Cambridge Assessment International Examination

Further Mathematics 9231/1 Paper 1 Further Pure Mathematics 1

Topical Past Year

from CAIE and other Exam Boards

For Syllabus Year 2020

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9231/1 Roots of Polynomial Equations

CAIE/9231/01/MJ02/Q5

The roots of the equation $x^3 - 3x^2 + 1 = 0$ are denoted by α , β , γ . Show that the equation whose roots are

$$\frac{\alpha}{\alpha-2}$$
, $\frac{\beta}{\beta-2}$, $\frac{\gamma}{\gamma-2}$

is
$$3y^3 - 9y^2 - 3y + 1 = 0$$
. [3]

Find the value of

(i)
$$(\alpha - 2)(\beta - 2)(\gamma - 2)$$
, [3]

(ii)
$$\alpha(\beta-2)(\gamma-2) + \beta(\gamma-2)(\alpha-2) + \gamma(\alpha-2)(\beta-2)$$
. [2]



[3]

CAIE/9231/01/ON02/Q2

The equation

$$x^4 + x^3 + Ax^2 + 4x - 2 = 0$$

where A is a constant, has roots α , β , γ , δ . Find a polynomial equation whose roots are

$$\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}, \frac{1}{\delta}.$$
 [2]

Given that

$$\alpha^2 + \beta^2 + \gamma^2 + \delta^2 = \frac{1}{\alpha^2} + \frac{1}{\beta^2} + \frac{1}{\gamma^2} + \frac{1}{\delta^2},$$

find the value of A.

$$2u^4 - 4u^3 - Au^2 - u - 1 = 0$$
; $A = -1$

CAIE/9231/01/MJ03/Q2

The equation

$$8x^3 + 12x^2 + 4x - 1 = 0$$

has roots α , β , γ . Show that the equation with roots $2\alpha + 1$, $2\beta + 1$, $2\gamma + 1$ is

$$y^3 - y - 1 = 0. ag{3}$$

The sum $(2\alpha + 1)^n + (2\beta + 1)^n + (2\gamma + 1)^n$ is denoted by S_n . Find the values of S_3 and S_{-2} . [5]

$$S_3 = 3$$
, $S_{-2} = 1$

CAIE/9231/01/ON03/Q6

Find the sum of the squares of the roots of the equation

$$x^3 + x + 12 = 0,$$

and deduce that only one of the roots is real.

[4]

The real root of the equation is denoted by α . Prove that $-3 < \alpha < -2$, and hence prove that the modulus of each of the other roots lies between 2 and $\sqrt{6}$. [5]



CAIE/9231/01/MJ04/Q110

The roots of the equation

$$x^3 - x - 1 = 0$$

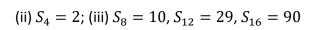
are α , β , γ , and

$$S_n = \alpha^n + \beta^n + \gamma^n.$$

(i) Use the relation $y = x^2$ to show that α^2 , β^2 , γ^2 are the roots of the equation

$$y^3 - 2y^2 + y - 1 = 0. ag{3}$$

- (ii) Hence, or otherwise, find the value of S_4 . [2]
- (iii) Find the values of S_8 , S_{12} and S_{16} . [9]



CAIE/9231/01/ON04/Q3

Given that

$$\alpha + \beta + \gamma = 0$$
, $\alpha^2 + \beta^2 + \gamma^2 = 14$, $\alpha^3 + \beta^3 + \gamma^3 = -18$,

find a cubic equation whose roots are α , β , γ .

[4]

Hence find possible values for α , β , γ .

[2]

$x^3 - 7x + 6 = 0$; 1, 2, -3

CAIE/9231/01/MJ05/Q4

Show that the sum of the cubes of the roots of the equation

$$x^3 + \lambda x + 1 = 0$$

is
$$-3$$
.

Show also that there is no real value of λ for which the sum of the fourth powers of the roots is negative.

CAIE/9231/01/ON05/Q5

In the equation

$$x^3 + ax^2 + bx + c = 0,$$

the coefficients a, b and c are real. It is given that all the roots are real and greater than 1.

- (i) Prove that a < -3.
- (ii) By considering the sum of the squares of the roots, prove that $a^2 > 2b + 3$. [2]
- (iii) By considering the sum of the cubes of the roots, prove that $a^3 < -9b 3c 3$. [4]

CAIE/9231/01/MJ06/Q11E

Obtain the sum of the squares of the roots of the equation

$$x^4 + 3x^3 + 5x^2 + 12x + 4 = 0.$$
 [2]

Deduce that this equation does not have more than 2 real roots. [3]

Show that, in fact, the equation has exactly 2 real roots in the interval -3 < x < 0. [5]

Denoting these roots by α and β , and the other 2 roots by γ and δ , show that $|\gamma| = |\delta| = \frac{2}{\sqrt{(\alpha\beta)}}$. [4]

CAIE/9231/01/ON06/Q6

The roots of the equation

$$x^3 + x + 1 = 0$$

are α , β , γ . Show that the equation whose roots are

$$\frac{4\alpha+1}{\alpha+1}$$
, $\frac{4\beta+1}{\beta+1}$, $\frac{4\gamma+1}{\gamma+1}$

is of the form

$$y^3 + py + q = 0,$$

where the numbers p and q are to be determined.

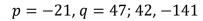
[5]

Hence find the value of

$$\left(\frac{4\alpha+1}{\alpha+1}\right)^n+\left(\frac{4\beta+1}{\beta+1}\right)^n+\left(\frac{4\gamma+1}{\gamma+1}\right)^n,$$

for n = 2 and for n = 3.

[4]



CAIE/9231/01/MJ07/Q7

The equation

$$x^3 + 3x - 1 = 0$$

has roots α , β , γ . Use the substitution $y = x^3$ to show that the equation whose roots are α^3 , β^3 , γ^3 is

$$y^3 - 3y^2 + 30y - 1 = 0.$$
 [2]

Find the value of
$$\alpha^9 + \beta^9 + \gamma^9$$
. [5]



CAIE/9231/01/ON07/Q4

The roots of the equation

$$x^3 - 8x^2 + 5 = 0$$

are α , β , γ . Show that

$$\alpha^2 = \frac{5}{\beta + \gamma}.$$
 [4]

It is given that the roots are all real. Without reference to a graph, show that one of the roots is negative and the other two roots are positive. [3]

CAIE/9231/01/MJ08/Q5

The equation

$$x^3 + x - 1 = 0$$

has roots α,β,γ . Show that the equation with roots $\alpha^3,\beta^3,\gamma^3$ is

$$y^3 - 3y^2 + 4y - 1 = 0.$$
 [4]

Hence find the value of $\alpha^6 + \beta^6 + \gamma^6$. [3]

CAIE/9231/01/ON08/Q120

The roots of the equation

$$x^4 - 5x^2 + 2x - 1 = 0$$

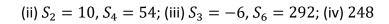
are α , β , γ , δ . Let $S_n = \alpha^n + \beta^n + \gamma^n + \delta^n$.

(i) Show that

$$S_{n+4} - 5S_{n+2} + 2S_{n+1} - S_n = 0. ag{2}$$

- (ii) Find the values of S_2 and S_4 . [3]
- (iii) Find the value of S_3 and hence find the value of S_6 . [6]
- (iv) Hence find the value of

$$\alpha^{2}(\beta^{4}+\gamma^{4}+\delta^{4})+\beta^{2}(\gamma^{4}+\delta^{4}+\alpha^{4})+\gamma^{2}(\delta^{4}+\alpha^{4}+\beta^{4})+\delta^{2}(\alpha^{4}+\beta^{4}+\gamma^{4}). \hspace{1.5cm} [3]$$



CAIE/9231/01/MJ09/Q1

The equation

$$x^4 - x^3 - 1 = 0$$

has roots α , β , γ , δ . By using the substitution $y = x^3$, or by any other method, find the exact value of $\alpha^6 + \beta^6 + \gamma^6 + \delta^6$. [5]

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CAIE/9231/01/ON09/Q5

The equation

$$x^3 + 5x + 3 = 0$$

has roots α , β , γ . Use the substitution $x = -\frac{3}{y}$ to find a cubic equation in y and show that the roots of this equation are $\beta\gamma$, $\gamma\alpha$, $\alpha\beta$.

Find the exact values of $\beta^2 \gamma^2 + \gamma^2 \alpha^2 + \alpha^2 \beta^2$ and $\beta^3 \gamma^3 + \gamma^3 \alpha^3 + \alpha^3 \beta^3$. [5]

CAIE/9231/11/12/MJ10/Q6

The equation

$$x^3 + x - 1 = 0$$

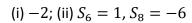
has roots α , β , γ . Use the relation $x = \sqrt{y}$ to show that the equation

$$y^3 + 2y^2 + y - 1 = 0$$

has roots
$$\alpha^2$$
, β^2 , γ^2 . [2]

Let $S_n = \alpha^n + \beta^n + \gamma^n$.

- (i) Write down the value of S_2 and show that $S_4 = 2$. [3]
- (ii) Find the values of S_6 and S_8 . [4]



CAIE/9231/13/MJ10/Q10

The equation

$$x^4 + x^3 + cx^2 + 4x - 2 = 0$$
,

where c is a constant, has roots α , β , γ , δ .

- (i) Use the substitution $y = \frac{1}{x}$ to find an equation which has roots $\frac{1}{\alpha}$, $\frac{1}{\beta}$, $\frac{1}{\gamma}$, $\frac{1}{\delta}$.
- (ii) Find, in terms of c, the values of $\alpha^2 + \beta^2 + \gamma^2 + \delta^2$ and $\frac{1}{\alpha^2} + \frac{1}{\beta^2} + \frac{1}{\gamma^2} + \frac{1}{\delta^2}$. [3]
- (iii) Hence find

$$\left(\alpha - \frac{1}{\alpha}\right)^2 + \left(\beta - \frac{1}{\beta}\right)^2 + \left(\gamma - \frac{1}{\gamma}\right)^2 + \left(\delta - \frac{1}{\delta}\right)^2$$

in terms of c. [2]

(iv) Deduce that when c = -3 the roots of the given equation are not all real. [3]

(i)
$$2y^4 - 4y^3 - cy^2 - y - 1 = 0$$
; (ii) $1 - 2c$, $4 + c$; (iii) $-c - 3$

CAIE/9231/01/ON10/Q7

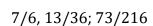
The roots of the equation $x^3 + 4x - 1 = 0$ are α , β and γ . Use the substitution $y = \frac{1}{1+x}$ to show that the equation $6y^3 - 7y^2 + 3y - 1 = 0$ has roots $\frac{1}{\alpha+1}$, $\frac{1}{\beta+1}$ and $\frac{1}{\gamma+1}$.

For the cases n = 1 and n = 2, find the value of

$$\frac{1}{(\alpha+1)^n} + \frac{1}{(\beta+1)^n} + \frac{1}{(\gamma+1)^n}.$$
 [2]

Deduce the value of
$$\frac{1}{(\alpha+1)^3} + \frac{1}{(\beta+1)^3} + \frac{1}{(\gamma+1)^3}.$$
 [2]

Hence show that
$$\frac{(\beta+1)(\gamma+1)}{(\alpha+1)^2} + \frac{(\gamma+1)(\alpha+1)}{(\beta+1)^2} + \frac{(\alpha+1)(\beta+1)}{(\gamma+1)^2} = \frac{73}{36}$$
. [3]



CAIE/9231/11/12/MJ11/Q2

The roots of the equation

$$x^3 + px^2 + qx + r = 0$$

are
$$\frac{\beta}{k}$$
, β , $k\beta$, where p , q , r , k and β are non-zero real constants. Show that $\beta = -\frac{q}{p}$. [4]

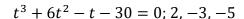
Deduce that
$$rp^3 = q^3$$
. [2]

CAIE/9231/13/MJ11/Q3

Find a cubic equation with roots α , β and γ , given that

$$\alpha + \beta + \gamma = -6,$$
 $\alpha^2 + \beta^2 + \gamma^2 = 38,$ $\alpha \beta \gamma = 30.$ [3]

Hence find the numerical values of the roots. [3]



[5]

[3]

CAIE/9231/11/12/ON11/Q1
The equation $x^3 + px + q = 0$ has a repeated root. Prove that $4p^3 + 27q^2 = 0$.

CAIE/9231/13/ON11/Q3

The equation

$$x^3 + 5x^2 - 3x - 15 = 0$$

has roots α , β , γ . Find the value of $\alpha^2 + \beta^2 + \gamma^2$.

Hence show that the matrix $\begin{pmatrix} 1 & \alpha & \beta \\ \alpha & 1 & \gamma \\ \beta & \gamma & 1 \end{pmatrix}$ is singular. [4]

CAIE/9231/11/12/MJ12/Q1

The roots of the cubic equation $x^3 - 7x^2 + 2x - 3 = 0$ are α , β , γ . Find the values of

(i)
$$\alpha^2 + \beta^2 + \gamma^2$$
, [2]

(ii)
$$\alpha^3 + \beta^3 + \gamma^3$$
. [3]



(i) 45; (ii) 310