# E1.10 Fourier Series and Transforms

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Main fact: Complicated time waveforms can be expressed as a sum of sine and cosine waves.

Why bother? Sine/cosine are the only bounded waves that stay the same when differentiated.

Any electronic circuit:
sine wave in ⇒ sine wave out (same frequency).

Joseph Fourier
1768-1830

Hard problem: Complicated waveform → electronic circuit→ output = ?

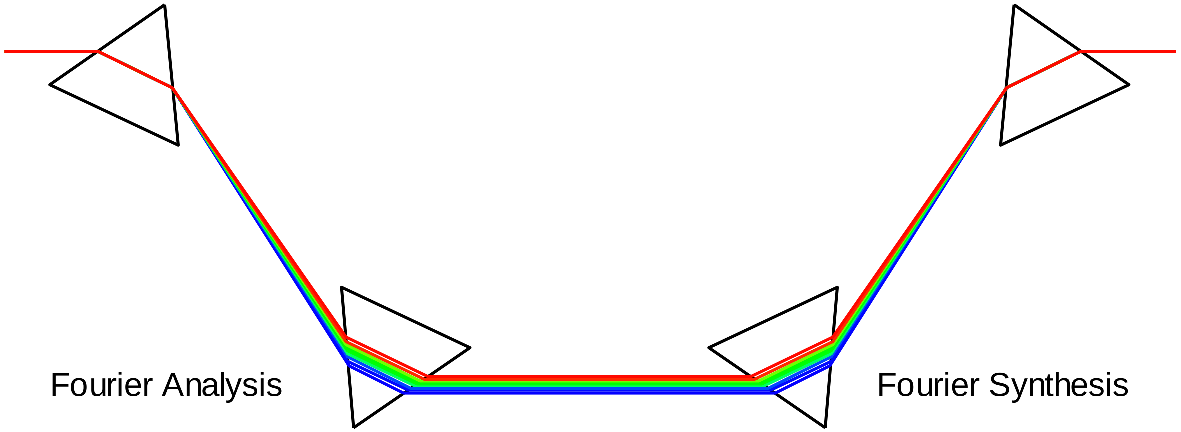
Easier problem: Complicated waveform → sum of sine waves
→ linear electronic circuit (⇒ obeys superposition)
→ add sine wave outputs → output = ?

Syllabus: Preliminary maths (1 lecture)
Fourier series for periodic waveforms (4 lectures)
Fourier transform for aperiodic waveforms (3 lectures)

# Optical Fourier Transform



A pair of prisms can split light up into its component frequencies (colours).
This is called Fourier Analysis.
A second pair can re-combine the frequencies.
This is called Fourier Synthesis.




We want to do the same thing with mathematical signals instead of light.

# Organization



- 8 lectures: feel free to ask questions
- Textbook: Riley, Hobson & Bence "Mathematical Methods for Physics and Engineering", ISBN:978052167971-8, Chapters [4], 12 & 13
- Lecture slides (including animations) and problem sheets + answers available via Blackboard or from my website: http://www.ee.ic.ac.uk/hp/staff/dmb/courses/E1Fourier/E1Fourier.htm
- Email me with any errors in slides or problems and if answers are wrong or unclear

# 1: Sums and Averages

A geometric series is a sum of terms that increase or decrease by a constant factor, $x$:

$$S = a + ax + ax^2 + \ldots + ax^n$$

The sequence of terms themselves is called a geometric progression.

We use a trick to get rid of most of the terms:

$$\begin{aligned} S &= a + ax + ax^2 + \ldots + ax^{n-1} + ax^n \\ xS &= \quad\; ax + ax^2 + ax^3 + \ldots \quad + ax^n + ax^{n+1} \end{aligned}$$

Now subtract the lines to get: $S - xS = (1 - x)\, S = a - ax^{n+1}$

Divide by $1 - x$ to get: $a =$ first term $\quad n + 1 =$ number of terms

$$S = a \times \frac{1-x^{n+1}}{1-x}$$

Example:

$$S = 3 + 6 + 12 + 24 \qquad [a = 3,\ x = 2,\ n + 1 = 4]$$

$$= 3 \times \frac{1-2^4}{1-2} = 3 \times \frac{-15}{-1} = 45$$

# Infinite Geometric Series



A finite geometric series: $S_n = a + ax + ax^2 + \cdots + ax^n = a\frac{1-x^{n+1}}{1-x}$

What is the limit as $n \to \infty$?

If $|x| < 1$ then $x^{n+1} \underset{n\to\infty}{\longrightarrow} 0$ which gives

$$S_\infty = a + ax + ax^2 + \cdots = a\frac{1}{1-x} = \frac{a}{1-x}$$

$a =$ first term

$x =$ factor

Example 1:

$0.4 + 0.04 + 0.004 + \ldots = \frac{0.4}{1-0.1} = 0.\dot{4}$ $[a = 0.4,\ x = 0.1]$

Example 2: (alternating signs)

$2 - 1.2 + 0.72 - 0.432 + \ldots = \frac{2}{1-(-0.6)} = 1.25$ $[a = 2,\ x = -0.6]$

Example 3:

$1 + 2 + 4 + \ldots \neq \frac{1}{1-2} = \frac{1}{-1} = -1$ $[a = 1,\ x = 2]$

The formula $S = a + ax + ax^2 + \ldots = \frac{a}{1-x}$ is only valid for $|x| < 1$

Using a $\sum$ sign, we can write the geometric series more compactly:

$$S_n = a + ax + ax^2 + \ldots + ax^n = \sum_{r=0}^{n} ax^r$$

[Note: $x^0 \triangleq 1$ in this context even when $x = 0$]

Here $r$ is a dummy variable: you can replace it with anything else

$$\sum_{r=0}^{n} ax^r = \sum_{k=0}^{n} ax^k = \sum_{\alpha=0}^{n} ax^\alpha$$

Dummy variables are undefined outside the summation so they sometimes get re-used elsewhere in an expression:

$$\sum_{r=0}^{3} 2^r + \sum_{r=1}^{2} 3^r = \left(1 \times \frac{1-2^4}{1-2}\right) + \left(3 \times \frac{1-3^2}{1-3}\right) = 15 + 12 = 27$$

The two dummy variables are both called $r$ but they have no connection with each other at all (or with any other variable called $r$ anywhere else).

We can derive the formula for the geometric series using $\sum$ notation:

$$S_n = \sum_{r=0}^{n} ax^r \text{ and } xS_n = \sum_{r=0}^{n} ax^{r+1}$$

We need to manipulate the second sum to involve $x^r$.

Use the substitution $s = r+1 \Leftrightarrow r = s-1$.
Substitute for $r$ everywhere it occurs (including both limits)

$$xS_n = \sum_{s=1}^{n+1} ax^s = \sum_{r=1}^{n+1} ax^r$$

It is essential to sum over exactly the same set of values when substituting for dummy variables.

Subtracting gives $(1-x)S_n = S_n - xS_n = \sum_{r=0}^{n} ax^r - \sum_{r=1}^{n+1} ax^r$

$r \in [1, n]$ is common to both sums, so extract the remaining terms:

$$\begin{aligned}(1-x)S_n &= ax^0 - ax^{n+1} + \sum_{r=1}^{n} ax^r - \sum_{r=1}^{n} ax^r \\ &= ax^0 - ax^{n+1} = a\left(1-x^{n+1}\right)\end{aligned}$$

Hence: $S_n = a\frac{1-x^{n+1}}{1-x}$

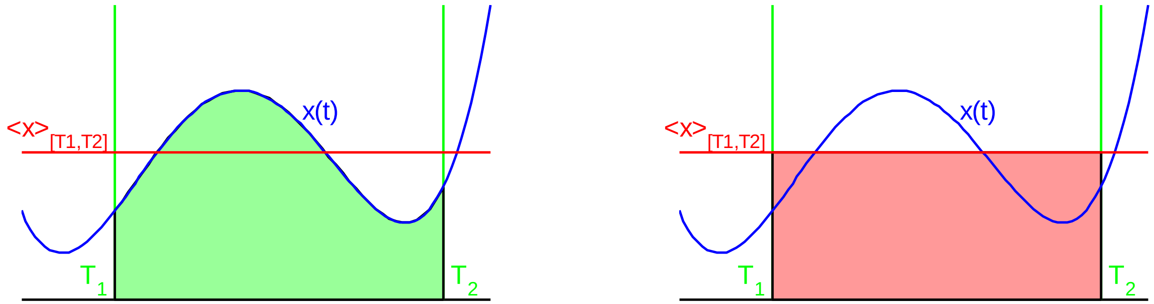
# Averages



If a signal varies with time, we can plot its waveform, $x(t)$.

The average value of $x(t)$ in the range $T_1 \le t \le T_2$ is

$$\langle x \rangle_{[T_1,T_2]} = \frac{1}{T_2 - T_1} \int_{t=T_1}^{T_2} x(t)dt$$




The area under the curve $x(t)$ is equal to the area of the rectangle defined by $0$ and $\langle x \rangle_{[T_1,T_2]}$.

Angle brackets alone, $\langle x \rangle$, denotes the average value over all time

$$\langle x(t) \rangle = \lim_{A,B\to\infty} \langle x(t) \rangle_{[-A,+B]}$$

# Average Properties



The properties of averages follow from the properties of integrals:

Addition: $\langle x(t)+y(t)\rangle = \langle x(t)\rangle + \langle y(t)\rangle$

Add a constant: $\langle x(t)+c\rangle = \langle x(t)\rangle + c$

Constant multiple: $\langle a\times x(t)\rangle = a\times\langle x(t)\rangle$

where the constants $a$ and $c$ do not depend on time.

For example:

$$\begin{aligned}\langle x(t)+y(t)\rangle_{[T_1,T_2]} &= \tfrac{1}{T_2-T_1}\int_{t=T_1}^{T_2}(x(t)+y(t))\,dt\\ &= \tfrac{1}{T_2-T_1}\int_{t=T_1}^{T_2}x(t)dt + \tfrac{1}{T_2-T_1}\int_{t=T_1}^{T_2}y(t)dt\\ &= \langle x(t)\rangle_{[T_1,T_2]} + \langle y(t)\rangle_{[T_1,T_2]}\end{aligned}$$

But beware: $\langle x(t)\times y(t)\rangle \neq \langle x(t)\rangle\times\langle y(t)\rangle$.

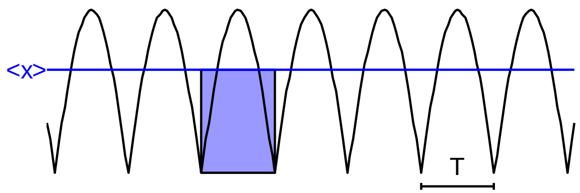
# Periodic Waveforms

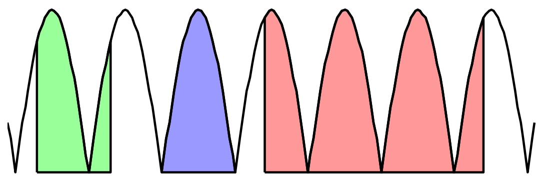


A periodic waveform with period $T$ repeats itself at intervals of $T$:

$$x(t+T) = x(t) \quad \Rightarrow \quad x(t \pm kT) = x(t) \text{ for any integer } k.$$

The smallest $T > 0$ for which $x(t+T) = x(t)\ \forall t$ is the fundamental period. The fundamental frequency is $F = \frac{1}{T}$.



For a periodic waveform, $\langle x(t) \rangle$ equals the average over one period.
It doesn't make any difference where in a period you start or how many whole periods you take the average over.

Example:

$$x(t) = |\sin t|$$

$$\langle x \rangle = \frac{1}{\pi} \int_{t=0}^{\pi} |\sin t|\ dt = \frac{1}{\pi} \int_{t=0}^{\pi} \sin t\, dt$$

$$= \frac{1}{\pi} \left[-\cos t\right]_0^{\pi} = \frac{1}{\pi}(1+1) = \frac{2}{\pi} \approx 0.637$$

# [proof that $x(t \pm kT) = x(t)$]

**Proof that $x(t+T) = x(t)\ \forall t \ \Rightarrow\ x(t \pm kT) = x(t)\ \forall t,\ \forall k \in \mathbb{Z}$**

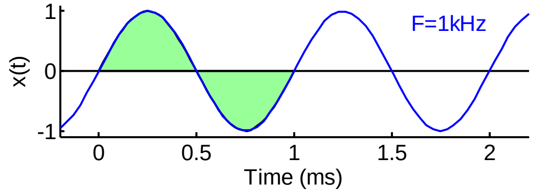
We use induction. Let $H_k$ be the hypothesis that $x(t+kT) = x(t)\ \forall t$. Under the assumption that $x(t+T) = x(t)\ \forall t$, we will show that if $H_k$ is true, then so are $H_{k+1}$ and $H_{k-1}$. Since we know that $H_0$ is definitely true, this implies that $H_k$ is true for all integers $k$, i.e. for all $k \in \mathbb{Z}$.

- ☐ Suppose $H_k$ is true, i.e. $x(\tau + kT) = x(\tau)\ \forall \tau$. Now set $\tau = t + T$. This gives $x(t+T+kT) = x(t+T)\ \forall t$. But, we assume that $x(t+T) = x(t)$, so $x\left(t + (k+1)T\right) = x(t+T+kT) = x(t+T) = x(T)\ \forall t$. Hence $H_{k+1}$is true.
- ☐ Now suppose $H_k$ is true as before but this time set $\tau = t - T$. Substituting this into $u(\tau + kT) = u(\tau)$ gives $u(t-T+kT) = u(t-T)$. Substituting it also into $u(\tau+T) = u(\tau)$ gives $u(t) = u(t-T)$. Finally, combining these two identities gives $u\left(t + (k-1)T\right) = u(t)$ which is $H_{k-1}$.

A sine wave, $x(t) = \sin 2\pi F t$, has a frequency $F$ and a period $T = \frac{1}{F}$

so that, $\sin\left(2\pi F\left(t + \frac{1}{F}\right)\right) = \sin(2\pi F t + 2\pi) = \sin 2\pi F t$.

$$\langle \sin 2\pi F t \rangle = \frac{1}{T}\int_{t=0}^{T} \sin(2\pi F t)\, dt \\ = 0$$




Also, $\langle \cos 2\pi F t \rangle = 0$ except for the case $F = 0$ since $\cos 2\pi 0 t \equiv 1$.

Hence: $\langle \sin 2\pi F t \rangle = 0$ and $\langle \cos 2\pi F t \rangle = \begin{cases} 0 & F \neq 0 \\ 1 & F = 0 \end{cases}$

Also:
$$\left\langle e^{i2\pi F t} \right\rangle = \langle \cos 2\pi F t + i \sin 2\pi F t \rangle \\ = \langle \cos 2\pi F t \rangle + i \langle \sin 2\pi F t \rangle \\ = \begin{cases} 0 & F \neq 0 \\ 1 & F = 0 \end{cases}$$

- Sum of geometric series (see RHB Chapter 4)
  - Finite series: $S = a \times \frac{1-x^{n+1}}{1-x}$
  - Infinite series: $S = \frac{a}{1-x}$ but only if $|x| < 1$
- Dummy variables
  - Commonly re-used elsewhere in expressions
  - Substitutions must cover exactly the same set of values
- Averages: $\langle x \rangle_{[T_1,T_2]} = \frac{1}{T_2-T_1} \int_{t=T_1}^{T_2} x(t)dt$
- Periodic waveforms: $x(t \pm kT) = x(t)$ for any integer $k$
  - Fundamental period is the smallest $T$
  - Fundamental frequency is $F = \frac{1}{T}$
  - For periodic waveforms, $\langle x \rangle$ is the average over any integer number of periods
  - $\langle \sin 2\pi Ft \rangle = 0$
  - $\langle \cos 2\pi Ft \rangle = \left\langle e^{i2\pi Ft} \right\rangle = \begin{cases} 0 & F \neq 0 \\ 1 & F = 0 \end{cases}$