Cambridge Assessment International Examination

Further Mathematics 9231/1

Paper 1

Further Pure Mathematics 1

Topical Past Year

from CAIE and other Exam Boards

For Syllabus Year 2020

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9231/1

Roots of Polynomial Equations

CAIE/9231/01/MJ02/Q5

The roots of the equation $x^3 - 3x^2 + 1 = 0$ are denoted by α, β, γ. Show that the equation whose roots are

$$\frac{\alpha}{\alpha - 2}, \frac{\beta}{\beta - 2}, \frac{\gamma}{\gamma - 2}$$

is $3y^3 - 9y^2 - 3y + 1 = 0$. [3]

Find the value of

(i) $(\alpha - 2)(\beta - 2)(\gamma - 2)$, [3]

(ii) $\alpha(\beta - 2)(\gamma - 2) + \beta(\gamma - 2)(\alpha - 2) + \gamma(\alpha - 2)(\beta - 2)$. [2]

(i) 3; (ii) 9

CAIE/9231/01/ON02/Q2
The equation $$x^4 + x^3 + Ax^2 + 4x - 2 = 0,$$ where A is a constant, has roots $\alpha, \beta, \gamma, \delta$. Find a polynomial equation whose roots are $$\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}, \frac{1}{\delta}.$$ [2] Given that $$\alpha^2 + \beta^2 + \gamma^2 + \delta^2 = \frac{1}{\alpha^2} + \frac{1}{\beta^2} + \frac{1}{\gamma^2} + \frac{1}{\delta^2},$$ find the value of A. [3]

$2u^4 - 4u^3 - Au^2 - u - 1 = 0; A = -1$

CAIE/9231/01/MJ03/Q2
The equation $$8x^3 + 12x^2 + 4x - 1 = 0$$ has roots α, β, γ. Show that the equation with roots $2\alpha + 1, 2\beta + 1, 2\gamma + 1$ is $$y^3 - y - 1 = 0.$$ [3] The sum $(2\alpha + 1)^n + (2\beta + 1)^n + (2\gamma + 1)^n$ is denoted by S_n. Find the values of S_3 and S_{-2}. [5]

$S_3 = 3, S_{-2} = 1$

CAIE/9231/01/ON03/Q6
Find the sum of the squares of the roots of the equation $$x^3 + x + 12 = 0,$$ and deduce that only one of the roots is real. [4] The real root of the equation is denoted by α. Prove that $-3 < \alpha < -2$, and hence prove that the modulus of each of the other roots lies between 2 and $\sqrt{6}$. [5]

$\sum \alpha^2 = -2$

CAIE/9231/01/MJ04/Q11O
The roots of the equation $$x^3 - x - 1 = 0$$ are α, β, γ, and $$S_n = \alpha^n + \beta^n + \gamma^n.$$ (i) Use the relation $y = x^2$ to show that α^2, β^2, γ^2 are the roots of the equation $$y^3 - 2y^2 + y - 1 = 0.$$ [3] (ii) Hence, or otherwise, find the value of S_4. [2] (iii) Find the values of S_8, S_{12} and S_{16}. [9]

(ii) $S_4 = 2$; (iii) $S_8 = 10, S_{12} = 29, S_{16} = 90$

CAIE/9231/01/ON04/Q3
Given that $$\alpha+\beta+\gamma=0, \quad \alpha^2+\beta^2+\gamma^2=14, \quad \alpha^3+\beta^3+\gamma^3=-18,$$ find a cubic equation whose roots are α, β, γ. [4] Hence find possible values for α, β, γ. [2]

$x^3-7x+6=0; 1, 2, -3$

CAIE/9231/01/MJ05/Q4
Show that the sum of the cubes of the roots of the equation $$x^3+\lambda x+1=0$$ is -3. [3] Show also that there is no real value of λ for which the sum of the fourth powers of the roots is negative. [3]

CAIE/9231/01/ON05/Q5
In the equation $$x^3 + ax^2 + bx + c = 0,$$ the coefficients a, b and c are real. It is given that all the roots are real and greater than 1. **(i)** Prove that $a < -3$. [1] **(ii)** By considering the sum of the squares of the roots, prove that $a^2 > 2b + 3$. [2] **(iii)** By considering the sum of the cubes of the roots, prove that $a^3 < -9b - 3c - 3$. [4]

CAIE/9231/01/MJ06/Q11E	
Obtain the sum of the squares of the roots of the equation $$x^4 + 3x^3 + 5x^2 + 12x + 4 = 0.$$	[2]
Deduce that this equation does not have more than 2 real roots.	[3]
Show that, in fact, the equation has exactly 2 real roots in the interval $-3 < x < 0$.	[5]
Denoting these roots by α and β, and the other 2 roots by γ and δ, show that $\lvert\gamma\rvert = \lvert\delta\rvert = \dfrac{2}{\sqrt{(\alpha\beta)}}$.	[4]

CAIE/9231/01/ON06/Q6

The roots of the equation

$$x^3 + x + 1 = 0$$

are α, β, γ. Show that the equation whose roots are

$$\frac{4\alpha+1}{\alpha+1}, \quad \frac{4\beta+1}{\beta+1}, \quad \frac{4\gamma+1}{\gamma+1}$$

is of the form

$$y^3 + py + q = 0,$$

where the numbers p and q are to be determined. [5]

Hence find the value of

$$\left(\frac{4\alpha+1}{\alpha+1}\right)^n + \left(\frac{4\beta+1}{\beta+1}\right)^n + \left(\frac{4\gamma+1}{\gamma+1}\right)^n,$$

for $n = 2$ and for $n = 3$. [4]

$p = -21, q = 47; 42, -141$

CAIE/9231/01/MJ07/Q7
The equation $$x^3 + 3x - 1 = 0$$ has roots α, β, γ. Use the substitution $y = x^3$ to show that the equation whose roots are $\alpha^3, \beta^3, \gamma^3$ is $$y^3 - 3y^2 + 30y - 1 = 0.$$ [2] Find the value of $\alpha^9 + \beta^9 + \gamma^9$. [5]

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CAIE/9231/01/ON07/Q4

The roots of the equation

$$x^3 - 8x^2 + 5 = 0$$

are α, β, γ. Show that

$$\alpha^2 = \frac{5}{\beta + \gamma}.$$ [4]

It is given that the roots are all real. Without reference to a graph, show that one of the roots is negative and the other two roots are positive. [3]

CAIE/9231/01/MJ08/Q5

The equation

$$x^3 + x - 1 = 0$$

has roots α, β, γ. Show that the equation with roots α^3, β^3, γ^3 is

$$y^3 - 3y^2 + 4y - 1 = 0.$$ [4]

Hence find the value of $\alpha^6 + \beta^6 + \gamma^6$. [3]

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CAIE/9231/01/ON08/Q12O
The roots of the equation $$x^4 - 5x^2 + 2x - 1 = 0$$ are $\alpha, \beta, \gamma, \delta$. Let $S_n = \alpha^n + \beta^n + \gamma^n + \delta^n$. **(i)** Show that $$S_{n+4} - 5S_{n+2} + 2S_{n+1} - S_n = 0.$$ [2] **(ii)** Find the values of S_2 and S_4. [3] **(iii)** Find the value of S_3 and hence find the value of S_6. [6] **(iv)** Hence find the value of $$\alpha^2(\beta^4 + \gamma^4 + \delta^4) + \beta^2(\gamma^4 + \delta^4 + \alpha^4) + \gamma^2(\delta^4 + \alpha^4 + \beta^4) + \delta^2(\alpha^4 + \beta^4 + \gamma^4).$$ [3]

(ii) $S_2 = 10$, $S_4 = 54$; (iii) $S_3 = -6$, $S_6 = 292$; (iv) 248

CAIE/9231/01/MJ09/Q1
The equation $$x^4 - x^3 - 1 = 0$$ has roots $\alpha, \beta, \gamma, \delta$. By using the substitution $y = x^3$, or by any other method, find the exact value of $\alpha^6 + \beta^6 + \gamma^6 + \delta^6$. [5]

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CAIE/9231/01/ON09/Q5
The equation $$x^3 + 5x + 3 = 0$$ has roots α, β, γ. Use the substitution $x = -\frac{3}{y}$ to find a cubic equation in y and show that the roots of this equation are $\beta\gamma, \gamma\alpha, \alpha\beta$. [4]   Find the exact values of $\beta^2\gamma^2 + \gamma^2\alpha^2 + \alpha^2\beta^2$ and $\beta^3\gamma^3 + \gamma^3\alpha^3 + \alpha^3\beta^3$. [5]

$y^3 - 5y^2 - 9 = 0$; 25, 152

CAIE/9231/11/12/MJ10/Q6
The equation $$x^3 + x - 1 = 0$$ has roots α, β, γ. Use the relation $x = \sqrt{y}$ to show that the equation $$y^3 + 2y^2 + y - 1 = 0$$ has roots $\alpha^2, \beta^2, \gamma^2$. [2]   Let $S_n = \alpha^n + \beta^n + \gamma^n$.   **(i)** Write down the value of S_2 and show that $S_4 = 2$. [3]   **(ii)** Find the values of S_6 and S_8. [4]

(i) -2; (ii) $S_6 = 1, S_8 = -6$

CAIE/9231/13/MJ10/Q10
The equation $$x^4 + x^3 + cx^2 + 4x - 2 = 0,$$ where c is a constant, has roots $\alpha, \beta, \gamma, \delta$.

(i) Use the substitution $y = \frac{1}{x}$ to find an equation which has roots $\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}, \frac{1}{\delta}$. [2]

(ii) Find, in terms of c, the values of $\alpha^2 + \beta^2 + \gamma^2 + \delta^2$ and $\frac{1}{\alpha^2} + \frac{1}{\beta^2} + \frac{1}{\gamma^2} + \frac{1}{\delta^2}$. [3]

(iii) Hence find

$$\left(\alpha - \frac{1}{\alpha}\right)^2 + \left(\beta - \frac{1}{\beta}\right)^2 + \left(\gamma - \frac{1}{\gamma}\right)^2 + \left(\delta - \frac{1}{\delta}\right)^2$$

in terms of c. [2]

(iv) Deduce that when $c = -3$ the roots of the given equation are not all real. [3]

(i) $2y^4 - 4y^3 - cy^2 - y - 1 = 0$; (ii) $1 - 2c$, $4 + c$; (iii) $-c - 3$

CAIE/9231/01/ON10/Q7
The roots of the equation $x^3 + 4x - 1 = 0$ are α, β and γ. Use the substitution $y = \frac{1}{1+x}$ to show that the equation $6y^3 - 7y^2 + 3y - 1 = 0$ has roots $\frac{1}{\alpha+1}$, $\frac{1}{\beta+1}$ and $\frac{1}{\gamma+1}$. [2] For the cases $n = 1$ and $n = 2$, find the value of $$\frac{1}{(\alpha+1)^n} + \frac{1}{(\beta+1)^n} + \frac{1}{(\gamma+1)^n}.$$ [2] Deduce the value of $\frac{1}{(\alpha+1)^3} + \frac{1}{(\beta+1)^3} + \frac{1}{(\gamma+1)^3}$. [2] Hence show that $\frac{(\beta+1)(\gamma+1)}{(\alpha+1)^2} + \frac{(\gamma+1)(\alpha+1)}{(\beta+1)^2} + \frac{(\alpha+1)(\beta+1)}{(\gamma+1)^2} = \frac{73}{36}$. [3]

7/6, 13/36; 73/216

CAIE/9231/11/12/MJ11/Q2
The roots of the equation $$x^3 + px^2 + qx + r = 0$$ are $\frac{\beta}{k}$, β, $k\beta$, where p, q, r, k and β are non-zero real constants. Show that $\beta = -\frac{q}{p}$. [4]
Deduce that $rp^3 = q^3$. [2]

CAIE/9231/13/MJ11/Q3
Find a cubic equation with roots α, β and γ, given that $$\alpha + \beta + \gamma = -6, \qquad \alpha^2 + \beta^2 + \gamma^2 = 38, \qquad \alpha\beta\gamma = 30.$$ [3]
Hence find the numerical values of the roots. [3]

$t^3 + 6t^2 - t - 30 = 0$; $2, -3, -5$

CAIE/9231/11/12/ON11/Q1	
The equation $x^3 + px + q = 0$ has a repeated root. Prove that $4p^3 + 27q^2 = 0$.	[5]

CAIE/9231/13/ON11/Q3	
The equation $$x^3 + 5x^2 - 3x - 15 = 0$$ has roots α, β, γ. Find the value of $\alpha^2 + \beta^2 + \gamma^2$.	[3]
Hence show that the matrix $\begin{pmatrix} 1 & \alpha & \beta \\ \alpha & 1 & \gamma \\ \beta & \gamma & 1 \end{pmatrix}$ is singular.	[4]

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CAIE/9231/11/12/MJ12/Q1	
The roots of the cubic equation $x^3 - 7x^2 + 2x - 3 = 0$ are α, β, γ. Find the values of	
(i) $\alpha^2 + \beta^2 + \gamma^2$,	[2]
(ii) $\alpha^3 + \beta^3 + \gamma^3$.	[3]

(i) 45; (ii) 310