)
f = x.^x;
chebfunpref.setDefaults('splitting',false)
```

8.2 `domain`: the default domain

Like Chebyshev polynomials themselves, chebfuns are defined by default on the domain $[-1,1]$ if no other domain is specified. However, this default choice of the default domain can be modified. For example, we can work with trigonometric functions on $[0,2\pi]$ conveniently like this:

```
chebfunpref.setDefaults('domain',[0 2*pi])
f = chebfun(@(t) sin(19*t));
g = chebfun(@(t) cos(20*t));
plot(f,g), axis equal, axis off
```



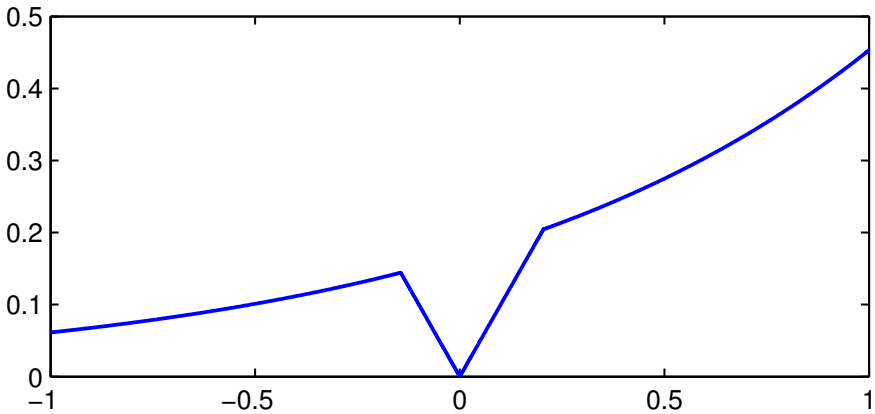
8.3 splitting: breaking into subintervals or not

Perhaps the preference that users wish to control most often is the choice of splitting off or on. Splitting off is the factory default.

In both splitting off and splitting on modes, a chebfun may consist of a number of pieces, called funs. For example, even in splitting off mode, the following sequence makes a chebfun with four funs:

```
chebfunpref.setDefaults('factory');
x = chebfun(@(x) x);
f = min(abs(x),exp(x)/6);
format short, f.ends
plot(f)
```

```
ans =
   -1.0000   -0.1443         0    0.2045    1.0000
```



One breakpoint is introduced at $x = 0$, where the constructor determines that $|x|$ has a zero, and two more breakpoints are introduced at -0.1443 and at 0.2045, where it recognizes that $|x|$ and $\exp(x)/6$ will intersect.

The difference between splitting off and splitting on pertains to additional breakpoints that may be introduced in the more basic chebfun construction process, when the constructor makes a chebfun solely by sampling point values. For example, suppose we try to make the same chebfun as above

from scratch, by sampling an anonymous function, in splitting off mode. We get a warning message:

```
ff = @(x) min(abs(x),exp(x)/6);
f = chebfun(ff);
```

```
Warning: Function not resolved using 65537 pts. Have you tried 'splitting on'?
```

With splitting on, Chebfun's built-in edge detector quickly finds the singular points and introduces breakpoints there:

```
f = chebfun(ff,'splitting','on');
f.ends
```

```
ans =
   -1.0000   -0.1443    0.0000    0.2045    1.0000
```

This example involves specific points of singularity, which the constructor has duly located. In addition to this, in splitting on mode the constructor will subdivide intervals recursively at non-singular points when convergence is not taking place fast enough. For example, with splitting off we cannot successfully construct a chebfun for the square root function on $[0, 1]$ (unless we use singular exponents as described in the next chapter):

```
f = chebfun(@(x) sqrt(x),[0 1]);
```

```
Warning: Function not resolved using 65537 pts. Have you tried 'splitting on'?
```

With splitting on, however, all is well:

```
f = chebfun(@(x) sqrt(x),[0 1],'splitting','on');
length(f)
format long, f((.1:.1:.5)'.^2)
```

```
ans =
   515
ans =
   0.100000000057989
   0.200000000011776
   0.300000000012712
   0.400000000008074
   0.500000000010248
```

Inspection reveals that Chebfun has broken the interval into a succession of pieces, each 100 times smaller than the next:

```
f.ends
```

```
ans =
  Columns 1 through 3
                  0   0.000000000100000   0.000000010000000
  Columns 4 through 6
   0.000001000000000   0.000100000000000   0.010000000000000
  Column 7
   1.000000000000000
```

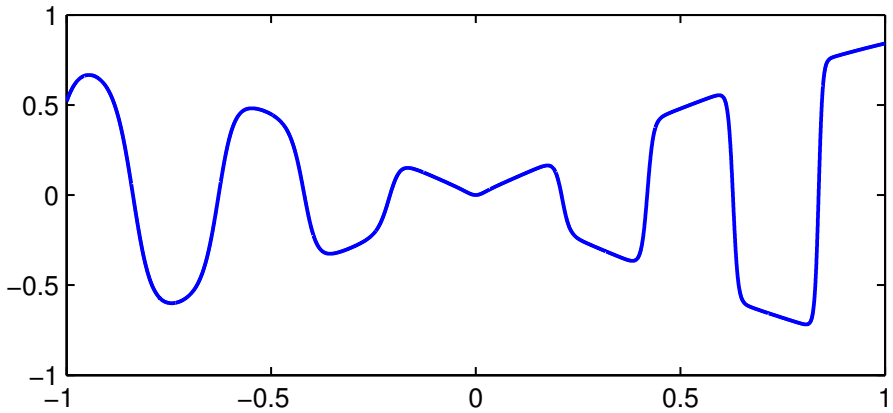
In this example the subdivisions have occurred near an endpoint, for the edge detector has determined that the difficulty of resolution lies there. For other functions, however, splitting will take place at midpoints. For example, here is a function that is complicated throughout $[-1, 1]$, especially for larger values of x.

```
ff = @(x) sin(x).*tanh(3*exp(x).*sin(15*x));
```

With splitting off, it gets resolved by a global polynomial of rather high degree.

```
f = chebfun(ff);
length(f)
plot(f)
```

```
ans =
        1346
```



With splitting on, the function is broken up into pieces, and there is some reduction in the overall length:

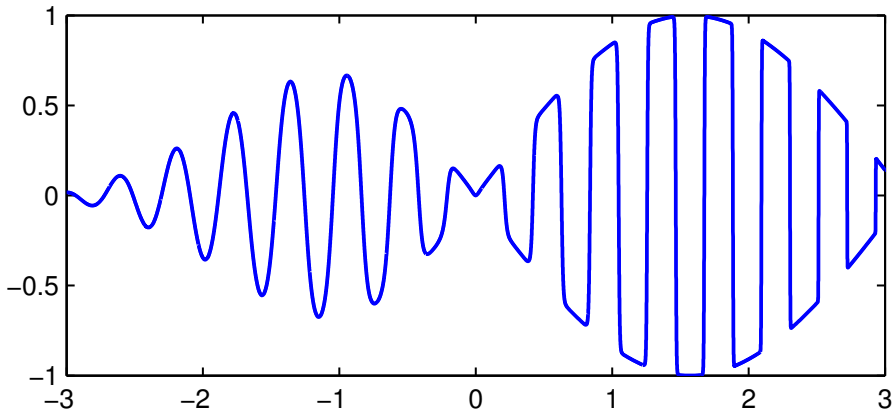
```
f = chebfun(ff,'splitting','on');
length(f)
format short, f.ends
```

```
ans =
   825
ans =
  Columns 1 through 7
   -1.0000   -0.7500   -0.5000   -0.2500         0    0.2500    0.3750
  Columns 8 through 12
    0.5000    0.6250    0.8125    0.8750    1.0000
```

When should one use splitting off, and when splitting on? If the goal is simply to represent complicated functions, especially when they are more complicated in some regions than others, splitting on sometimes has advantages. An example is given by the function above posed on $[-3, 3]$ instead of $[-1, 1]$. With splitting off, the global polynomial has a degree in the tens of thousands:

```
f3 = chebfun(ff,[-3 3]);
length(f3)
plot(f3)
```

```
ans =
       16009
```



With splitting on the representation is much more compact:

```
f3 = chebfun(ff,[-3 3],'splitting','on');
length(f3)
```

```
ans =
        2828
```

On the other hand, splitting off mode has advantages of robustness. In particular, operations involving derivatives generally work better when functions are represented by global polynomials, and chebops for the most part require this. Also, for educational purposes, it is very convenient that Chebfun can be used so easily to study the properties of pure polynomial representations even of very high degree.

8.4 `splitLength`: length limit in splitting on mode

When intervals are subdivided in splitting on mode, as just illustrated, the parameter `splitLength` determines where this will happen. With the factory value `splitLength=160`, splitting will take place if a polynomial of length 160 proves insufficient to resolve a fun. (Actually, when Chebfun uses Chebyshev points of the second kind as it does by default, this number is rounded down to 1 more than a power of 2.) Let us confirm for the chebfun `f` constructed a moment ago that the lengths of the individual funs are all less than or equal to 160 (actually 129):

```
f.funs
```

```
ans =
  Columns 1 through 4
    [66x1 bndfun]    [78x1 bndfun]    [90x1 bndfun]    [90x1 bndfun]
  Columns 5 through 8
    [124x1 bndfun]    [31x1 bndfun]    [96x1 bndfun]    [62x1 bndfun]
  Columns 9 through 11
    [79x1 bndfun]    [74x1 bndfun]    [35x1 bndfun]
```

Alternatively, suppose we wish to allow individual funs to have length up to 513. We can do that like this:

```
f = chebfun(ff,'splitting','on','splitLength',513);
length(f)
format short, f.ends
f.funs
```

```
ans =
        1183
ans =
   -1.0000         0    0.5000    0.7500    1.0000
ans =
    [353x1 bndfun]    [307x1 bndfun]    [246x1 bndfun]    [277x1 bndfun]
```

8.5 maxLength: maximum length

As just mentioned, in splitting off mode, the constructor tries to make a global chebfun from the given string or anonymous function. For a function like $|x|$ or sign(x), this will typically not be possible and we must give up somewhere. The parameter `maxLength`, set to $2^{16} + 1$ in the factory, determines this giving-up point.

For example, here's what happens normally if we try to make a chebfun for sign(x).

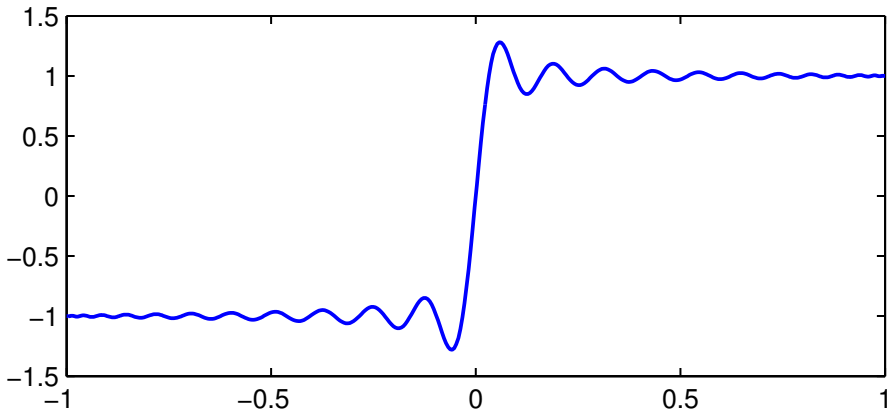
```
f = chebfun('sign(x)');
```

```
Warning: Function not resolved using 65537 pts. Have you tried 'splitting on'?
```

Suppose we wish to examine the interpolant to this function through 50 points instead of 65537. One way is like this:

```
f = chebfun('sign(x)',50);
length(f)
plot(f)
```

```
ans =
    50
```



Notice that no warning message is produced since we have asked explicitly for exactly 50 points. On the other hand we could also change the default maximum to this number (or more precisely the default degree to one less than this number), giving the same effect though now with another warning message:

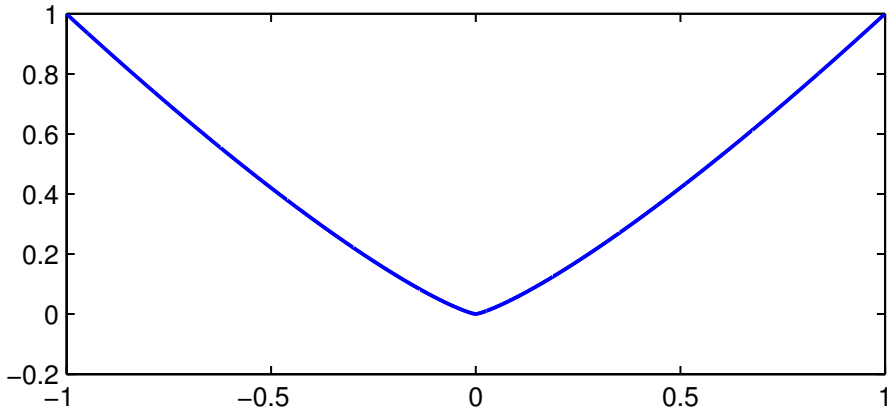
```
f = chebfun('sign(x)','maxLength',50);
length(f)
```

```
Warning: Function not resolved using 50 pts. Have you tried 'splitting on'?
ans =
    33
```

Perhaps more often one might wish to adjust this preference to enable use of especially high degrees. On the machines of 2014, Chebfun is perfectly capable of working with polynomials of degrees in the millions. The function $|x|^{5/4}$ on $[-1,1]$ provides an example, for it is smooth enough to be resolved by a global polynomial, provided it is of rather high degree:

```
tic
f = chebfun('abs(x).^1.25','maxLength',1e6);
lengthf = length(f)
format long, sumf = sum(f)
plot(f)
toc
```

```
lengthf =
      105185
sumf =
   0.888888888881000
Elapsed time is 0.796616 seconds.
```



(More efficient ways of resolving this function, by eliminating the singularity, are described in Chapter 9.)

8.6 `minSamples`: minimum number of sample points

At the other end of the spectrum, the preference `minSamples` determines the minimum number of points at which a function is sampled during the chebfun construction process, and the factory value of this parameter is 17. This does not mean that all chebfuns have length at least 17. For example, if f is a cubic, then it will be sampled at 17 points, Chebyshev expansion coefficients will be computed, and 13 of these will be found to be of negligible size and discarded. So the resulting chebfun is a cubic, even though the constructor never sampled at fewer than 17 points.

```
f = chebfun('x.^3');
lengthf = length(f)
```

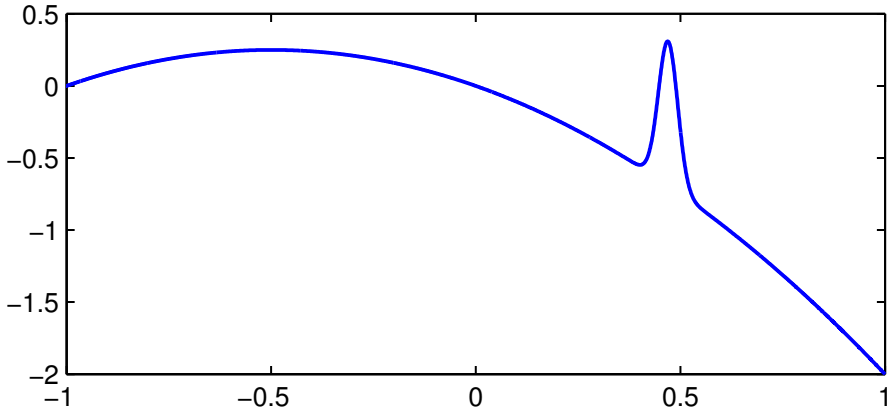
```
lengthf =
     4
```

More generally a function is sampled at $17, 33, 65, \ldots$ points until a set of Chebyshev coefficients are obtained with a tail judged to be negligible.

Like any process based on sampling, this one can fail. For example, here is a success:

```
f = chebfun('-x -x.^2 + exp(-(30*(x-.47)).^2)');
length(f)
plot(f)
```

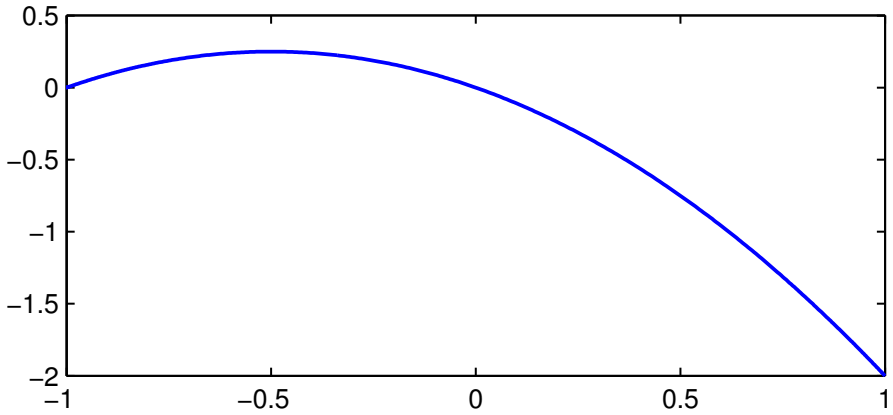
```
ans =
   309
```



But if we change the exponent to 4, we get a failure:

```
f = chebfun('-x -x.^2 + exp(-(30*(x-.47)).^4)');
length(f)
plot(f)
```

```
ans =
     3
```

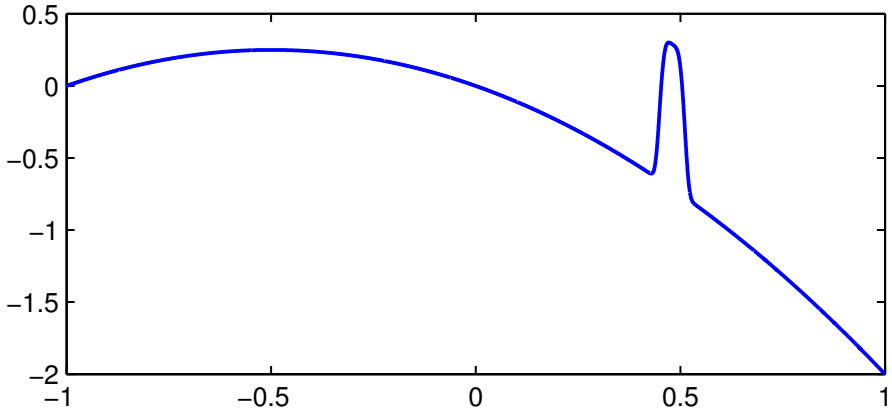


What has happened can be explained as follows. The function being sampled has a narrow spike near $x = 0.47$, and the closest grid points lie near 0.383 and 0.556. In the case of the exponent 2, we note that at $x = 0.383$ and $x = 0.556$, $\exp(-(30(x - .47)^2))$ takes values of about 0.001, which are easily large enough to be noticed by the Chebfun constructor. On the other hand in the case of exponent 4, the values at these points shrink to less than 10^{-19}, which is below machine precision. So in the latter case the constructor thinks it has a quadratic and does not try a finer grid.

If we increase `minSamples`, the correct chebfun is found:

```
f = chebfun('-x -x.^2 + exp(-(30*(x-.48)).^4)','minSamples',33);
length(f)
plot(f)
```

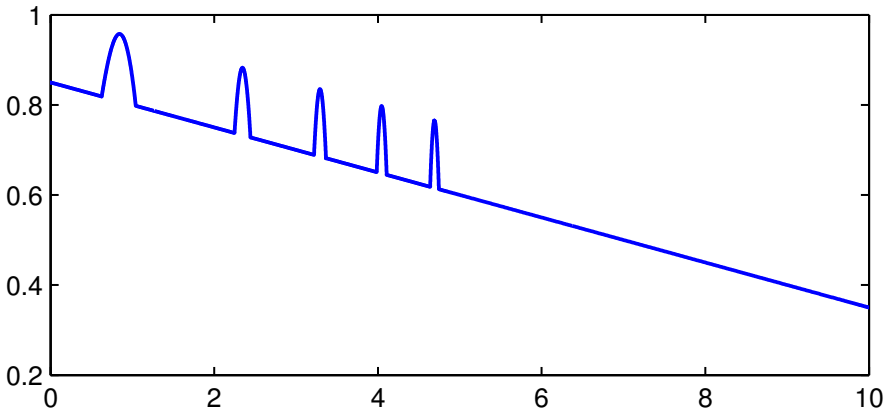
```
ans =
        1021
```



Incidentally, if the value of `minSamples` specified is not one greater than a power of 2, it is rounded up to the next such value.

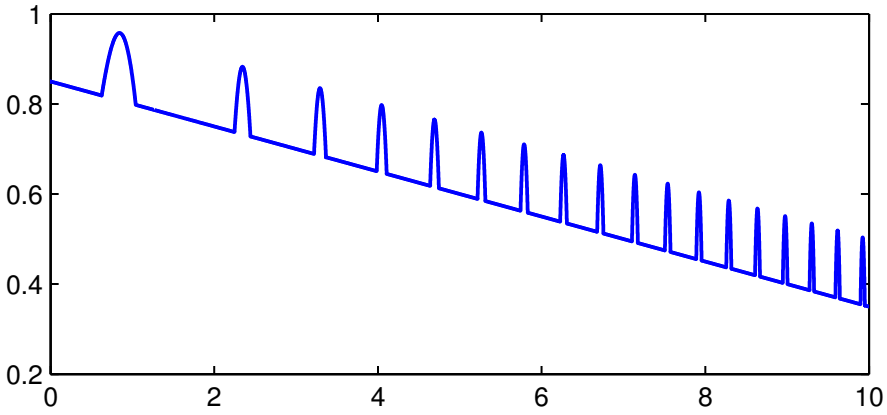
The factory value `minSamples=17` was chosen as a compromise between efficiency and reliability. (Until Version 5, the choice was |minSamples=9'.) In practice it rarely seems to fail, but perhaps it is most vulnerable when applied in splitting on mode to functions with narrow spikes. For example, the following chebfun is missing most of the spikes that should be there:

```
ff = @(x) max(.85,sin(x+x.^2)) - x/20;
f = chebfun(ff,[0,10],'splitting','on');
plot(f)
```



Increasing `minSamples` fills them in:

```
f = chebfun(ff,[0,10],'splitting','on','minsamples',33);
plot(f)
```



8.7 `resampling`: exploiting nested grids or not

We now turn to a particularly interesting preference for Chebfun geeks, relating to the very idea of what it means to sample a function.

When a chebfun is constructed, a function is normally sampled at $17, 33, 65, \ldots$ Chebyshev points until convergence is achieved. (We are speaking here of the process for Chebyshev points of the second kind; for first-kind points the details are different.) Now Chebyshev grids are nested, so the 33-point grid, for example, only contains 16 points that are not in the 17-point grid. By default, the Chebfun constructor takes advantage of this property so as not to recompute values that have already been computed. (The default went the other way until 2009.)

For example, here is a chebfun constructed in the usual factory mode:

```
ff = @(x) besselj(x,exp(x))
tic, f = chebfun(ff,[0 8]); toc
length(f)
```

```
ff =
    @(x)besselj(x,exp(x))
Elapsed time is 0.038966 seconds.
ans =
        3777
```

There is little difference, even in the timing, if we set 'resampling on', so that previously computed values are not reused:

```
tic, f = chebfun(ff,[0 8],'resampling','on'); toc
length(f)
```

```
Elapsed time is 0.049952 seconds.
ans =
        3782
```

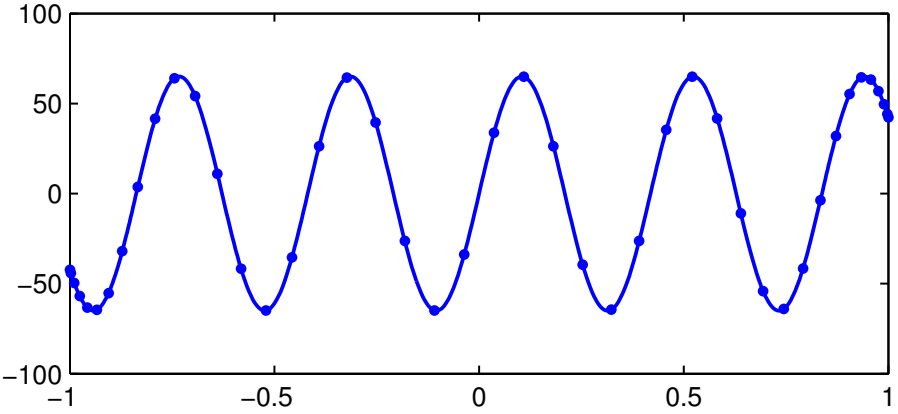
One might wonder why 'resampling on' is an option at all, but in fact, it introduces some very interesting possibilities. What if the "function" being sampled is not actually a fixed function, but depends on the grid? For example, consider this prescription:

```
ff = @(x) length(x)*sin(15*x);
```

The values of f at any particular point will depend on the length of the vector in which it is embedded! What will happen if we try to make a chebfun, disabling the "sampleTest" feature that is usually applied by the constructor as a safety test? The constructor tries the 17-point Chebyshev grid, then the 33-point grid, then the 65-point grid. On the last of these it finds the Chebyshev coefficients are sufficiently small, and proceeds to truncate to length 44. We end up with a chebfun of length 44 that precisely matches the function $65\sin(15x)$.

```
f = chebfun(ff,'sampleTest',0,'resampling','on');
length(f)
max(f)
plot(f,'.-')
```

```
ans =
    44
ans =
    65
```



This rather bizarre example encourages us to play further. What if we change length(x)*sin(15*x) to sin(length(x)*x)? Now there is no convergence, for no matter how fine the grid is, the function is underresolved.

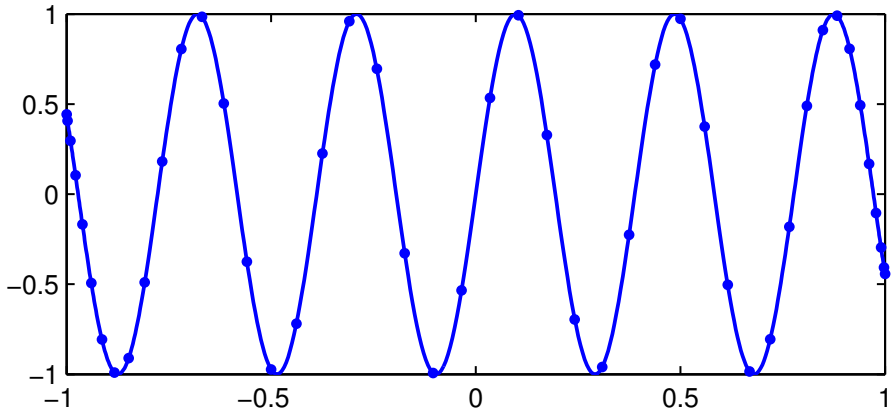
```
hh = @(x) sin(length(x)*x);
h = chebfun(hh,'sampleTest',0,'resampling','on');
```

```
Warning: Function not resolved using 65537 pts. Have you tried 'splitting on'?
```

Here is an in-between case where convergence is achieved on the grid of length 65, and the resulting chebfun then trimmed to length 46.

```
kk = @(x) sin(length(x).^(2/3)*x);
k = chebfun(kk,'sampleTest',0,'resampling','on');
length(k)
plot(k,'.-')
```

```
ans =
    46
```



Are such curious effects of any use? Yes indeed, they are at the heart of Chebop. When the chebop system solves a differential equation by a command like `u = L\f`, for example, the chebfun `u` is determined by a "sampling" process in which a matrix problem obtained by Chebyshev spectral discretization is solved on grids of increasing sizes. The matrices change with the grids, so the sample values for u are crucially grid-dependent. Without resampling, chebops would not work.

8.8 `eps`: Chebfun constructor tolerance

One of the controllable preferences is all too tempting: you can weaken the tolerance used in constructing a chebfun. The chebfunpref parameter `eps` is set by default to machine precision:

```
p = chebfunpref;
p.eps
```

```
ans =
     2.220446049250313e-16
```

However, one can change this with a command like `chebfunpref.setDefaults('eps',1e-6)`.

There are cases where weakening the tolerance makes a big difference. For example, this happens in certain applications in 2D and in certain applications involving differential equations. (Indeed, the Chebfun differential equations commands have their own tolerance control strategies.) However, Chebfun does such a good job at resolving many functions that the `eps`-adjustment feature is not as useful as you might imagine, and we recommend that users not change `eps` unless they are having real problems with standard precision.

8.9 Chebyshev grids of first or second kind

Beginning with Version 5, Chebfun includes capabilities for carrying out almost all computations with Chebyshev points of either the first kind ($\cos((j+1/2)\pi/(n+1))$, $0 \le j \le n$, implemented in the `chebtech1` class) or the second kind ($\cos(j\pi/n)$, $0 \le j \le n$, implemented in the `chebtech2` class). These capabilities were included to further our research into the pros and cons of different

kinds of algorithms, and most users can ignore this choice entirely. You can query which kind of Chebyshev points is in use with

```
t = chebkind
```

```
t =
     2
```

and you can set it with, for example,

```
chebkind(1)
```

or

```
chebkind 1
```

An equivalent would be the command

```
chebfunpref.setDefaults('tech',@chebtech1)
```

Let us return to factory settings:

```
chebfunpref.setDefaults('factory')
```

8.10 Rectangular or ultraspherical spectral discretizations

Chebfun's factory default for spectral discretizations is rectangular collocation in Chebyshev points of the second kind, which corresponds to

```
chebioppref.setDefaults('discretization','colloc2')
```

To change the preference to first-kind points, one can execute

```
chebioppref.setDefaults('discretization','colloc1')
```

and for Olver-Townsend ultraspherical discretizations,

```
chebioppref.setDefaults('discretization','ultraspherical')
```

Let us undo these changes:

```
chebioppref.setDefaults('factory')
```

8.11 Chebfun2 preferences

The Chebfun2 preference that users may be most interested in is `MaxRank`, which determines the maximum rank of a low-rank approximation used to represent a function on a rectangle. The current factory default is 512, and this can be changed for example with

```
chebfunpref.setDefaults({'cheb2Prefs','maxRank'},1024);
```

Let us undo this change:

```
chebfunpref.setDefaults('factory')
```

8.12 Additional preferences

Information about additional Chebfun preferences can be found by executing `chebfunpref` or `help chebfunpref`. In general the most reliable values to use in setting preferences are are `1` or `true' and |0` or `false` (not `'on'` and `'off'`).

For example, `'sampleTest'` controls whether a function is evaluated at an extra point as a safety check of convergence. With the default `'on'` value, this test is indeed carried out.

Another example is that `'blowup'` relates to the construction of chebfuns that diverge to infinity, as described in Chapter 9. `blowup=0` is used for no singularities, `blowup=1` if for functions with poles (blowups with a negative integer power) and `blowup=2` for functions with branch points (blowups with an arbitrary power).

9. Infinite Intervals, Infinite Function Values, and Singularities

Lloyd N. Trefethen, November 2009, latest revision June 2014

Contents

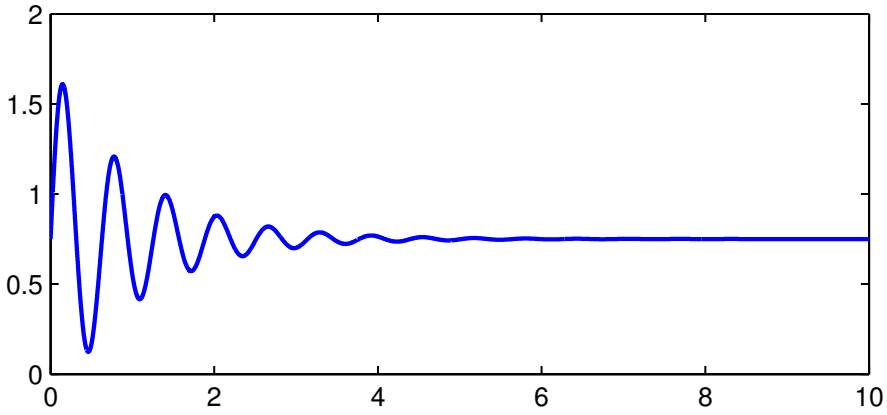
This chapter presents some features of Chebfun that are less robust than what is described in the first eight chapters. With classic bounded chebfuns on a bounded interval $[a, b]$, you can do amazingly complicated things often without encountering any difficulties. Now we are going to let the intervals and the functions diverge to infinity – but please lower your expectations! These features are not always as accurate or reliable.

9.1 Infinite intervals

If a function converges reasonably rapidly to a constant at ∞, you can define a corresponding chebfun. Here are a couple of examples on $[0, \infty]$. First we plot a function and find its maximum:

```
f = chebfun('0.75 + sin(10*x)./exp(x)',[0 inf]);
LW = 'linewidth'; MS = 'markersize';
plot(f,LW,1.6)
maxf = max(f)
```

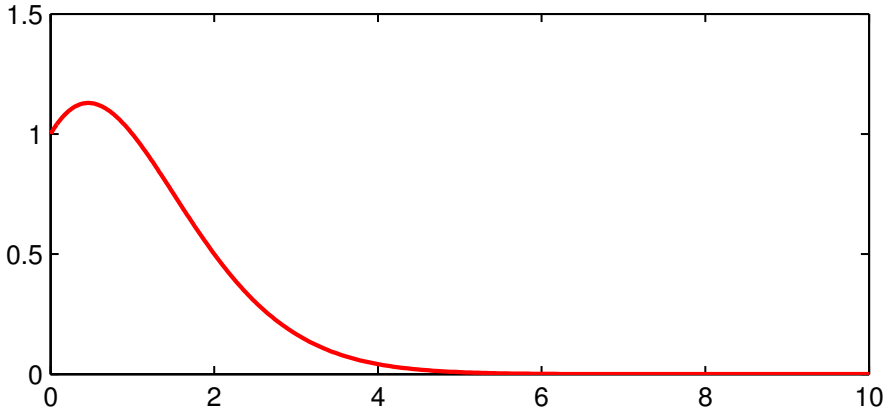
```
maxf =
   1.608912750768336
```



Next we plot another function and integrate it from 0 to ∞:

```
g = chebfun('1./(gamma(x+1))',[0 inf]);
sumg = sum(g)
plot(g,'r',LW,1.6)
```

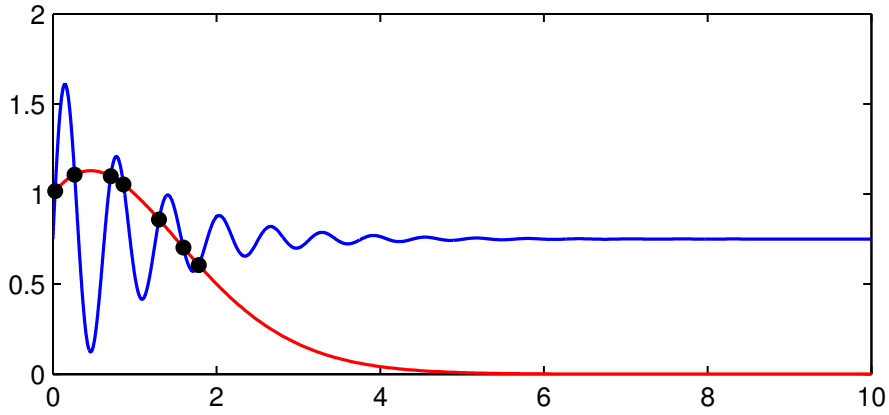
```
sumg =
   2.266534507699834
```



Where do f and g intersect? We can find out using `roots`:

```
plot(f,'b',g,'r',LW,1.2), hold on
r = roots(f-g)
plot(r,f(r),'.k',MS,18)
```

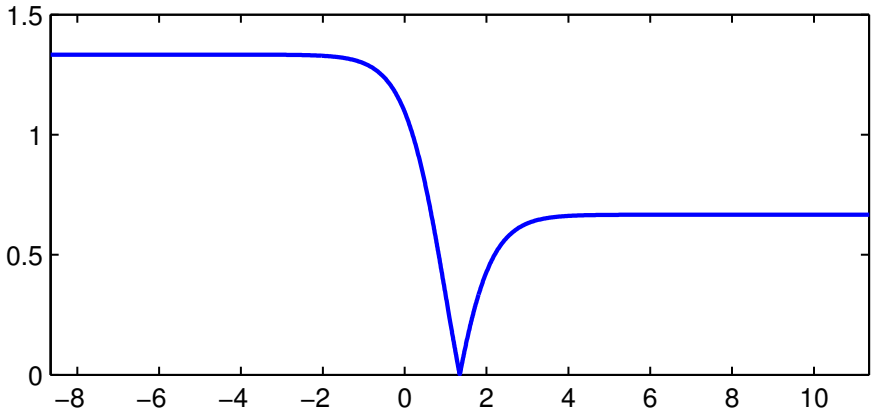
```
r =
   0.027639744894513
   0.265714132607450
   0.706922132176979
   0.862331877000826
   1.297442594652156
   1.594466987072374
   1.781855556974647
```



Here's an example on $(-\infty, \infty)$ with a calculation of the location and value of the minimum:

```
g = chebfun(@(x) tanh(x-1),[-inf inf]);
g = abs(g-1/3);
clf, plot(g,LW,1.6)
[minval,minpos] = min(g)
```

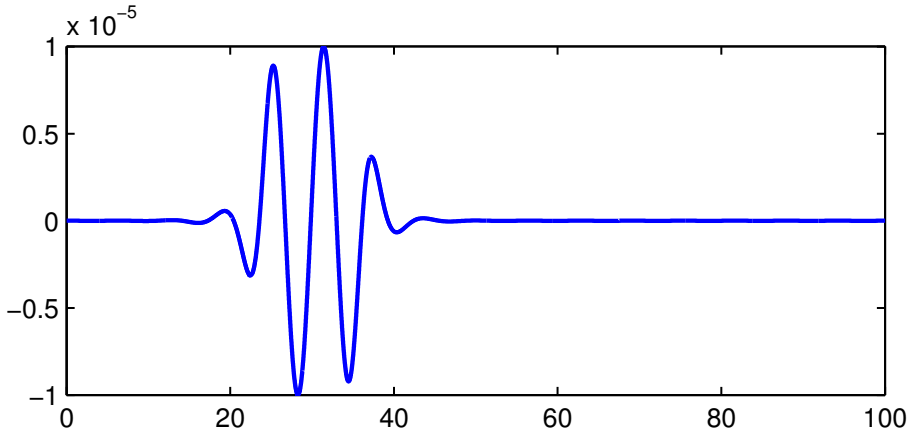
```
minval =
     0
minpos =
   1.346573590279974
```



Notice that a function on an infinite domain is by default plotted on an interval like $[0, 10]$ or $[-10, 10]$. You can use an extra `'interval'` flag to plot on other intervals, as shown by this example of a function of small norm whose largest values are near $x = 30$:

```
hh = @(x) cos(x)./(1e5+(x-30).^6);
h = chebfun(hh,[0 inf]);
plot(h,LW,1.6,'interval',[0 100])
normh = norm(h)
```

```
normh =
     2.441961683577728e-05
```



Chebfun provides a convenient tool for the numerical evaluation of integrals over infinite domains:

```
g = chebfun('(2/sqrt(pi))*exp(-x.^2)',[0 inf]);
sumg = sum(g)
```

```
sumg =
   0.999999999999992
```

The cumsum operator applied to this integrand gives us the error function, which matches the MATLAB `erf` function reasonably well:

```
errorfun = cumsum(g)
disp('          erf                errorfun')
for n = 1:6, disp([erf(n) errorfun(n)]), end
```

```
errorfun =
   chebfun column (1 smooth piece)
       interval       length   endpoint values
[       0,     Inf]       97         0        1
Epslevel = 2.220446e-16.  Vscale = 1.000000e+00.
          erf                errorfun
   0.842700792949715   0.842700792948944
   0.995322265018953   0.995322265017458
   0.999977909503001   0.999977909500824
   0.999999984582742   0.999999984579919
   0.999999999998463   0.999999999995027
   1.000000000000000   0.999999999995981
```

One should be cautious in evaluating integrals over infinite intervals, however, for as mentioned in Section 1.5, the accuracy is sometimes disappointing, especially for functions that do not decay very quickly:

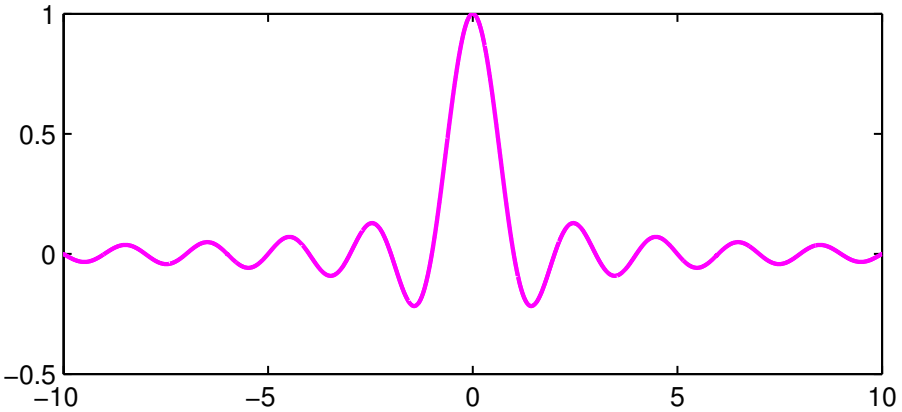
```
sum(chebfun('(1/pi)./(1+s.^2)',[-inf inf]))
```

```
ans =
   0.999999999998234
```

Here's an example of a function whose wiggles decay too slowly to be fully resolved:

```
sinc = chebfun('sin(pi*x)./(pi*x)',[-inf inf]);
plot(sinc,'m',LW,1.6,'interval',[-10 10])
```

```
Warning: Function not resolved using 65537 pts. Have you tried 'splitting on'?
```



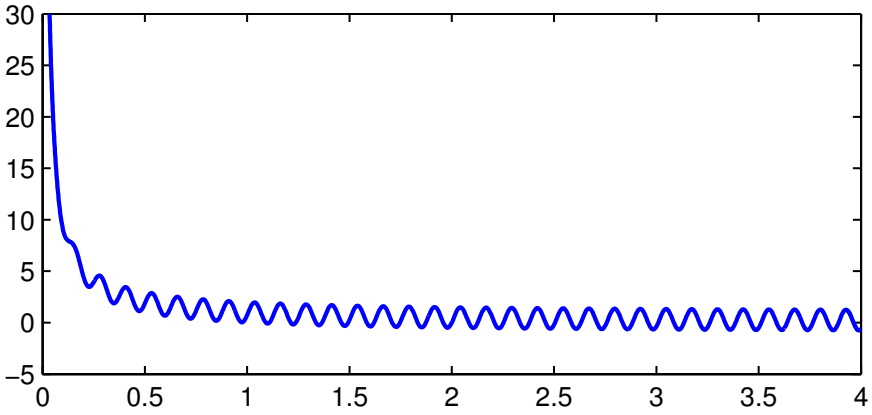
Chebfun's capability of handling infinite intervals was introduced by Rodrigo Platte in 2008-09. The details of the implementation then changed considerably with the introduction of version 5 in 2014.

The use of mappings to transform an unbounded domain to a bounded one is an idea that has been employed many times over the years. One of the references we have benefitted especially from, which also contains pointers to other works in this area, is the book [Boyd 2001].

9.2 Poles

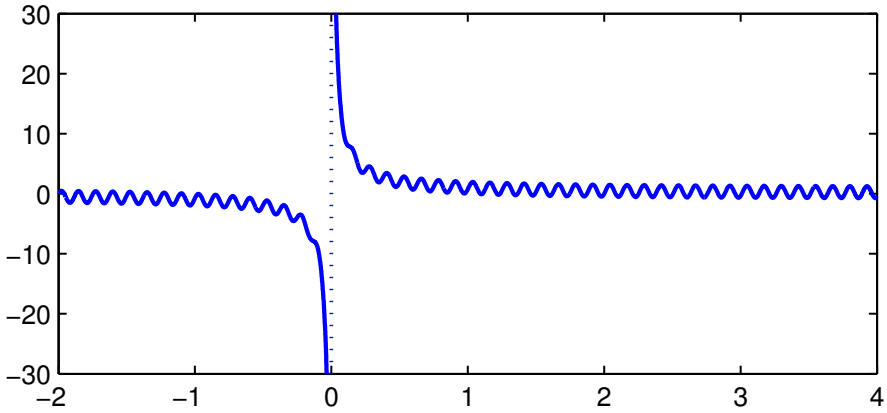
Chebfun can handle certain "vertical" as well as "horizontal" infinities – especially, functions that blow up according to an integer power, i.e., with a pole. If you know the nature of the blowup, it is a good idea to specify it using the `'exps'` flag. For example, here's a function with a simple pole at 0. We use `'exps'` to tell the constructor that the function looks like x^{-1} at the left endpoint and x^0 (i.e., smooth) at the right endpoint.

```
f = chebfun('sin(50*x) + 1./x',[0 4],'exps',[-1,0]);
plot(f,LW,1.6), ylim([-5 30])
```



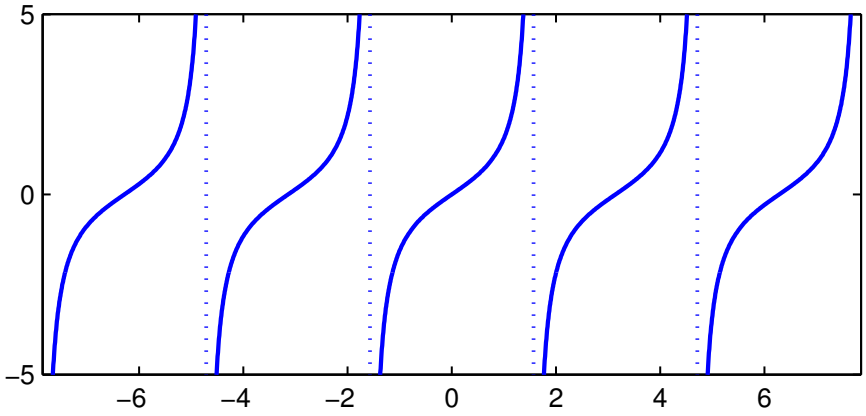
Here's the same function but over a domain that contains the singularity in the middle. We tell the constructor where the pole is and what the singularity looks like:

```
f = chebfun('sin(50*x) + 1./x',[-2 0 4],'exps',[0,-1,0]);
plot(f,LW,1.6), ylim([-30 30])
```



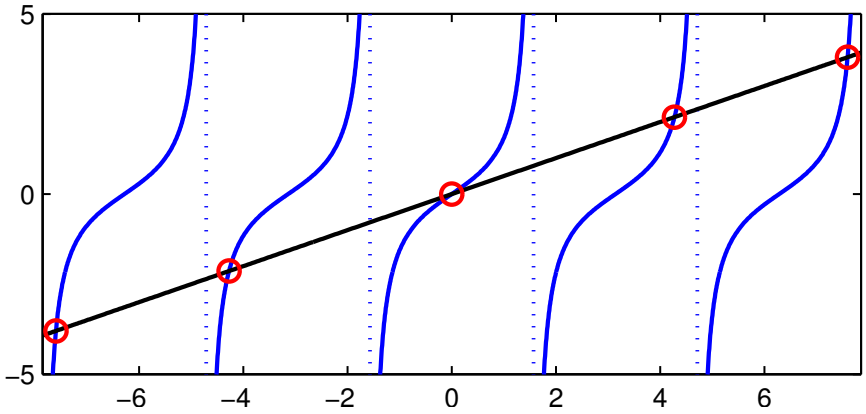
Here's the tangent function:

```
f = chebfun('tan(x)', pi*((-5/2):(5/2)), 'exps', -ones(1,6));
plot(f,LW,1.6), ylim([-5 5])
```



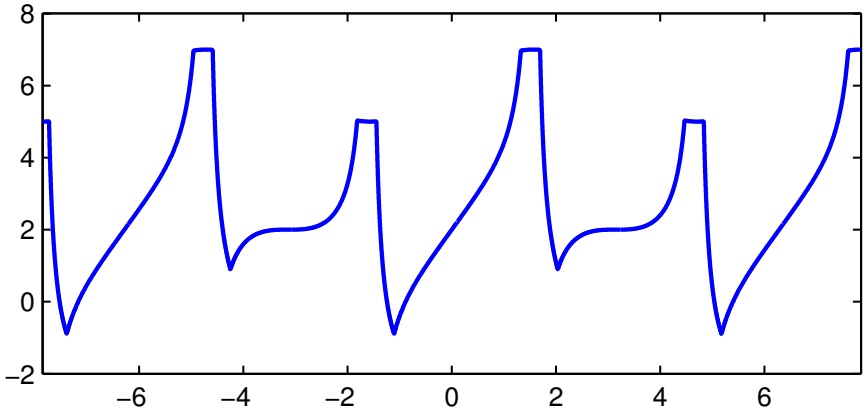
Rootfinding works as expected:

```
x2 = chebfun('x/2',pi*(5/2)*[-1 1]);
hold on, plot(x2,'k',LW,1.6)
r = roots(f-x2,'nojump'); plot(r,x2(r),'or',LW,1.6,'markersize',8)
```



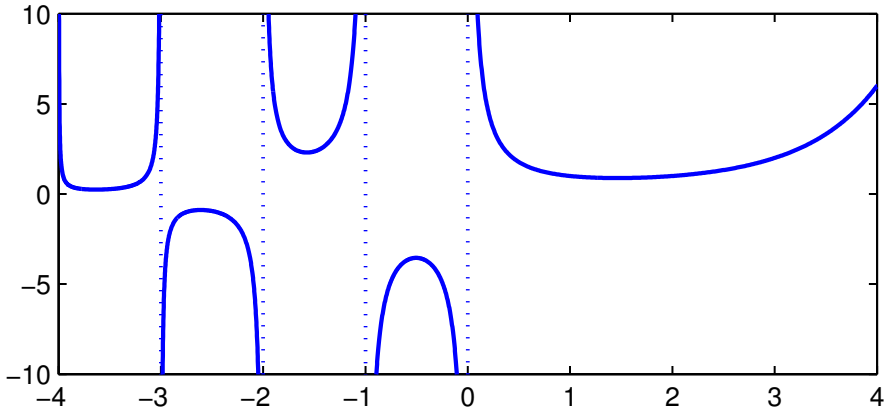
And we can manipulate the function in various other familiar ways:

```
g = sin(2*x2)+min(abs(f+2),6);
hold off, plot(g,LW,1.6)
```



If you don't know what singularities your function may have, Chebfun has some ability to find them if the flags `'blowup'` and `'splitting'` are on:

```
gam = chebfun('gamma(x)',[-4 4],'splitting','on','blowup',1);
plot(gam,LW,1.6), ylim([-10 10])
```

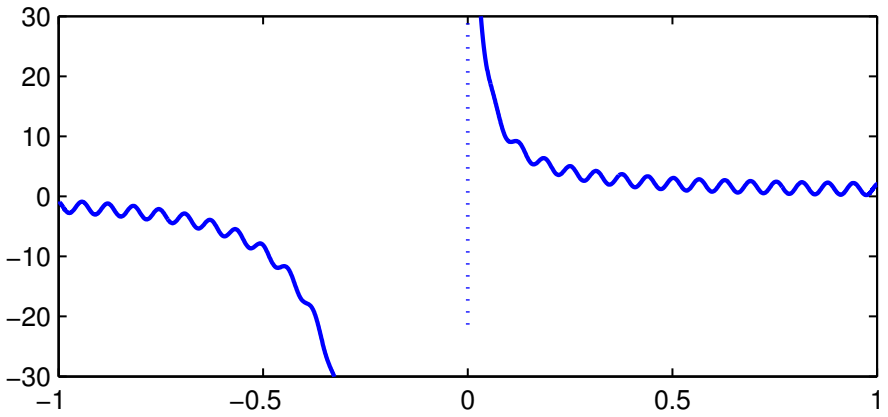


But it's always better to specify the breakpoints and powers if you know them:

```
gam = chebfun('gamma(x)',[-4:0 4],'exps',[-1 -1 -1 -1 -1 0]);
```

It's also possible to have poles of different strengths on two sides of a singularity. In this case, you specify two exponents at each internal breakpoint rather than one:

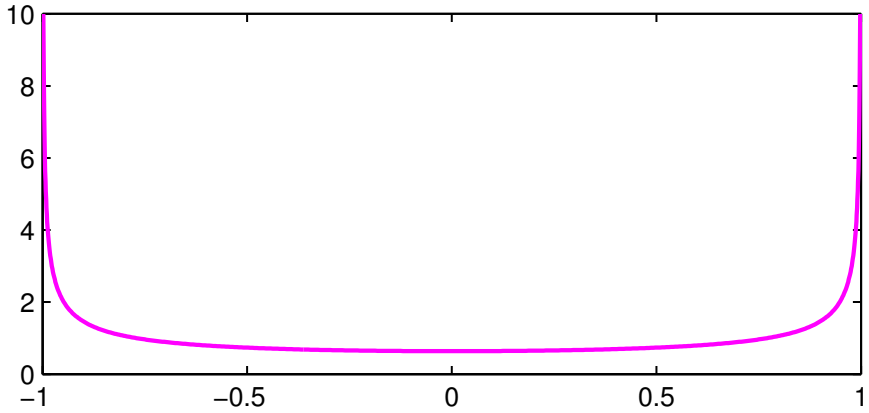
```
f = chebfun(@(x) cos(100*x)+sin(x).^(-2+sign(x)),[-1 0 1],'exps',[0 -3 -1 0]);
plot(f,LW,1.6), ylim([-30 30])
```



9.3 Singularities other than poles

Less reliable but also sometimes useful is the possibility of working with functions with algebraic singularities that are not poles. Here's a function with inverse square root singularities at each end:

```
w = chebfun('(2/pi)./(sqrt(1-x.^2))','exps',[-.5 -.5]);
plot(w,'m',LW,1.6), ylim([0 10])
```



The integral is 2:

```
sum(w)
```

```
ans =
     2
```

We pick this example because Chebyshev polynomials are the orthogonal polynomials with respect to this weight function, and Chebyshev coefficients are defined by inner products against Chebyshev polynomials with respect to this weight. For example, here we compute inner products of $x^4 + x^5$ against the Chebyshev polynomials $T_0, \ldots, T_5$. (The integrals in these inner products are calculated by Gauss-Jacobi quadrature using methods due to Hale and Townsend; for more on this subject see the command `jacpts`.)

```
x = chebfun('x');
T = chebpoly(0:5)';
f = x.^4 + x.^5;
chebcoeffs1 = T*(w.*f)
```

```
chebcoeffs1 =
   0.750000000000000
   0.625000000000000
   0.500000000000000
   0.312500000000000
   0.125000000000000
   0.062500000000000
```

Here for comparison are the Chebyshev coefficients as obtained from `chebcoeffs`:

```
chebcoeffs2 = flipud(chebcoeffs(f)')
```

```
chebcoeffs2 =
   0.375000000000000
   0.625000000000000
   0.500000000000000
   0.312500000000000
   0.125000000000000
   0.062500000000000
```

Notice the excellent agreement except for coefficient a_0. As mentioned in Section 4.1, in this special case the result from the inner product must be multiplied by 1/2.

You can specify singularities for functions that don't blow up, too. For example, suppose we want to work with $(x\exp(x))^{1/2}$ on the interval $[0, 2]$. A first try fails completely:

```
ff = @(x) sqrt(x.*exp(x));
d = [0,2];
f = chebfun(ff,d)
```

```
Warning: Function not resolved using 65537 pts. Have you tried 'splitting on'?
f =
   chebfun column (1 smooth piece)
       interval       length   endpoint values
[       0,       2]    65537   3.6e-15      3.8
Epslevel = 6.814598e-11.  Vscale = 3.844231e+00.
```

We could turn splitting on and resolve the function by many pieces, as illustrated in Section 8.3:

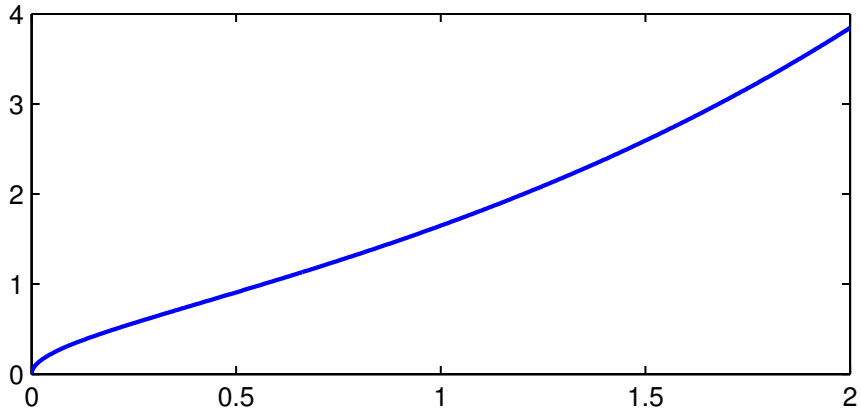
```
f = chebfun(ff,d,'splitting','on')
```

```
f =
   chebfun column (6 smooth pieces)
       interval       length   endpoint values
[       0,   2e-10]       43   1.2e-07  1.4e-05
[   2e-10,   2e-08]       62   1.4e-05  0.00014
[   2e-08,   2e-06]       91   0.00014   0.0014
[   2e-06,  0.0002]      111    0.0014    0.014
[  0.0002,    0.02]      123     0.014     0.14
[    0.02,       2]       84      0.14      3.8
Epslevel = 4.572597e-10.  Vscale = 3.844231e+00.  Total length = 514.
```

A better representation, however, is constructed if we tell Chebfun about the singularity at $x = 0$:

```
f = chebfun(ff,d,'exps',[.5 0])
plot(f,LW,1.6)
```

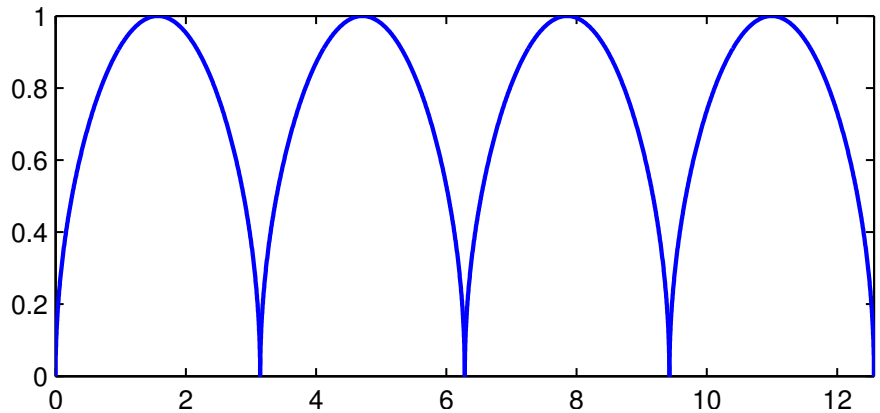
```
f =
   chebfun column (1 smooth piece)
       interval       length   endpoint values   endpoint exponents
[       0,       2]       12         0      3.8         [0.5      0]
Epslevel = 1.000000e-14.  Vscale = 3.844231e+00.
```



Under certain circumstances Chebfun will introduce singularities like this of its own accord. For example, just as `abs(f)` introduces breakpoints at roots of `f`, `sqrt(abs(f))` introduces breakpoints and also singularities at such roots:

```
theta = chebfun('t',[0,4*pi]);
f = sqrt(abs(sin(theta)))
plot(f,LW,1.6)
sumf = sum(f)
```

```
f =
   chebfun column (4 smooth pieces)
       interval       length   endpoint values   endpoint exponents
[       0,     3.1]       19         0        0         [0.5      0.5]
[     3.1,     6.3]       19         0        0         [0.5      0.5]
[     6.3,     9.4]       19         0        0         [0.5      0.5]
[     9.4,      13]       19         0        0         [0.5      0.5]
Epslevel = 1.408355e-15.  Vscale = 1.  Total length = 76.
sumf =
   9.585121877884735
```



If you have a function that blows up but you don't know the nature of the singularities, even whether they are poles or not, Chebfun will try to figure them out automatically if you run in `'blowup 2'` mode. Here's an example

```
f = chebfun('x.*(1+x).^(-exp(1)).*(1-x).^(-pi)','blowup',2)
```

```
f =
   chebfun column (1 smooth piece)
       interval       length   endpoint values   endpoint exponents
[      -1,       1]       30      -Inf      Inf          [-2.7      -3.1]
Epslevel = 1.000000e-14.  Vscale = Inf.
```

Notice that the `'exps'` field shows values close to $-e$ and $-\pi$, as is confirmed by looking at the numbers to higher precision:

```
get(f, 'exps')
```

```
ans =
  -2.718281828460000  -3.141592653590000
```

The treatment of blowups in Chebfun was initiated by Mark Richardson in an MSc thesis at Oxford [Richardson 2009], then further developed by Richardson in collaboration with Rodrigo Platte and Nick Hale.

9.4 Another approach to singularities

Chebfun version 4 offered an alternative `singmap` approach to singularities based on mappings of the x variable. This is no longer available in version 5.

9.5 References

[Boyd 2001] J. P. Boyd, *Chebyshev and Fourier Spectral Methods*, 2nd ed., Dover, 2001.

[Richardson 2009] M. Richardson, *Approximating Divergent Functions in the Chebfun System*, thesis, MSc in Mathematical Modelling and Scientific Computing, Oxford University, 2009.

10. Nonlinear ODEs and Chebgui

Lloyd N. Trefethen, November 2009, latest revision June 2014

Contents

Chapter 7 described the chebop capability for solving linear ODEs (ordinary differential equations) by the backslash command. We will now describe extensions of chebops to nonlinear problems, as well as other methods for nonlinear ODEs:

o Initial-value problems: `ode45`, `ode113`, `ode15s`

o Boundary-value problems: `bvp4c`, `bvp5c`

o Both kinds of problems via chebops: nonlinear backslash (=|solvebvp|)

In this chapter we outline the use of these methods; for fuller details, see the `help` documentation and especially the online Chebfun Examples. The last of the methods listed, nonlinear backslash or `solvebvp`, represents a "pure Chebfun" approach in which Newton's method is applied on chebfuns, with the necessary derivative operators calculated by Chebfun's built-in capabilities of Automatic Differentiation (AD). This is the main Chebfun method recommended for solving boundary-value problems.

We use the abbreviations IVP for initial-value problem and BVP for boundary-value problem.

For time-dependent PDEs, try `help pde15s`.

10.1 `ode45`, `ode15s`, `ode113`

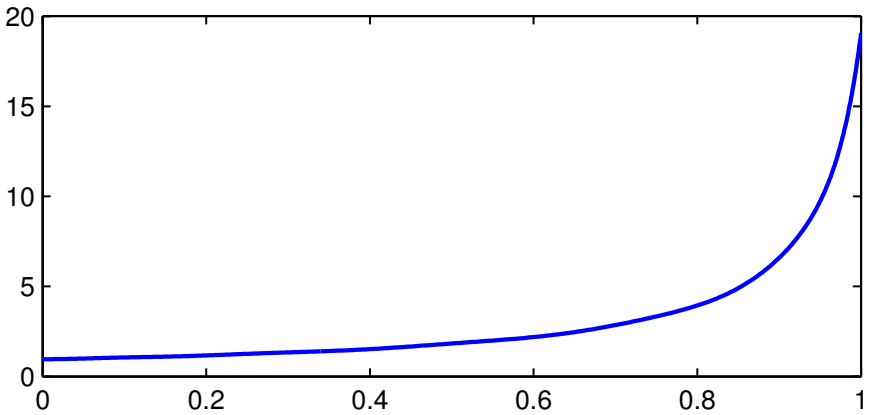
MATLAB has a highly successful suite of ODE IVP solvers introduced originally by Shampine and Reichelt [Shampine & Reichelt 1997]. The codes are called `ode23`, `ode45`, `ode113`, `ode15s`, `ode23s`, `ode23t`, and `ode23tb`, and are adapted to various mixes of accuracy requirements and stiffness.

Chebfun includes versions of `ode45` (for medium accuracy), `ode113` (for high accuracy), and `ode15s` (for stiff problems) originally created by Toby Driscoll and Rodrigo Platte. These codes operate

by calling their MATLAB counterparts, then converting the result to a chebfun. Thanks to the Chebfun framework of dealing with functions, their use is very natural and simple.

For example, here is a solution of $u' = u^2$ over $[0, 1]$ with initial condition $u(0) = 0.95$.

```
fun = @(t,u) u.^2;
u = chebfun.ode45(fun, [0, 1], 0.95);
LW = 'linewidth'; lw = 1.6;
plot(u,LW,lw)
```



The first argument to `ode45` defines the equation, the second defines the domain for the independent variable, and the third provides the initial condition.

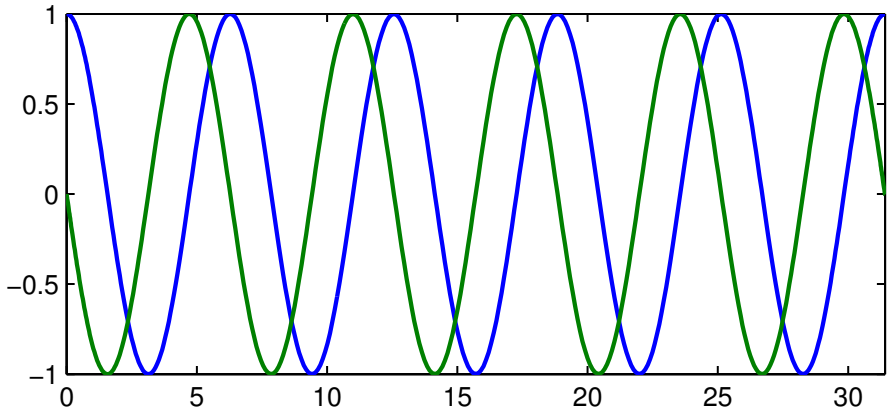
To find out where the solution takes the value 10, for example, we can write

```
roots(u-10)
```

```
ans =
   0.952926047089523
```

As a second example, let us consider the linear second-order equation $u'' = -u$, whose solutions are sines and cosines. We convert this to first-order form by using a vector v with $v(1) = u$ and $v(2) = u'$ and solve the problem again using `ode45`:

```
fun = @(t, v) [v(2); -v(1)];
v = chebfun.ode45(fun, [0 10*pi], [1 0]);
plot(v,LW,lw)
ylim([-1 1])
```



Here are the minimum and maximum values attained by u:

```
u = v(:,1); uprime = v(:,2);
minandmax(u)
```

```
ans =
  -1.000068010175644
   1.000039406904860
```

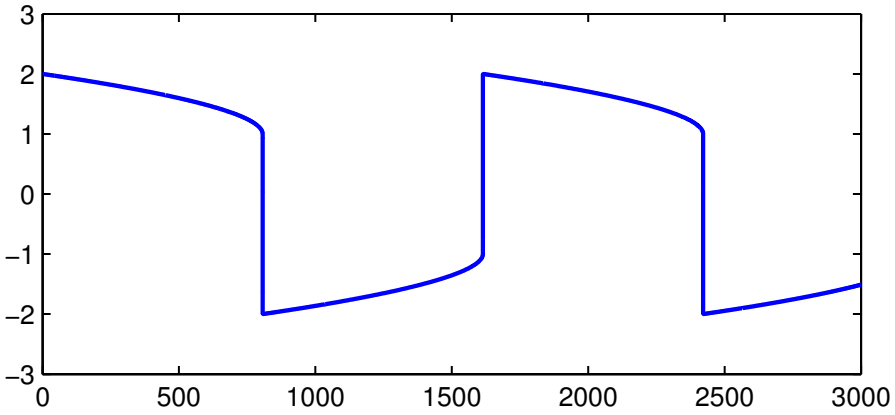
Evidently the accuracy is only around five digits. The reason is that the chebfun `ode45` code uses the same default tolerances as the original `ode45`. We can tighten the tolerance using the standard MATLAB `odeset` command, switching also to `ode113` since it is more efficient for high-accuracy computations:

```
opts = odeset('abstol',3e-14,'reltol',3e-14);
v = chebfun.ode113(fun, [0 10*pi], [1 0], opts);
minandmax(v(:,1))
```

```
ans =
  -0.999999999999994
   1.000000000000000
```

As a third example we solve the van der Pol equation for a nonlinear oscillator. Following the example in the MATLAB ODE documentation, we take $u'' = 1000(1-u^2)u' - u$ with initial conditions $u = 2$, $u' = 0$. This is a highly stiff problem whose solution contains very rapid transitions, so we use `ode15s` in "splitting on" mode:

```
opts = odeset('abstol',1e-8,'reltol',1e-8);
fun = @(t,v) [v(2); 1000*(1 - v(1)^2)*v(2) - v(1)];
chebfunpref.setDefaults('splitting','on')
v = chebfun.ode15s(fun, [0 3000], [2 0], opts);
chebfunpref.setDefaults('factory')
u = v(:,1); plot(u,LW,lw)
```



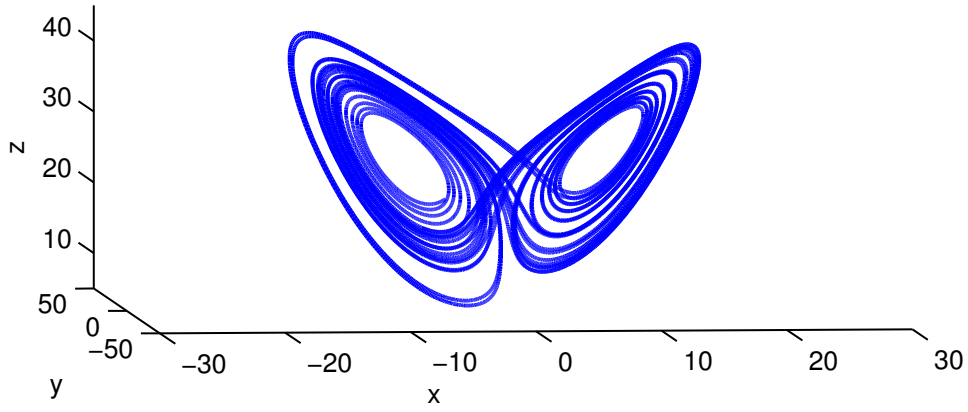
Here is a pretty good estimate of the period of the oscillator:

```
diff(roots(u))
```

```
ans =
   1.0e+02 *
   8.072000511670380
   8.072001250877227
```

Finally here is an illustration of the Lorenz equations:

```
fun = @(t,u) [10*(u(2)-u(1)); 28*u(1)-u(2)-u(1)*u(3); u(1)*u(2)-(8/3)*u(3)];
u = chebfun.ode15s(fun, [0 30], [-5 -7 21], opts);
plot3(u(:,1), u(:,2), u(:,3)), view(-5,9)
axis([-30 30 -50 50 5 45])
xlabel x, ylabel y, zlabel z
```

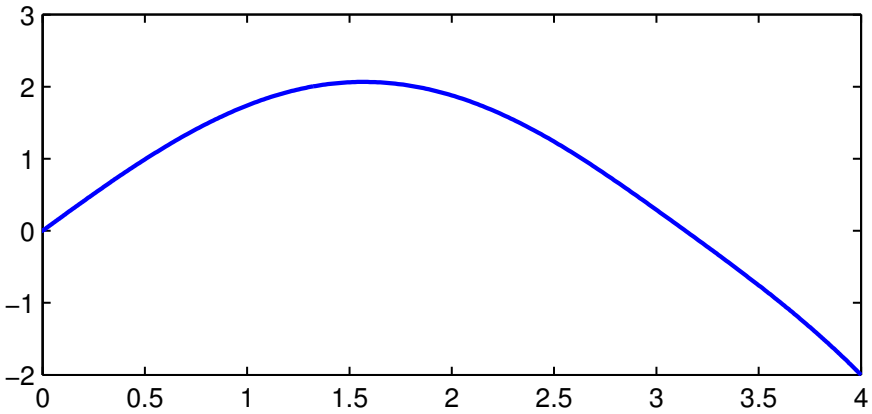


10.2 `bvp4c`, `bvp5c`

MATLAB also has well-established codes `bvp4c` and `bvp5c` for solving BVPs, and these too have been replicated in Chebfun. Again the Chebfun usage becomes somewhat simpler than the original. In particular, there is no need to call `bvpinit`; the initial guess and associated mesh are both determined by an input initial guess u_0.

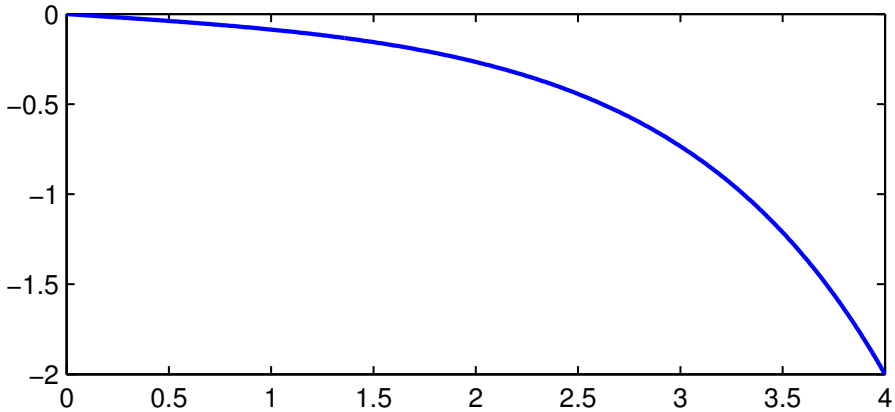
For example, here is the problem labeled `twoode` in the MATLAB `bvp4c` documentation. The domain is $[0,4]$, the equation is $u'' + |u| = 0$, and the boundary conditions are $u(0) = 0$, $u(4) = -2$. We get one solution from the initial condition $u = 1$:

```
twoode = @(x,v) [v(2); -abs(v(1))];
twobc = @(va,vb) [va(1); vb(1)+2];
d = [0,4];
one = chebfun(1, d);
v0 = [one 0*one];
v = bvp4c(twoode, twobc, v0);
u = v(:,1); plot(u,LW,lw)
```



The initial guess $u = -1$ gives another valid solution:

```
v0 = [-one 0*one];
v = bvp4c(twoode,twobc,v0);
u = v(:,1); plot(u,LW,lw)
```

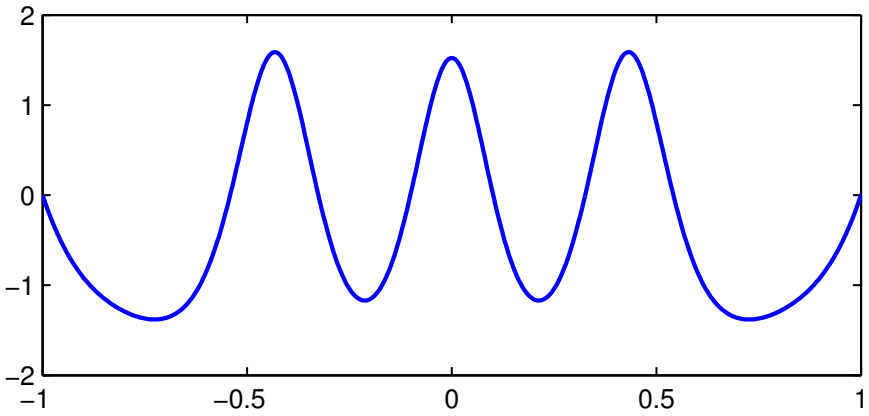


Here is an example with a variable coefficient, a problem due to George Carrier described in Sec. 9.7 of the book [Bender & Orzsag 1978]. On $[-1,1]$, we seek a function u satisfying

$$\varepsilon u'' + 2(1-x^2)u + u^2 = 1, \qquad u(-1) = u(1) = 0.$$

with $\varepsilon = 0.01$. Here is a solution with `bvp4c`, just one of many solutions of this problem.

```
ep = 0.01;
ode = @(x,v) [v(2); (1-v(1)^2-2*(1-x^2)*v(1))/ep];
bc = @(va,vb) [va(1); vb(1)];
d = [-1,1];
one = chebfun(1,d);
v0 = [0*one 0*one];
v = bvp4c(ode, bc, v0);
u = v(:,1); plot(u, LW, lw)
```



10.3 Automatic differentiation

The options described in the last two sections rely on standard numerical discretizations, whose results are then converted to Chebfun form. It is natural, however, to want to be able to solve ODEs fully within the Chebfun context, operating always at the level of functions. If the ODE is nonlinear, this will lead to Newton iterations for functions, also known as Newton-Kantorovich iterations. As with any Newton method, this will require a derivative, which in this case becomes a linear operator: an infinite-dimensional Jacobian, or more properly a *Frechet derivative*.

Chebfun contains features for making such explorations possible. This means that with Chebfun, you can explore Newton iterations at the function level. The enabling tool is Chebfun Automatic Differentiation (AD), introduced by Asgeir Birkisson and Toby Driscoll [Birkisson 2014, Birkisson & Driscoll 2011].

To illustrate Chebfun AD, consider the sequence of computations

```
x = chebfun('x', [0 1]);
u = x.^2;
v = exp(x) + u.^3;
w = u + diff(v);
```

Suppose we ask, how does one of these variables depend on another one earlier in the sequence? If the function u is perturbed by an infinitesimal function du, for example, what will the effect be on v?

As mathematicians we can answer this question as follows. The variation takes the form $dv/du = 3u^2 = 3x^4$. In other words, dv/du is the linear operator that multiplies a function on $[0,1]$ by $3x^4$.

In Chebfun, to compute this operator, we need to select a variable to act as a basis variable for derivative computations, and seed its derivative. (This procedure has changed with Version 5.) To compute derivatives with respect to u, we convert it to an object known as an `adchebfun` and redo the computations:

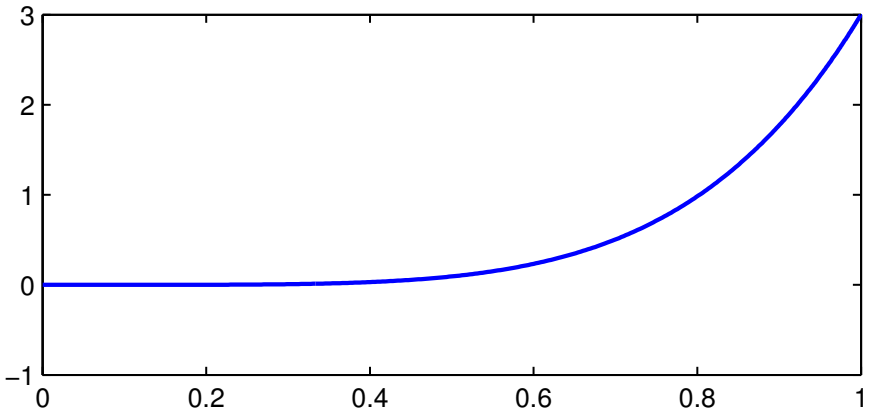
```
u = adchebfun(u);
v = exp(x) + u.^3;
w = u + diff(v);
```

We can now obtain the derivative of v with respect to u by accessing the `.jacobian` field of v:

```
dvdu = v.jacobian;
```

The result `dvdu` is a linear chebop of the kind discussed in Chapter 7. For example, `dvdu*x` is $3x^4$ times x, or $3x^5$:

```
plot(dvdu*x, LW, lw)
```



Notice that `dvdu` is a multiplication operator, acting on a function just by pointwise multiplication. (The technical term is *multiplier operator*.)

What about dw/du? To do this on paper we must think a little more carefully and compute

$$\frac{dw}{du} = \frac{\partial w}{\partial u} + \frac{\partial w}{\partial v}\frac{\partial v}{\partial u} = I + D(3u^2) = I + D(3x^4),$$

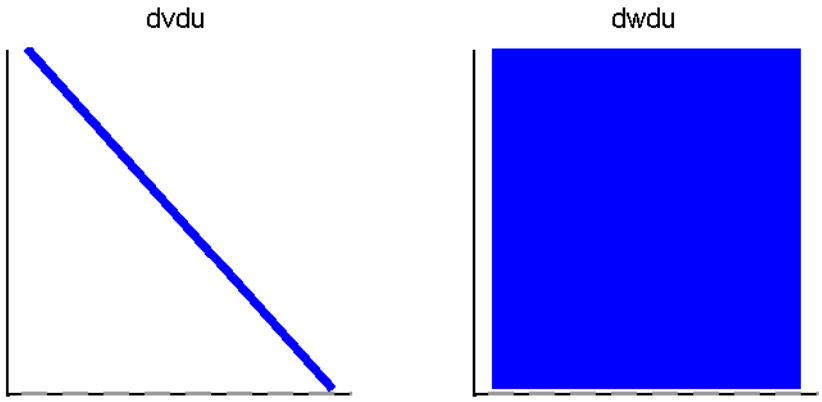
where I is the identity operator and D is the differentiation operator with respect to x. If we apply dw/du to x, for example, the result will be $x + (3x^5)' = x + 15x^4$. The following computation confirms that Chebfun reaches this result automatically.

```
dwdu = w.jacobian;
norm(dwdu*x - (x+15*x.^4))
```

```
ans =
     8.422556560155997e-16
```

We can use use the overloaded `spy` command to see at a glance that the first of our Frechet derivatives is a multiplier operator while the others are non-diagonal:

```
subplot(1,2,1), spy(linop(dvdu)), title dvdu
subplot(1,2,2), spy(linop(dwdu)), title dwdu
```



We now look at how AD enables Chebfun users to solve nonlinear ODE problems with backslash, just like the linear ones solved in Chapter 7.

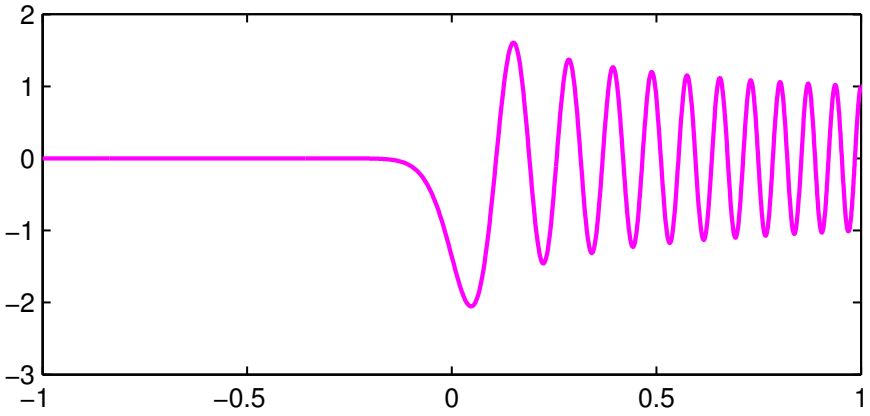
10.4 Nonlinear backslash and `solvebvp`

In Chapter 7, we realized linear operators as chebops constructed by commands like these:

```
L = chebop(-1, 1);
L.op = @(x,u) 0.0001*diff(u,2) + x.*u;
```

We could then solve a BVP:

```
L.lbc = 0;
L.rbc = 1;
u = L\0;
clf, plot(u, 'm', LW, lw)
```



What's going on in such a calculation is that `L` is a prescription for constructing matrices of arbitrary dimensions which are spectral approximations to the operator in question. When backslash is executed, the problem is solved on successively finer grids until convergence is achieved.

The object `L` we have created is a chebop, with these fields:

```
disp(L)
```

```
   Linear operator:
      0.0001*diff(u,2)+x.*u = 0
   operating on chebfun objects defined on:
      [-1 1]
   with
    left boundary conditions:
      u-BC = 0
    right boundary conditions:
      @(u)u-BC = 0
```

Notice that one of the fields is called `init`, which may hold an initial guess for an iteration if one is specified. If a guess is not specified, then a low-order polynomial function is used that matches the boundary conditions. To solve a nonlinear ODE, Chebfun uses a Newton or damped Newton iteration starting at the given initial guess. Each step of the iteration requires the solution of a linear problem specified by a Jacobian operator (Frechet derivative) evaluated at the current estimated solution. This is provided by the AD facility, and the linear problem is then solved by chebops.

In Section 7.9 we hand-coded our own Newton iteration to solve the nonlinear BVP

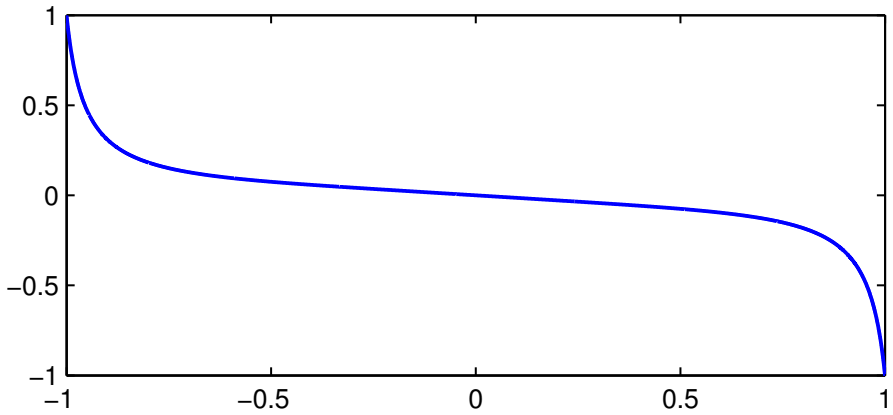
$$0.001u'' - u^3 = 0, \qquad u(-1) = 1, \quad u(1) = -1.$$

However, since the required Jacobian information is now computed by AD, construction of the Jacobian operator J is taken care of by `linearize(L,u)`, which returns the derivative of the operator J when it is linearized around the function u. Compare the code below to that in Section 7.9.

```
L = chebop(@(x,u) 0.001*diff(u,2) - u.^3);
L.lbc = 1; L.rbc = -1;
u = chebfun('-x');  nrmdu = Inf;
while nrmdu > 1e-10
   r = L*u;
   J = linearize(L,u);
   du = -(J\r);
   u = u + du;  nrmdu = norm(du)
end
clf, plot(u)
```

```
nrmdu =
   0.260668532007021
nrmdu =
   0.164126069559936
```

```
nrmdu =
   0.098900892365438
nrmdu =
   0.053787171683933
nrmdu =
   0.021518152858428
nrmdu =
   0.003586696693251
nrmdu =
     8.951602488748837e-05
nrmdu =
     5.357404846072875e-08
nrmdu =
     2.500515198784863e-14
```



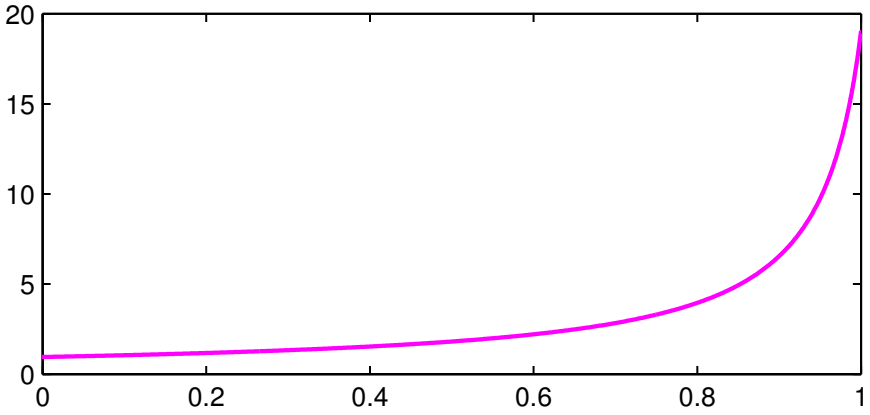
However, it is not necessary to construct such Newton iterations by hand! The code above is a much simplified version of what happens under-the-hood when |nonlinear backslash' is called to solve nonlinear differential equations. A few examples of this are demonstrated below.

Let us reconsider some of the examples of the last three sections. First in Section 10.1 we had the nonlinear IVP

$$u' = u^2, \qquad u(0) = 0.95.$$

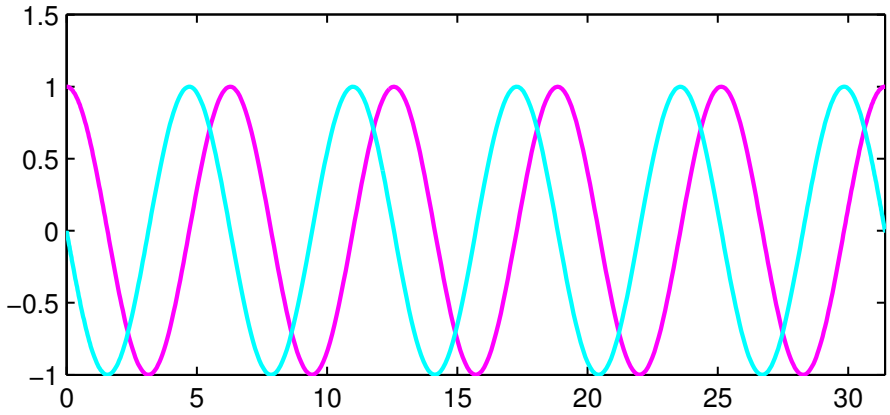
This can be solved in chebop formulation like this:

```
N = chebop(0, 1);
N.op = @(x,u) diff(u) - u.^2;
N.lbc = 0.95;
u = N\0;
plot(u,'m',LW,lw)
```



Next came the linear equation $u'' = -u$. With chebops, there is no need to reformulate the problem as a first-order system. There are two boundary conditions at the left, which can be imposed by making `N.lbc` a function returning an array.

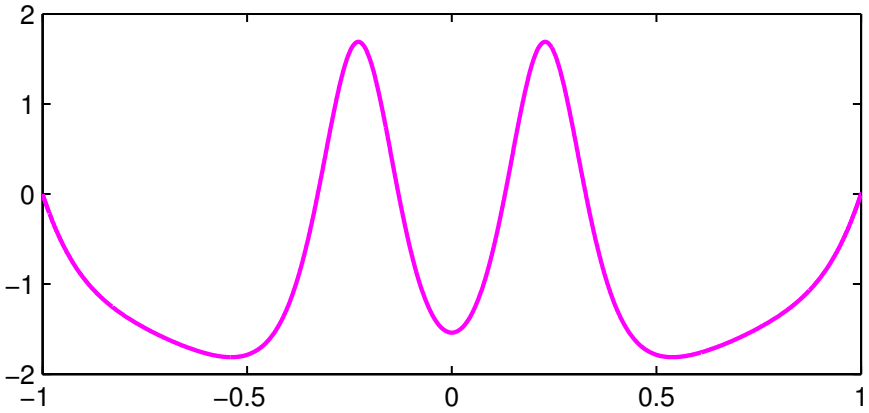
```
N = chebop(0, 10*pi);
N.op = @(x,u) diff(u,2) + u;
N.lbc = @(u) [u-1; diff(u)];
u = N\0;
plot(u, 'm', diff(u), 'c', LW, lw)
```



The van der Pol problem of Section 10.1 cannot be solved by chebops; the stiffness causes failure of the Newton iteration.

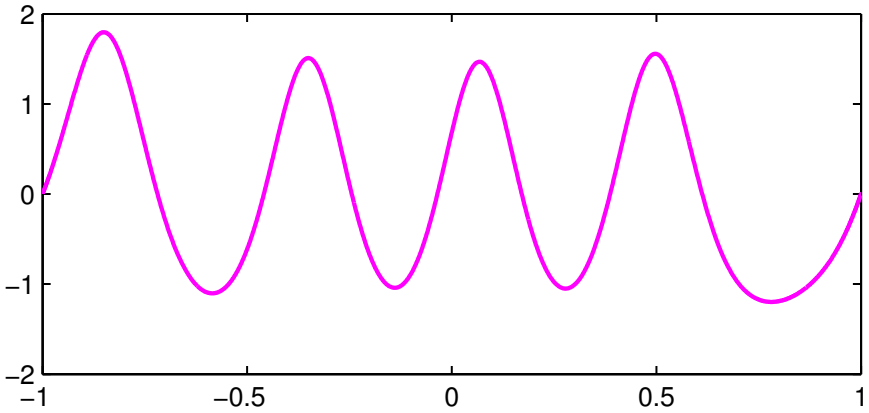
Here is the Carrier problem of section 10.2:

```
ep = 0.01;
N = chebop(-1, 1);
N.op = @(x,u) ep*diff(u,2) + 2*(1 - x.^2).*u + u.^2;
N.bc = 'dirichlet';
x = chebfun('x');
N.init = 2*(x.^2 - 1).*(1 - 2./(1 + 20*x.^2));
u = N\1; plot(u, 'm', LW, lw)
```



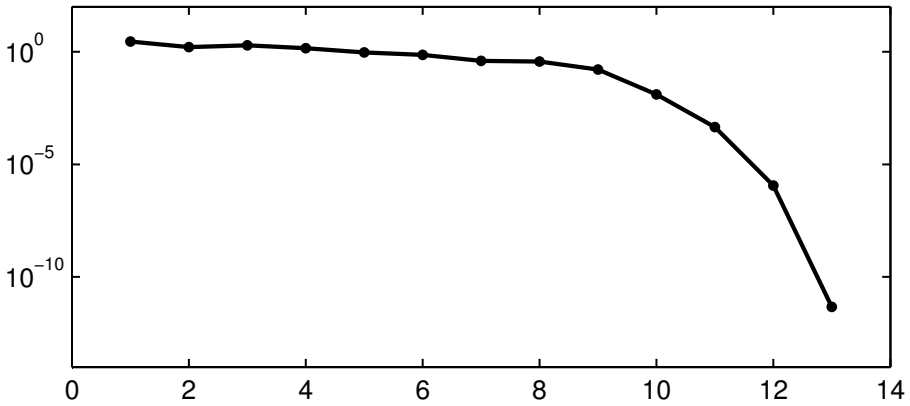
We get a different solution from the one we got before! This one is correct too; the Carrier problem has many solutions. If we multiply this solution by $2\sin(x/2)$ and take the result as a new initial guess, we converge to a third solution:

```
N.init= u.*sin(pi*x/2);
[u, info] = solvebvp(N, 1);
plot(u,'m',LW,lw)
```



This time, we called the method `solvebvp` with two output arguments. The second output is a MATLAB struct, which contains data showing the norms of the updates during the Newton iteration, revealing in this case a troublesome initial phase followed by eventual rapid convergence.

```
nrmdu = info.normDelta;
semilogy(nrmdu,'.-k',LW,lw), ylim([1e-14,1e2])
```



Another way to get information about the Newton iteration with nonlinear backlash is by setting

```
cheboppref.setDefaults('plotting','on')
```

or

```
cheboppref.setDefaults('display','iter')
```

Type `help cheboppref` for details. Here we shall not pursue this option and thus return the system to its factory state:

```
cheboppref.setDefaults('plotting','off')
cheboppref.setDefaults('display','none')
```

The heading of this section refers to the command `solvebvp`. When you apply backslash to a nonlinear chebop, it invokes the overloaded MATLAB command `mldivide`; this in turn calls a command `solvebvp` to do the actual work. By calling `solvebvp` directly, you can control the computation in ways not accessible through backslash. This situation is just like the relationship in standard MATLAB between \ and `linsolve`. See the help documentation for details.

10.5 Graphical user interface: Chebgui

Chebfun includes a GUI (Graphical User Interface) for solving all kinds of ODE, time-dependent PDE, and eigenvalue problems interactively. We will not describe it here, but we encourage the reader to type `chebgui` and give it a try. Be sure to note the `Demo` menu, which contains dozens of preloaded examples, both scalars and systems. Perhaps most important of all is the |Export to m-file" button, which produces a Chebfun m-file corresponding to whatever problem is loaded into the GUI. This feature enables one to get going quickly and interactively, then switch to a Chebfun program to adjust the fine points.

```
chebgui
```

Part II
Functions of two variables (Chebfun2)

11. Chebfun2: Getting Started

Alex Townsend, March 2013, latest revision June 2014

Contents

11.1 What is a chebfun2?

Chebfun2 is the part of Chebfun that deals with functions of two variables defined on a rectangle $[a, b] \times [c, d]$. Just like Chebfun in 1D, it is an extremely convenient tool for all kinds of computations including algebraic manipulation, optimization, integration, and rootfinding. It also extends to vector-valued functions of two variable, so that one can perform vector calculus.

A chebfun2, with a lower-case "c", is a MATLAB object, the 2D analogue of a chebfun. The syntax for chebfun2 objects is similar to the syntax for matrices in MATLAB, and Chebfun2 objects have many MATLAB commands overloaded. For instance, `trace(f)` returns the sum of the diagonal entries when f is a matrix and the integral of $f(x, x)$ when f is a chebfun2.

Chebfun2 builds on Chebfun's univariate representations and algorithms. Algorithmic details are given in [Townsend & Trefethen 2013b].

The implementation of Chebfun2 exploits the observation that many functions of two variables can be well approximated by low rank approximants. A rank 1 function is of the form $u(y)v(x)$, and a rank k function can be written as the sum of k rank 1 functions. Smooth functions tend to be well approximated by functions of low rank. Chebfun2 determines low rank function approximations automatically by means of an algorithm that can be viewed as an iterative application of Gaussian elimination with complete pivoting [Townsend & Trefethen 2013]. The underlying function representations are related to work by Carvajal, Chapman and Geddes [Carvajal, Chapman, & Geddes 2008] and others including Bebendorf [Bebendorf 2008], Hackbusch, Khoromskij, Oseledets, and Tyrtyshnikov.

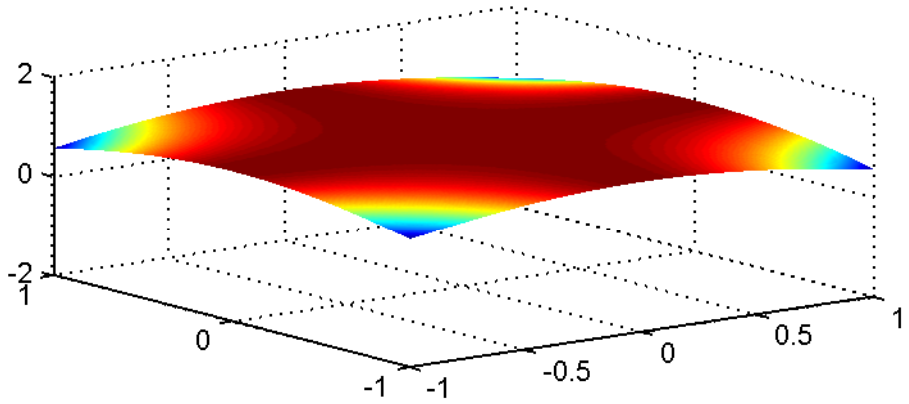
11.2 What is a chebfun2v?

Chebfun2 can represent scalar valued functions, such as $\exp(x+y)$, and vector valued functions, such as $[\exp(x+y); \cos(x-y)]$. A vector valued function is called a chebfun2v, and chebfun2v objects are useful for computations of vector calculus. For information about chebfun2v objects and vector calculus, see Chapters 14 and 15 of this guide.

11.3 Constructing chebfun2 objects

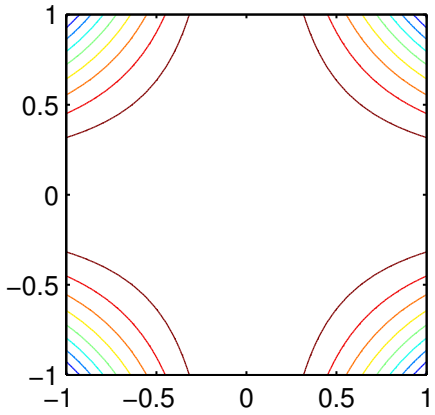
A chebfun2 is constructed by supplying the Chebfun2 constructor with a function handle or string. The default rectangular domain is $[-1,1]\times[-1,1]$. (An example showing how to specify a different domain is given at the end of this chapter.) For example, here we construct and plot a chebfun2 representing $\cos(xy)$ on $[-1,1]\times[-1,1]$.

```
f = chebfun2(@(x,y) cos(x.*y));
plot(f), zlim([-2 2])
```



There are several commands for plotting a chebfun2, including `plot`, `contour`, `surf`, and `mesh`. Here is a contour plot of f:

```
contour(f), axis square
```



One way to find the rank of the approximant used to represent f is like this:

```
length(f)
```

```
ans =
     7
```

Alternatively, more information can be given by displaying the chebfun2 object:

```
f
```

```
f =
   chebfun2 object: (1 smooth surface)
       domain                 rank       corner values
[  -1,   1] x [  -1,   1]        7     [0.54 0.54 0.54 0.54]
vertical scale =   1
```

The corner values are the values of the chebfun2 at $(-1,-1)$, $(-1,1)$, $(1,-1)$, and $(1,1)$, in that order. The vertical scale is used by operations to aim for close to machine precision relative to that number.

11.4 Basic operations

Once we have a chebfun2, we can compute quantities such as its definite double integral:

```
sum2(f)
```

```
ans =
   3.784332281468732
```

This matches well the exact answer obtained by calculus:

```
exact = 3.784332281468732
```

```
exact =
   3.784332281468732
```

We can also evaluate a chebfun2 at a point (x, y), or along a line. When evaluating along a line a chebfun is returned because the answer is a function of one variable.

Evaluation at a point:

```
x = 2*rand - 1; y = 2*rand - 1;
f(x,y)
```

```
ans =
   0.989047610535538
```

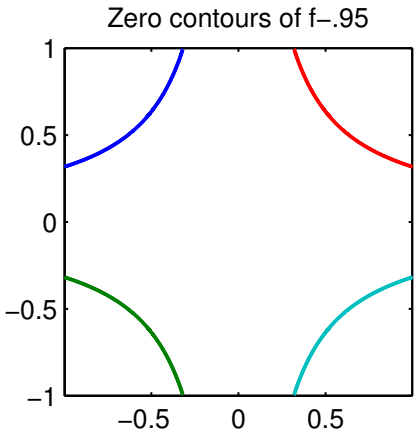
Evaluation along the line $y = \pi/6$:

```
f(:,pi/6)
```

```
ans =
   chebfun row (1 smooth piece)
       interval       length   endpoint values
[      -1,       1]        11      0.87     0.87
Epslevel = 8.096942e-15.  Vscale = 1.000000e+00.
```

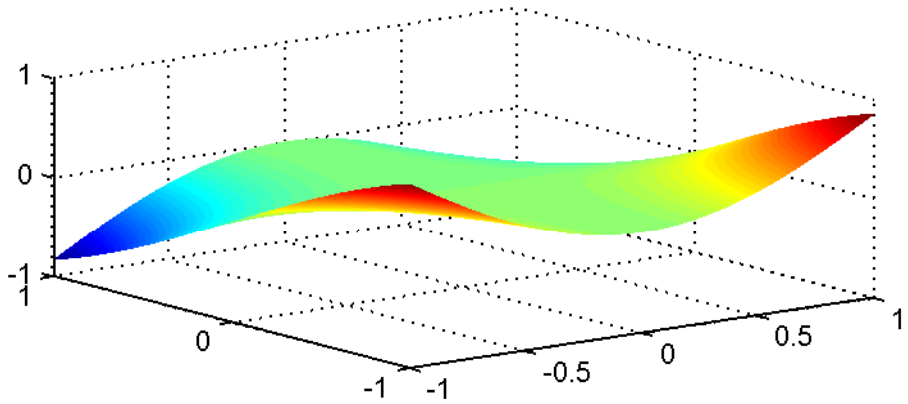
There are plenty of other questions that may be of interest. For instance, what are the zero contours of $f(x,y) - .95$?

```
r = roots(f-.95);
plot(r), axis square, title('Zero contours of f-.95')
```



What is the partial derivative $\partial f/\partial y$?

```
fy = diff(f,1,1);
plot(fy)
```



The syntax for the `diff` command can cause confusion because we are following the matrix syntax in MATLAB. Chebfun2 also offers `diffx(f,k)` and `diffy(f,k)`, which differentiate $f(x,y)$ k times with respect to the first first and second variable, respectively.

What is the mean value of $f(x, y)$ on $[-1, 1] \times [-1, 1]$?

```
mean2(f)
```

```
ans =
   0.946083070367183
```

11.5 Chebfun2 methods

There are over 100 methods that can be applied to chebfun2 objects. For a complete list type:

```
methods chebfun2
```

```
Methods for class chebfun2:

abs               fevalm            min2              size
cdr               flipdim           minandmax2        sph2cart
chebcoeffs2       fliplr            minus             sphere
chebfun2          flipud            mldivide          sqrt
chebpolyplot      fred              mrdivide          squeeze
chebpolyval2      get               mtimes            std
chol              grad              norm              std2
complex           gradient          pivotplot         subsref
conj              horzcat           pivots            sum
contour           imag              plot              sum2
contourf          integral          plotcoeffs        surf
cos               integral2         plotcoeffs2       surface
cosh              isempty           plus              surfacearea
ctranspose        isequal           pol2cart          svd
cumprod           isreal            potential         tan
cumsum            iszero            power             tand
cumsum2           jacobian          prod              tanh
dblquad           lap               qr                times
del2              laplacian         quad2d            trace
diag              ldivide           quiver            transpose
diff              length            quiver3           uminus
diffx             log               rank              uplus
diffy             lu                rdivide           vertcat
discriminant      max               real              volt
disp              max2              restrict          waterfall
display           mean              roots
ellipsoid         mean2             simplify
exp               median            sin
feval             min               sinh

Static methods:

chebpts2          outerProduct      vals2coeffs
coeffs2vals       paduaVals2coeffs
```

Most of these commands have been overloaded from MATLAB. More information about a Chebfun2 command can be found with `help`; for instance

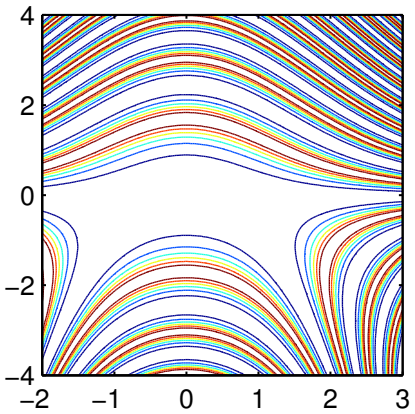
```
help chebfun2/max2
```

```
 MAX2   Global maximum of a CHEBFUN2.
    Y = MAX2(F) returns the global maximum of F over its domain.

    [Y, X] = MAX2(F) returns the global maximum in Y and its location X.

   This command may be faster if the OPTIMIZATION TOOLBOX is installed.

  See also MIN2, MINANDMAX2.
```

11.6 Object composition

So far, in this chapter, chebfun2 objects have been constructed explicitly via a command of the form `chebfun2(...)`. Another way to construct new chebfun2 objects is by composing them together with operations such as `+`, `-`, `.*`, and `.^`. For instance,

```
x = chebfun2(@(x,y) x, [-2 3 -4 4]);
y = chebfun2(@(x,y) y, [-2 3 -4 4]);

f = 1./( 2 + cos(.25 + x.^2.*y + y.^2) );
contour(f), axis square
```



11.7 Analytic functions

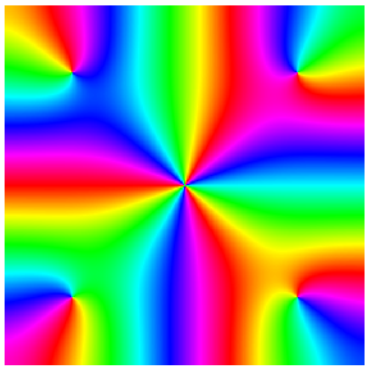
An analytic function $f(z)$ can be thought of as a complex valued function of two real variables, $f(x, y) = f(x+iy)$. If the Chebfun2 constructor is given an anonymous function with one argument, it assumes that argument is a complex variable. For instance,

```
f = chebfun2(@(z) sin(z));
f(1+1i), sin(1+1i)
```

```
ans =
  1.298457581415977 + 0.634963914784736i
ans =
  1.298457581415977 + 0.634963914784736i
```

These functions can be visualised by using a technique known as phase portrait plots. Given a complex number $z = re^{i\theta}$, the phase $e^{i\theta}$ can be represented by a colour. We follow Wegert's colour recommendations [Wegert 2012], using red for a phase i, then yellow, green, blue, and violet as the phase moves clockwise around the unit circle. For example,

```
f = chebfun2(@(z) sin(z)-sinh(z),2*pi*[-1 1 -1 1]);
plot(f)
```



Many properties of analytic functions can be visualised by these types of plots, such as the location of zeros and their multiplicities. Can you work out the multiplicity of the root at z=0 from this plot?

At present, since Chebfun2 only represents smooth functions, a trick is required to draw pictures like these for functions with poles [Trefethen 2013]. For functions with branch points or essential singularities, it is currently not possible in Chebfun2 to draw phase plots.

11.8 References

[Bebendorf 2008] M. Bebendorf, *Hierarchical Matrices: A Means to Efficiently Solve Elliptic Boundary Value Problems*, Springer, 2008.

[Carvajal, Chapman, & Geddes 2008] O. A. Carvajal, F. W. Chapman and K. O. Geddes, Hybrid symbolic-numeric integration in multiple dimensions via tensor-product series, *Proceedings of ISSAC'05*, M. Kauers, ed., ACM Press, 2005, pp.84-91.

[Townsend & Trefethen 2013] A. Townsend and L. N. Trefethen, Gaussian elimination as an iterative algorithm, *SIAM News*, March 2013.

[Townsend & Trefethen 2013b] A. Townsend and L. N. Trefethen, An extension of Chebfun to two dimensions, *SIAM Journal on Scientific Computing*, 35 (2013), C495-C518.

[Trefethen 2013] L. N. Trefethen, Phase Portraits for functions with poles,

`http://www2.maths.ox.ac.uk/chebfun/examples/complex/html/PortraitsWithPoles.shtml`

[Wegert 2012] E. Wegert, *Visual Complex Functions: An Introduction with Phase Portraits*, Birkhauser/Springer, 2012.

12. Chebfun2: Integration and Differentiation

Alex Townsend, March 2013, latest revision June 2014

Contents

12.1 `sum` and `sum2`

We have already seen the `sum2` command, which returns the definite double integral of a chebfun2 over its domain of definition. The sum command is a little different and integrates with respect to one variable at a time. For instance, the following commands integrate over the y variable:

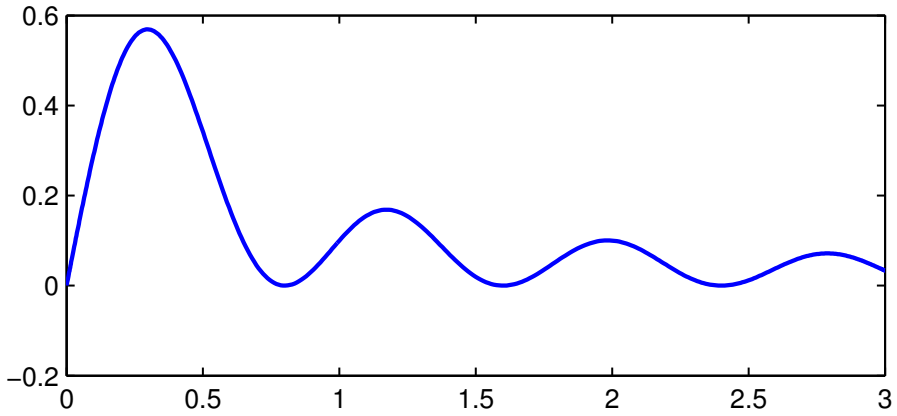
```
f = chebfun2(@(x,y) sin(10*x.*y),[0 pi/4 0 3]);
sum(f)
```

```
ans =
   chebfun row (1 smooth piece)
       interval       length   endpoint values
[       0,    0.79]       35  -1.8e-15     0.13
Epslevel = 8.354003e-15.  Vscale = 2.172798e+00.
```

A chebfun is returned because the result depends on x and hence, is a function of one variable. Similarly, we can integrate over the x variable, and plot the result.

```
LW = 'linewidth';
sum(f,2), plot(sum(f,2),LW,1.6)
```

```
ans =
   chebfun column (1 smooth piece)
       interval       length   endpoint values
[       0,       3]       35  -5.6e-16    0.033
Epslevel = 7.345480e-15.  Vscale = 5.688372e-01.
```



A closer look reveals that `sum(f)` returns a row chebfun while `sum(f,2)` returns a column chebfun. This distinction is a reminder that `sum(f)` is a function of x while `sum(f,2)` is a function of y. If we integrate over y and then x the result is the double integral of f.

```
sum2(f)
sum(sum(f))
```

```
ans =
   0.377914013520379
ans =
   0.377914013520379
```

It is interesting to compare the execution times involved for computing the double integral by different commands. Chebfun2 does very well for smooth functions. Here we see an example in which it is faster than the MATLAB `quad2d` command.

```
F = @(x,y) exp(-(x.^2 + y.^2 + cos(4*x.*y)));
tol = 3e-14;
tic, I = quad2d(F,-1,1,-1,1,'AbsTol',tol); t = toc;
fprintf('QUAD2D:  I = %17.15f  time = %6.4f secs\n',I,t)
tic, I = sum(sum(chebfun2(F))); t = toc;
fprintf('CHEBFUN2/SUMSUM:  I = %17.15f  time = %6.4f secs\n',I,t)
tic, I = sum2(chebfun2(F)); t = toc;
fprintf('CHEBFUN2/SUM2:  I = %17.15f  time = %6.4f secs\n',I,t)
```

```
QUAD2D:  I = 1.399888131932670  time = 0.0851 secs
CHEBFUN2/SUMSUM:  I = 1.399888131932670  time = 0.0229 secs
CHEBFUN2/SUM2:  I = 1.399888131932670  time = 0.0178 secs
```

Chebfun2 is not designed specifically for numerical quadrature (or more properly, "cubature"), and careful comparisons with existing software have not been carried out. Low rank function approximations have been previously used for numerical quadrature by Carvajal, Chapman, and Geddes [Carvajal, Chapman & Geddes 2005].

12.2 `norm`, `mean`, and `mean2`

The L^2-norm of a function $f(x, y)$ can be computed as the square root of the double integral of f^2. In Chebfun2 the command `norm(f)`, without any additional arguments, computes this quantity. For example,

```
f = chebfun2('exp(-(x.^2 + y.^2 +4*x.*y))');
norm(f), sqrt(sum2(f.^2))
```

```
ans =
   2.819481057146934
ans =
   2.819481057146935
```

Here is another example. This time we compute the norms of $f(x, y)$, $\cos(f(x, y))$, and $f(x, y)^5$.

```
f = chebfun2(@(x,y) exp(-1./( sin(x.*y) + x ).^2));
norm(f), norm( cos(f) ), norm( f.^5 )
```

```
ans =
   0.462652919760561
ans =
   1.950850368197068
ans =
   0.060896016071932
```

Just as `sum2` performs double integration, `mean2` computes the average value of $f(x, y)$ over both variables:

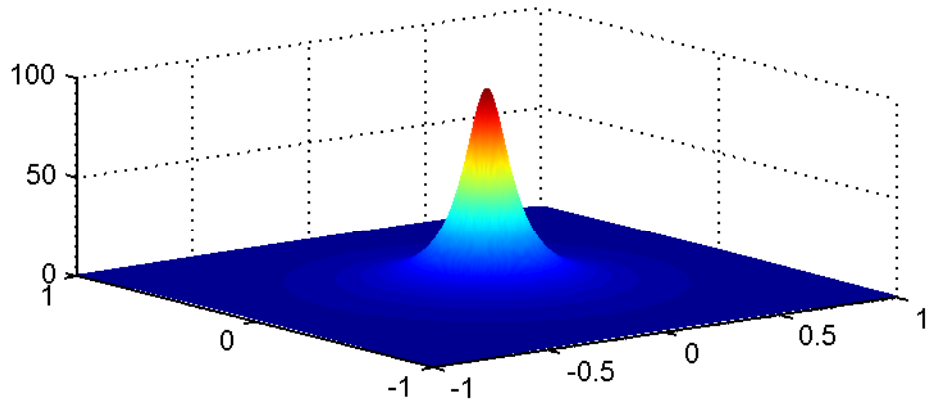
```
help chebfun2/mean2
```

```
 MEAN2   Mean of a CHEBFUN2
    V = MEAN2(F) returns the mean of a CHEBFUN:

                          d  b
                         /  /
    V = 1/(d-c)/(b-a)    |  |   f(x,y) dx dy
                         /  /
                        c  a

   where the domain of F is [a,b] x [c,d].

  See also MEAN, STD2.
```

For example, here is the average value of a 2D Runge function.

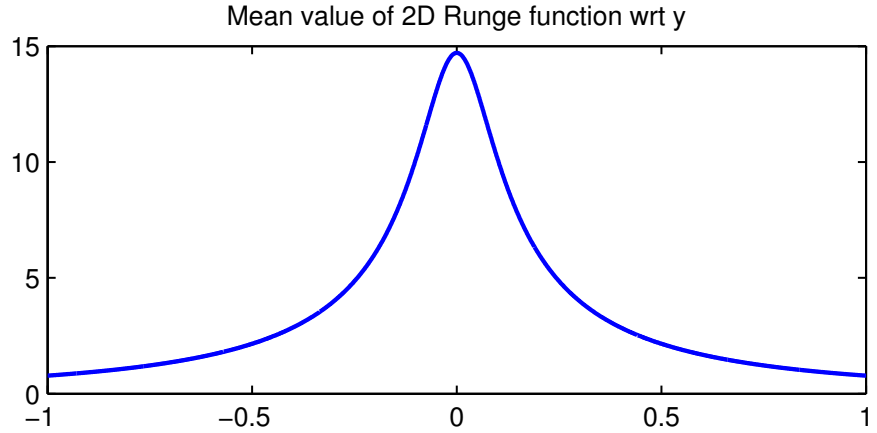
```
runge = chebfun2( @(x,y) 1./( .01 + x.^2 + y.^2 )) ;
plot(runge)
mean2(runge)
```

```
ans =
   3.796119578934829
```



The command `mean` computes the average along one variable. The output of `mean(f)` is a function of one variable represented by a chebfun, and so we can plot it.

```
plot(mean(runge),LW,1.6)
title('Mean value of 2D Runge function wrt y')
```



If we average over the y variable and then the x variable, we obtain the mean value over the whole domain.

```
mean(mean(runge))      % compare with mean2(runge)
```

```
ans =
   3.796119578934826
```

12.3 cumsum and cumsum2

The command cumsum2 computes the double indefinite integral, which is a function of two variables, and returns a chebfun2.

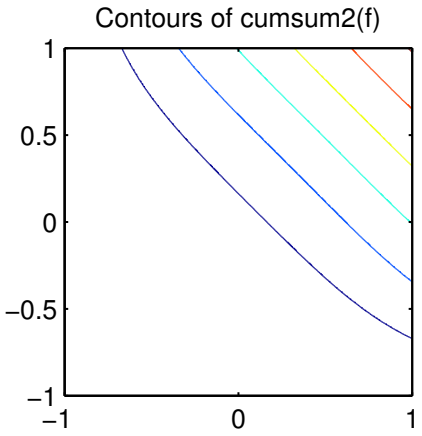
```
help chebfun2/cumsum2
```

```
 CUMSUM2   Double indefinite integral of a CHEBFUN2.
    F = CUMSUM2(F) returns the double indefinite integral of a CHEBFUN2. That is
                     y  x
                    /  /
    CUMSUM2(F) =   |  |   f(x,y) dx dy   for  (x,y) in [a,b] x [c,d],
                  /  /
                 c  a

    where [a,b] x [c,d] is the domain of f.

  See also CUMSUM, SUM, SUM2.
```

On the other hand, cumsum(f) computes the indefinite integral with respect to just one variable, also returning a chebfun2. Again, the indefinite integral in the y variable and then the x variable is the same as the double indefinite integral, as we can check numerically.

```
f = chebfun2(@(x,y) exp(-(x.^2 + 3*x.*y+y.^2) ));
contour(cumsum2(f),'numpts',400), axis equal
title('Contours of cumsum2(f)'), axis([-1 1 -1 1])
norm( cumsum(cumsum(f),2) - cumsum2(f) )
```

```
ans =
     0
```



12.4 Complex encoding

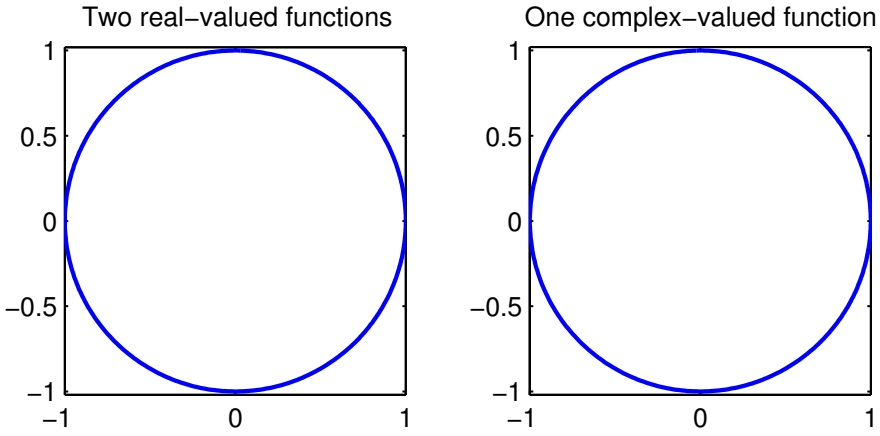
As is well known, a pair of real scalar functions f and g can be encoded as a complex function $f+ig$. This trick can be useful for simplifying many operations, but at the same time may be confusing.

For instance, instead of representing the unit circle with two real-valued functions, we can represent it with one complex-valued function:

```
c1 = chebfun(@(t) cos(t),[0 2*pi]);          % first real-valued function
c2 = chebfun(@(t) sin(t),[0 2*pi]);          % second real-valued function
c = chebfun(@(t) cos(t)+1i*sin(t),[0 2*pi]); % one complex function
```

Here are two ways to make a plot of a circle.

```
subplot(1,2,1), plot(c1,c2,LW,1.6)
axis equal, title('Two real-valued functions')
subplot(1,2,2), plot(c,LW,1.6)
axis equal, title('One complex-valued function')
```



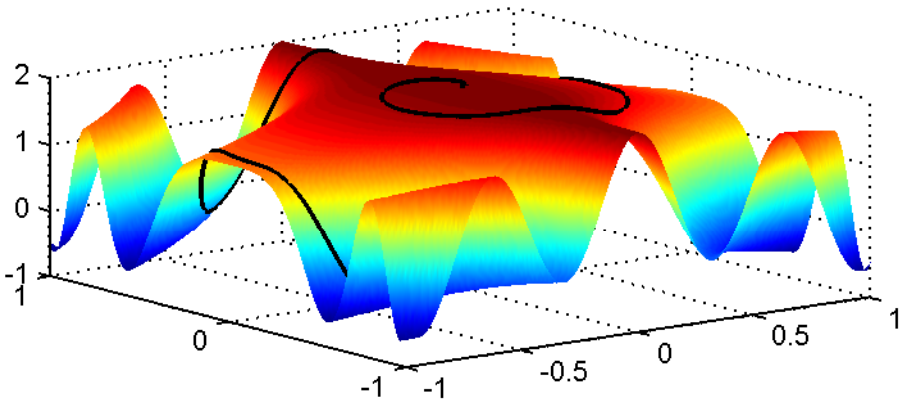
This complex encoding trick is used in a number of places in Chebfun2. Specifically, it's used to encode the path of integration for a line integral (see section 12.5, below), to represent zero contours of a chebfun2 (see Chapter 13), and to represent trajectories in vector fields (see Chapter 14).

We hope users become comfortable with using complex encodings, though they are not required for the majority of Chebfun2 functionality.

12.5 Integration along curves

Chebfun2 can compute the integral of $f(x, y)$ along a curve $(x(t), y(t))$. It uses the complex encoding trick and encode the curve $(x(t), y(t))$ as a complex valued chebfun $x(t) + iy(t)$. For instance, what is the area under the following curve?

```
clf
f = chebfun2(@(x,y) cos(10*x.*y.^2) + exp(-x.^2)); % chebfun2 object
C = chebfun(@(t) t.*exp(10i*t),[0 1]);             % spiral curve
plot(f), hold on
plot3(real(C),imag(C),f(C),'k','linewidth',2)
```



We can compute this by restricting f to the curve and then integrating

```
sum(f(C))
```

```
ans =
   1.613596461872283
```

12.6 diff

In MATLAB the `diff` command calculates finite differences of a matrix along its columns (by default) or rows. For a chebfun2 the same syntax represents partial differentiation $\partial f/\partial y$ (by default) or $\partial f/\partial x$. This command has the following syntax:

```
help chebfun2/diff
```

```
 DIFF   Derivative of a CHEBFUN2s.
    DIFF(F) is the derivative of F along the y direction.

    DIFF(F, N) is the Nth derivative of F in the y direction.

    DIFF(F, N, DIM) is the Nth derivative of F along the dimension DIM.
      DIM = 1 (default) is the derivative in the y-direction.
      DIM = 2 is the derivative in the x-direction.

    DIFF(F, [NX NY]) is the partial derivative of NX of F in the first variable,
    and NY of F in the second derivative. For example, DIFF(F,[1 2]) is
    d^3F/dxd^2y.

  See also GRADIENT, SUM, PROD.
```

Here we use `diff` to check that the Cauchy-Riemann equations hold for an analytic function.

```
f = chebfun2(@(x,y) sin(x+1i*y));   % a holomorphic function
u = real(f); v = imag(f);           % real and imaginary parts
norm(diff(u) - (-diff(v,1,2)))
norm(diff(u,1,2) - diff(v))         % Do the Cauchy-Riemann eqns hold?
```

```
ans =
     1.060890001425453e-14
ans =
     2.569762378645554e-14
```

12.7 Integration in three variables

Chebfun2 also works pretty well for integration in three variables. The idea is to integrate over two of the variables using Chebfun2 and the remaining variable using Chebfun. We have selected a tolerance of 10^{-6} for this example because the default tolerance in the MATLAB `integral3` command is also 10^{-6}.

```
r = @(x,y,z) sqrt(x.^2 + y.^2 + z.^2);
t = @(x,y,z) acos(z./r(x,y,z)); p = @(x,y,z) atan(y./x);
f = @(x,y,z) sin(5*(t(x,y,z) - r(x,y,z))) .* sin(p(x,y,z)).^2;

I = @(z) sum2(chebfun2(@(x,y) f(x,y,z),[-2 2 .5 2.5])); % integrate out x,y
tic, I = sum(chebfun(@(z) I(z),[1 2],'vectorize')); t = toc;
fprintf(' Chebfun2:  I = %16.14f  time = %5.3f secs\n',I,t)
tic, I = integral3(f,-2,2,.5,2.5,1,2); t = toc;         % compare with MATLAB
fprintf('Integral3:  I = %16.14f  time = %5.3f secs\n',I,t)
```

```
 Chebfun2:  I = -0.48056569408898  time = 1.272 secs
Integral3:  I = -0.48056569417568  time = 0.286 secs
```

12.8 References

[Carvajal, Chapman & Geddes 2005] O. A. Carvajal, F. W. Chapman and K. O. Geddes, Hybrid symbolic-numeric integration in multiple dimensions via tensor-product series, *Proceedings of ISSAC'05*, M. Kauers, ed., ACM Press, 2005, pp. 84–91.

13. Chebfun2: Rootfinding and Optimisation

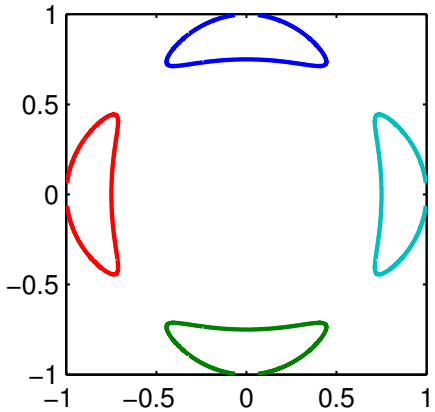
Alex Townsend, March 2013, latest revision June 2014

Contents

13.1 Zero contours

Chebfun2 comes with the capability to compute the zero contours of a function of two variables. For example, we can compute a representation of Trott's curve, an example from algebraic geometry [Trott 1997].

```
x = chebfun2(@(x,y) x); y = chebfun2(@(x,y) y);
trott = 144*(x.^4+y.^4)-225*(x.^2+y.^2) + 350*x.^2.*y.^2+81;
r = roots(trott);
LW = 'linewidth'; MS = 'markersize';
plot(r,LW,1.6), axis([-1 1 -1 1]), axis square
```



The zero curves are represented as complex valued chebfuns (see Chapter 5 of the guide). For example,

```
r(:,1)
```

```
ans =
   chebfun column (1 smooth piece)
       interval       length   endpoint values
[      -1,       1]      576    complex values
Epslevel = 1.000000e-05.  Vscale = 9.980190e-01.
```

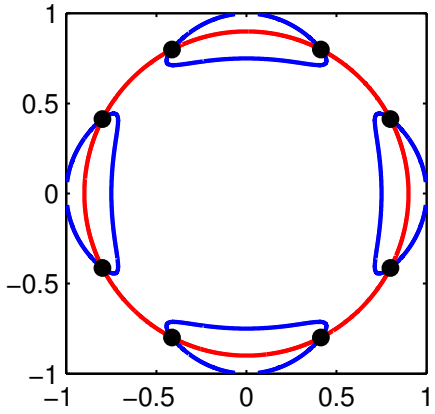
The zero contours of a function are computed by Chebfun2 to plotting accuracy and they are typically not accurate to machine precision.

13.2 `roots`

Chebfun2 also comes with the capability of finding zeros of bivariate systems, i.e., the solutions to $f(x, y) = g(x, y) = 0$. If the `roots` command is supplied with one chebfun2, it computes the zero contours of that function, as in the last section. However, if it is supplied with two chebfun2 objects, as in `roots(f,g)`, then it computes the roots of the bivariate system. Generically, these are isolated points.

What points on the Trott's curve intersect with the circle of radius 0.9?

```
g = chebfun2(@(x,y) x.^2 + y.^2 - .9^2);  % circle of radius 0.9
r = roots(trott,g);
plot(roots(trott),'b',LW,1.6), hold on
plot(roots(g),'r',LW,1.6)
plot(r(:,1),r(:,2),'.k',LW,1.6,MS,20)  % point intersections
axis([-1 1 -1 1]), axis square, hold off
```

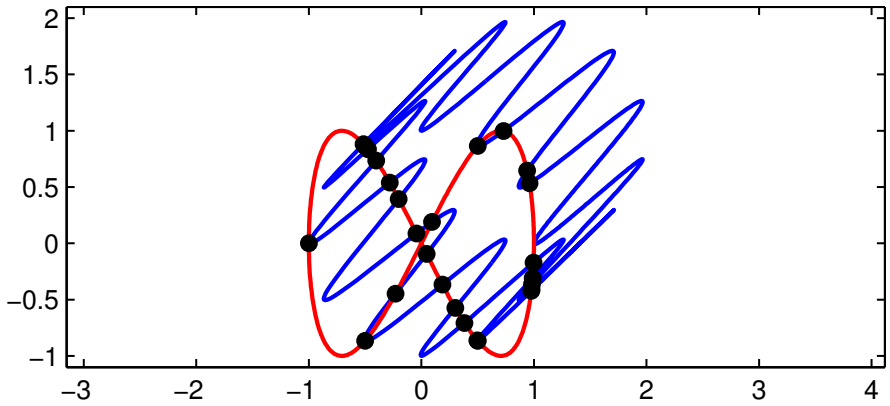


The solution to bivariate polynomial systems and intersections of curves, are typically computed to full machine precision.

13.3 Intersections of curves

The problem of determining the intersections of real parameterised complex curves can be expressed as a bivariate rootfinding problem. For instance, here are the intersections between the 'splat' curve [Guettel Example 2010] and a 'figure-of-eight' curve.

```
t = chebfun('t',[0,2*pi]);
sp = exp(1i*t) + (1+1i)*sin(6*t).^2;     % splat curve
figof8 = cos(t) + 1i*sin(2*t);           % figure of eight curve
plot(sp,LW,1.6), hold on
plot(figof8,'r',LW,1.6), axis equal

d = [0 2*pi 0 2*pi];
f = chebfun2(@(s,t) sp(t)-figof8(s),d);  % rootfinding
r = roots(real(f),imag(f));              % calculate intersections
spr = sp(r(:,2));
plot(real(spr),imag(spr),'.k',MS,20), ylim([-1.1 2.1])
hold off
```

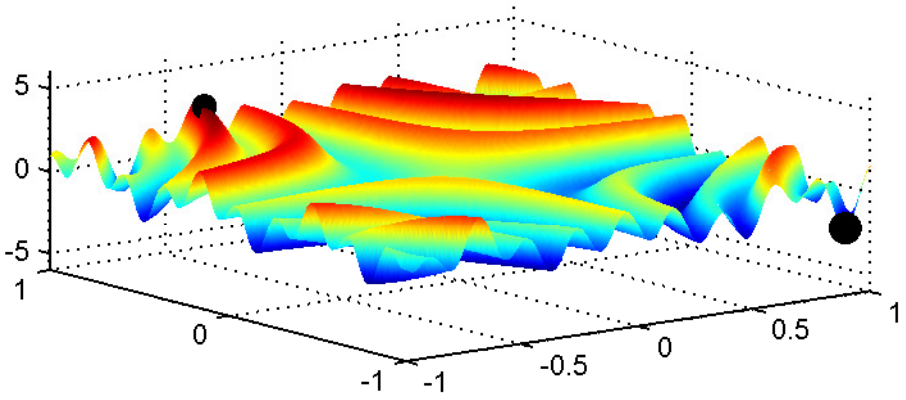


Chebfun2 rootfinding is based on an algorithm described in [Nakatsukasa, Noferini & Townsend 2013].

13.4 Global optimisation: `max2`, `min2`, and `minandmax2`

Chebfun2 also provides functionality for global optimisation. Here is a non-trivial example, where we plot the computed minimum and maximum as black dots.

```
f = chebfun2(@(x,y) sin(30*x.*y) + sin(10*y.*x.^2) + exp(-x.^2-(y-.8).^2));
[mn mnloc] = min2(f);
[mx mxloc] = max2(f);
plot(f), hold on
plot3(mnloc(1),mnloc(2),mn,'.k',MS,40)
plot3(mxloc(1),mxloc(2),mx,'.k',MS,30)
zlim([-6 6]), hold off
```



If both the global maximum and minimum are required, it is roughly twice as fast to compute them at the same time by using the `minandmax2` command. For instance,

```
tic; [mn mnloc] = min2(f);  [mx mxloc] = max2(f); t=toc;
fprintf('min2 and max2 separately = %5.3fs\n',t)
tic; [Y X] = minandmax2(f); t=toc;
fprintf('minandmax2 command = %5.3fs\n',t)
```

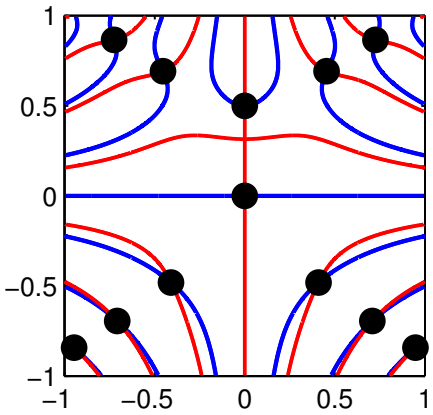
```
min2 and max2 separately = 0.292s
minandmax2 command = 0.142s
```

The commands `min2`, `max2`, and `minandmax2` run faster if the MATLAB Optimisation Toolbox is available.

13.5 Critical points

The critical points of smooth function of two variables can be located by finding the zeros of $\partial f/\partial y = \partial f/\partial x = 0$. This is a rootfinding problem. For example,

```
f = chebfun2(@(x,y) (x.^2-y.^3+1/8).*sin(10*x.*y));
r = roots(gradient(f));                        % critical points
plot(roots(diff(f,1,2)),'b',LW,1.6), hold on  % plot zero contours of f_x
plot(roots(diff(f)),'r')                       % plot zero contours of f_y
plot(r(:,1),r(:,2),'k.','MarkerSize',30)       % plot extrema
axis([-1,1,-1,1]), axis square
```



There is a new command here called gradient that computes the gradient vector and represents it as a chebfun2v object. The roots command then solves for the isolated roots of the bivariate polynomial system represented in the chebfun2v representing the gradient. For more information about the gradient command see Chapter 14 of this guide.

13.6 Infinity norm

The ∞-norm of a function is the maximum absolute value in its domain. It can be computed by passing an optional argument to the norm command.

```
f = chebfun2(@(x,y) sin(30*x.*y));
norm(f,inf)
```

```
ans =
   0.999999999999995
```

13.7 References

[Guettel Example 2010] S. Guettel,

`http://www2.maths.ox.ac.uk/chebfun/examples/geom/html/Area.shtml`

[Nakatsukasa, Noferini & Townsend 2013] Y. Nakatsukasa, V. Noferini and A. Townsend, "Computing the common zeros of two bivariate functions via Bezout resultants", *Numerische Mathematik*, to appear.

[Trott 2007] M. Trott, Applying Groebner Basis to Three Problems in Geometry, *Mathematica in Education and Research*, 6 (1997), pp.15-28.

14. Chebfun2: Vector Calculus

Alex Townsend, March 2013, latest revision June 2014

Contents

14.1 What is a chebfun2v?

Chebfun2 objects represent vector-valued functions. We use a lower case letter like f for a chebfun2 and an upper case letter like F for a chebfun2v.

Chebfun2 represents a vector-valued function $F(x,y) = (f(x,y); g(x,y))$ by approximating each component by a low rank approximant. There are two ways to form a chebfun2v: either by explicitly calling the constructor, or by vertical concatenation of two chebfun2 objects. Here are these two alternatives:

```
d = [0 1 0 2];
F = chebfun2v(@(x,y) sin(x.*y), @(x,y) cos(y), d);  % calling the constructor
f = chebfun2(@(x,y) sin(x.*y), d);
g = chebfun2(@(x,y) cos(y), d);
G = [f;g]                                           % vertical concatenation
```

```
G =
   chebfun2v object (Column vector) containing:
   chebfun2 object: (1 smooth surface)
       domain                 rank       corner values
[   0,   1] x [   0,   2]        7     [3.3e-32 -1.5e-16 -1.8e-16 0.91]
vertical scale =   1
   chebfun2 object: (1 smooth surface)
       domain                 rank       corner values
[   0,   1] x [   0,   2]        1     [   1    1 -0.42 -0.42]
vertical scale =   1
```

Note that displaying a chebfun2v shows that it is a vector of two chebfun2 objects.

14.2 Algebraic operations

Chebfun2 objects are useful for performing 2D vector calculus. The basic algebraic operations are scalar multiplication, vector addition, dot product and cross product.

Scalar multiplication is the product of a scalar function with a vector function:

```
f = chebfun2(@(x,y) exp(-(x.*y).^2/20), d);
f*F
```

```
ans =
   chebfun2v object (Column vector) containing:
   chebfun2 object: (1 smooth surface)
       domain                 rank       corner values
[   0,   1] x [   0,   2]        8     [7.3e-32 -2.1e-16 -2.7e-16 0.74]
vertical scale = 0.89
   chebfun2 object: (1 smooth surface)
       domain                 rank       corner values
[   0,   1] x [   0,   2]        7     [   1    1 -0.42 -0.34]
vertical scale =   1
```

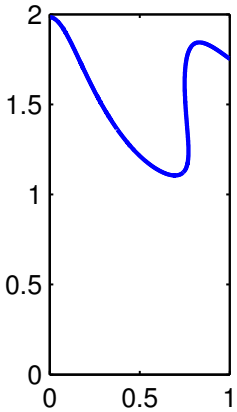
Vector addition yields another chebfun2v and satisfies the parallelogram law:

```
plaw = abs((2*norm(F)^2 + 2*norm(G)^2) - (norm(F+G)^2 + norm(F-G)^2));
fprintf('Parallelogram law holds with error = %10.5e\n',plaw)
```

```
Parallelogram law holds with error = 0.00000e+00
```

The dot product combines two vector functions into a scalar function. If the dot product of two chebfun2v objects takes the value zero at some (x,y), then the vector-valued functions are orthogonal at (x,y). For example, the following code segment determines a curve along which two vector-valued functions are orthogonal:

```
LW = 'linewidth';
F = chebfun2v(@(x,y) sin(x.*y), @(x,y) cos(y),d);
G = chebfun2v(@(x,y) cos(4*x.*y), @(x,y) x + x.*y.^2,d);
plot(roots(dot(F,G)),LW,1.6), axis equal, axis(d)
```



The cross product for 2D vector fields works as follows.

```
help chebfun2v/cross
```

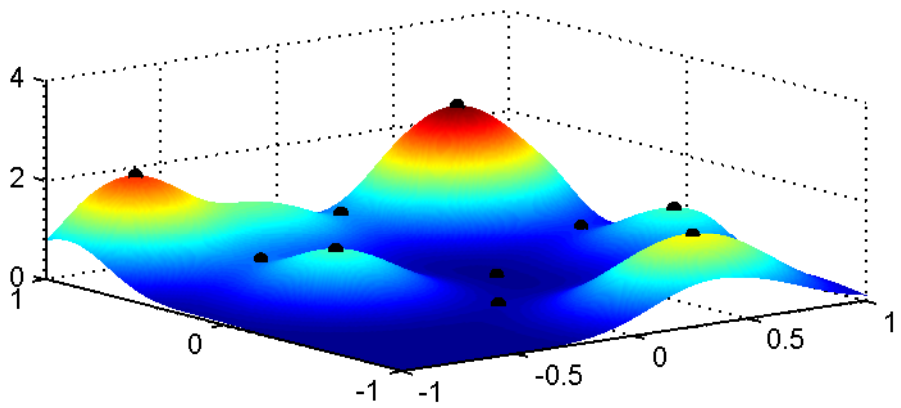
```
 CROSS   Vector cross product.
    CROSS(F, G) returns the cross product of the CHEBFUN2V objects F and G. If F
    and G both have two components, then it returns the CHEBFUN2 representing
        CROSS(F,G) = F(1) * G(2) - F(2) * G(1)
    where F = (F(1); F(2)) and G = (G(1); G(2)). If F and G have three
    components then it returns the CHEBFUN2V representing the 3D cross
    product.
```

14.3 Differential operators

Vector calculus also involves various differential operators defined on scalar- or vector-valued functions such as gradient, curl, divergence, and Laplacian.

The gradient of a chebfun2 representing a scalar function $f(x, y)$ is, geometrically, the direction and magnitude of steepest ascent of f. If the gradient of f is 0 at (x, y), then f has a critical point at (x, y). Here are the critical points of a sum of Gaussian bumps:

```
f = chebfun2(0);
rng('default')
for k = 1:10
    x0 = 2*rand-1; y0 = 2*rand-1;
    f = f + chebfun2(@(x,y) exp(-10*((x-x0).^2 + (y-y0).^2)));
end
plot(f), hold on
r = roots(gradient(f));
plot3(r(:,1),r(:,2),f(r(:,1),r(:,2)),'k.','markersize',20)
zlim([0 4]), hold off
```



The curl of 2D vector function is a scalar function defined as follows.

```
help chebfun2v/curl
```

```
 CURL  curl of a CHEBFUN2V
    S = CURL(F) returns the CHEBFUN2 of the curl of F. If F is a CHEBFUN2V with
    two components then it returns the CHEBFUN2 representing
          CURL(F) = F(2)_x - F(1)_y,
    where F = (F(1),F(2)).  If F is a CHEBFUN2V with three components then it
    returns the CHEBFUN2V representing the 3D curl operation.
```

If the chebfun2v F describes a vector velocity field of fluid flow, for example, then `curl(F)` is the scalar function equal to twice the angular speed of a particle in the flow at each point. A particle moving in a gradient field has zero angular speed and hence, the curl of the gradient is zero. We can check this numerically:

```
norm(curl(gradient(f)))
```

```
ans =
     3.449144990253006e-15
```

The divergence of a chebfun2v is defined as follows.

```
help chebfun2v/divergence
```

```
 DIVERGENCE   Divergence of a CHEBFUN2V.
    DIVERGENCE(F) returns the divergence of the CHEBFUN2V i.e.,
        divergence(F) = F_x + F_y.
```

This measures a vector field's distribution of sources or sinks. The Laplacian is closely related and is the divergence of the gradient,

```
norm(laplacian(f) - divergence(gradient(f)))
```

```
ans =
     0
```

14.4 Line integrals

Given a vector field F, we can compute the line integral along a curve with the command `integral`, defined as follows.

```
help chebfun2v/integral
```

```
 INTEGRAL   Line integration of a CHEBFUN2V.

    INTEGRAL(F, C) computes the line integral of F along the curve C, i.e.,

                              /
        INTEGRAL(F, C) =      |  < F(r), dr >
                              /
                             C

    where the curve C is parameterised by the complex curve r(t).
```

The gradient theorem says that if F is a gradient field, then the line integral along a smooth curve only depends on the end points of that curve. We can check this numerically:

```
f = chebfun2(@(x,y) cos(10*x.*y.^2) + exp(-x.^2));  % chebfun2
F = gradient(f);                                    % gradient (chebfun2v)
C = chebfun(@(t) t.*exp(10i*t),[0 1]);              % spiral curve
v = integral(F,C);ends = f(cos(10),sin(10))-f(0,0); % line integral
abs(v-ends)                                         % gradient theorem
```

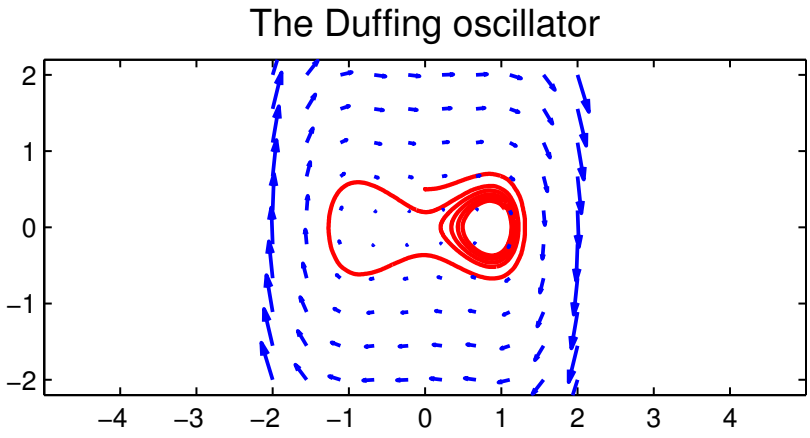
```
ans =
     1.332267629550188e-15
```

14.5 Phase diagram

A phase diagram is a graphical representation of a system of trajectories for a two variable autonomous dynamical system. Chebfun2 plots phase diagrams with `quiver` command, which has been overloaded to plot the vector field. Note that there is a potential terminological ambiguity in that a "phase portrait" can also refer to a portrait of a complex-valued function (see section 11.7 of the guide).

In addition, Chebfun2 makes it easy to compute and plot individual trajectories of a vector field. If F is a chebfun2v, then `ode45(F,tspan,y0)` computes a trajectory of the autonomous system $dx/dt = f(x, y)$, $dy/dt = g(x, y)$, where f and g are the first and second components of F. Given a prescribed time interval and initial conditions, this command returns a complex-valued chebfun representing the trajectory in the form $x(t) + iy(t)$. For example:

```
d = 0.04; a = 1; b = -.75;
F = chebfun2v(@(x,y)y, @(x,y)-d*y - b*x - a*x.^3, [-2 2 -2 2]);
[t y] = ode45(F,[0 40],[0,.5]);
plot(y,'r',LW,1.6), hold on,
quiver(F,'b'), axis equal
title('The Duffing oscillator','FontSize',14), hold off
```


The Duffing oscillator
2
1
0
−1
−2
−4
−3
−2
−1
0
1
2
3
4

15. Chebfun2: 2D Surfaces in 3D Space

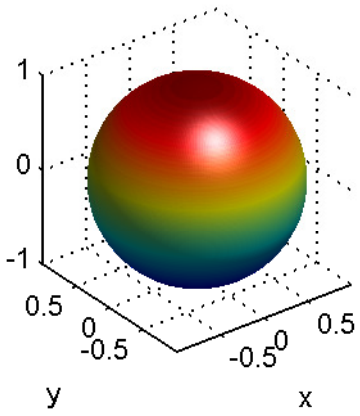
Alex Townsend, March 2013, latest revision June 2014

Contents

15.1 Representing parametric surfaces

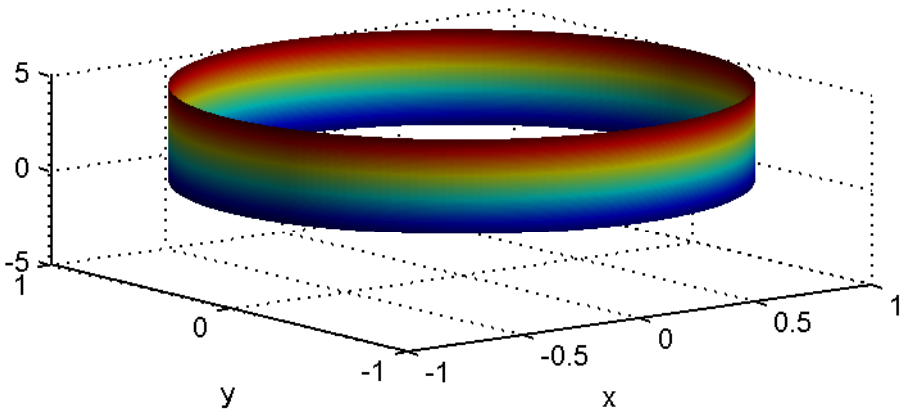
In Chapter 14, we explored chebfun2v objects with two components, but Chebfun2 can also work with functions with three components, i.e., functions from a rectangle in R^2 into R^3. For example, we can represent the unit sphere via spherical coordinates as follows:

```
th = chebfun2(@(th,phi) th, [0 pi 0 2*pi]);
phi = chebfun2(@(th,phi) phi, [0 pi 0 2*pi]);

x = sin(th).*cos(phi);
y = sin(th).*sin(phi);
z = cos(th);

F = [x;y;z];
surf(F), camlight, axis equal
```



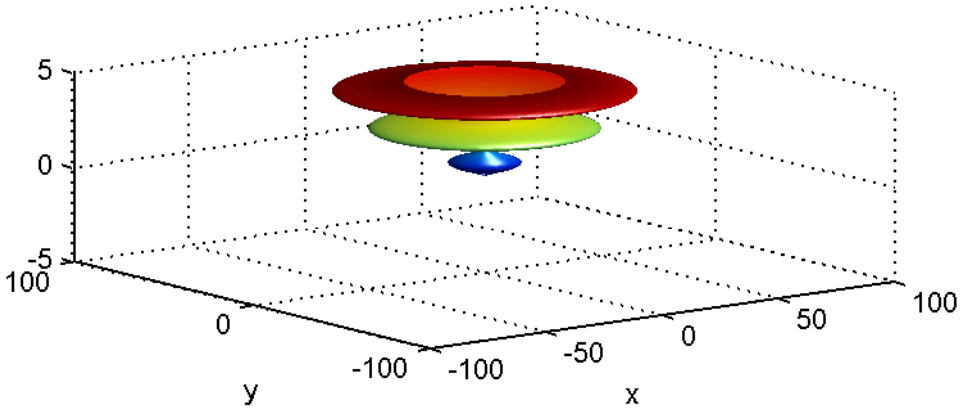
Above, we have formed a chebfun2v with three components by vertical concatenation of chebfun2 objects. However, for familiar surfaces such as cylinders, spheres, and ellipsoids Chebfun2 has overloads of the commands `cylinder`, `sphere`, and `ellipsoid` to generate these surfaces more easily. For example, a cylinder of radius 1 and height 5 can be constructed like this:

```
h = 5;
r = chebfun(@(th) 1+0*th,[0 h]);
F = cylinder(r);
surf(F), camlight
```



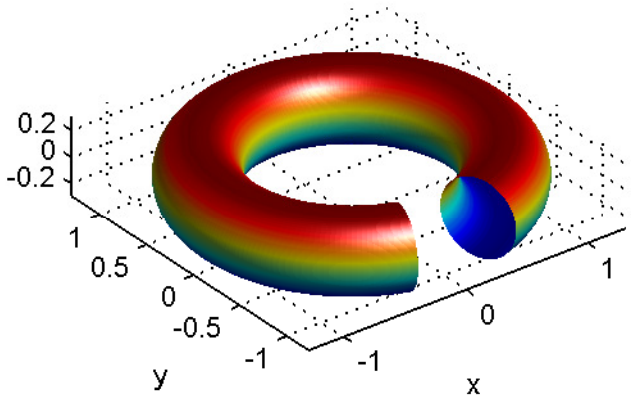
An important class of parametric surfaces are surfaces of revolution, which are formed by revolving a curve in the left half plane about the z-axis. The `cylinder` command can be used to generate surfaces of revolution. For example:

```
f = chebfun(@(t) (sin(pi*t)+1.1).*t.*(t-10),[0 5]);
F = cylinder(f);
surf(F), camlight
```



Here as another example is a torus with a gap in it.

```
x = chebfun2(@(x,y) x); y = chebfun2(@(x,y) y);
theta = 0.9*pi*x; phi = pi*y;
F = [-(1+.3*cos(phi)).*sin(theta);
      (1+.3*cos(phi)).*cos(theta);
      .3*sin(phi)];
surf(F), axis equal, camlight
```

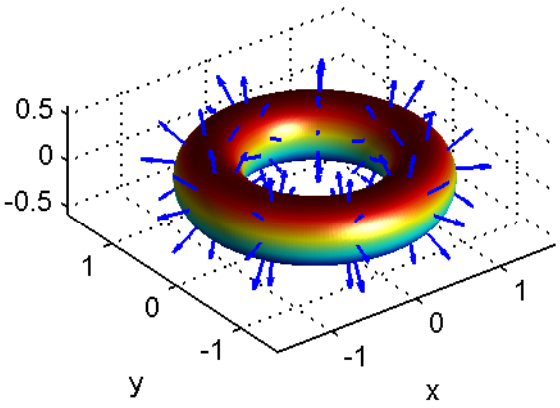


15.2 Surface normals and the divergence theorem

Given a chebfun2v representing a surface, the normal can be computed by the Chebfun2 `normal` command. Here are the normal vectors of another torus:

```
r1 = 1; r2 = 1/3;   % inner and outer radius
d = [0 2*pi 0 2*pi];
u = chebfun2(@(u,v) u,d);
v = chebfun2(@(u,v) v,d);
F = [-(r1+r2*cos(v)).*sin(u); (r1+r2*cos(v)).*cos(u); r2*sin(v)];  % torus

surf(F), camlight, hold on
```

```
quiver3(F(1),F(2),F(3),normal(F,'unit'),'numpts',10)
axis equal, hold off
```



Once we have the surface normal vectors we can compute, for instance, the volume of the torus by applying the divergence theorem:

$$\int\int_V\int div(G)dV = \int_S\int G \cdot d\mathbf{S},$$

where $div(G) = 1$. Instead of integrating over the 3D volume, which is currently not possible in Chebfun2, we integrate over the 2D surface:

```
G = F./3;  % full 3D divergence of G is 1 because F = [x;y;z].
integral2(dot(G,normal(F)))
exact = 2*pi^2*r1*r2.^2
```

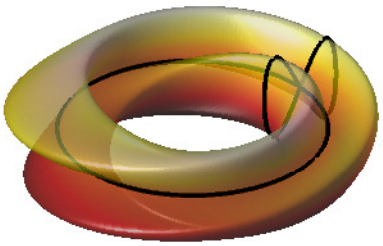
```
ans =
   2.193245422464302
exact =
   2.193245422464302
```

Chebfun2v objects with three components come with a warning. Chebfun2 works with functions of two real variables and therefore, operations such as curl and divergence (in 2D) have little physical meaning to the represented 3D surface. The reason we can compute the volume of the torus (above) is because we are using the divergence theorem and circumventing the 3D divergence.

To finish this section we represent the Klein Bagel. The solid black line shows the parameterisation seam and is displayed with the syntax `surf(F,'-')`. See [Platte 2013] for more on parameterised surfaces.

```
u = chebfun2(@(u,v) u, [0 2*pi 0 2*pi]);
v = chebfun2(@(u,v) v, [0 2*pi 0 2*pi]);
x=(3+cos(u/2).*sin(v)-sin(u/2).*sin(2*v)).*cos(u);
y=(3+cos(u/2).*sin(v)-sin(u/2).*sin(2*v)).*sin(u);
z=sin(u/2).*sin(v)+cos(u/2).*sin(2*v);
```

```
surf([x;y;z],'-k','FaceAlpha',.6), camlight left, colormap(hot)
axis tight equal off
```



15.3 References

[Platte 2013] R. Platte, Parameterizable surfaces, Chebfun2 Example:

http://www2.maths.ox.ac.uk/chebfun/examples/geom/html/ParametricSurfaces.shtml