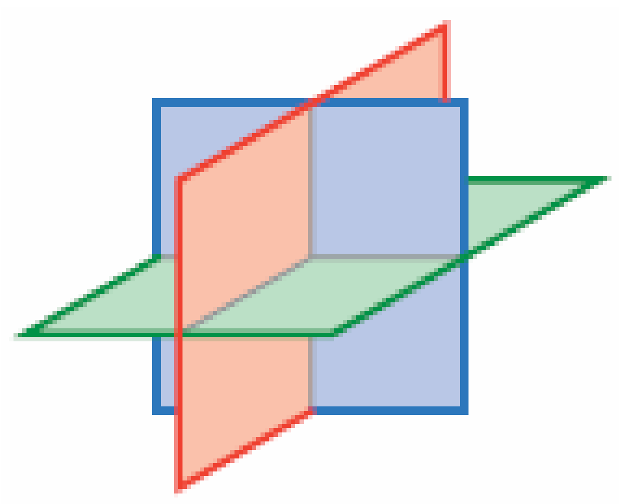
Déjà Vu, It's Algebra 2!

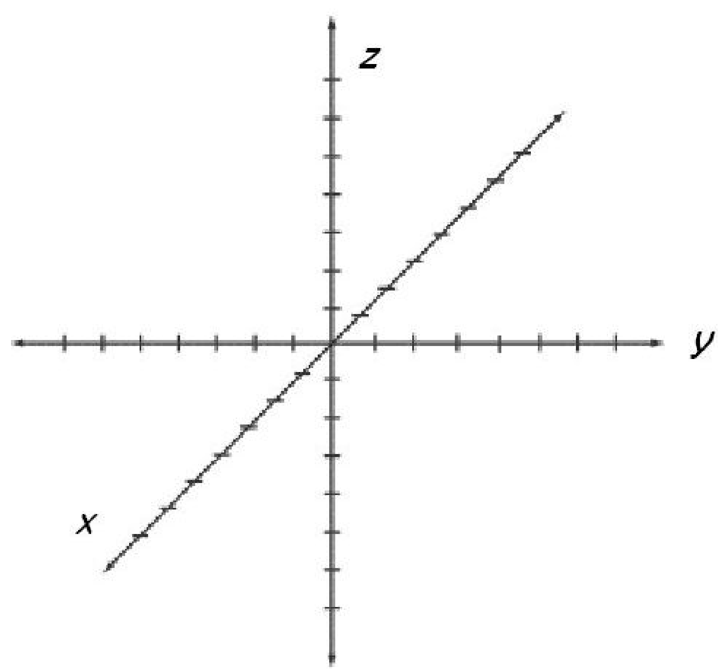
Lesson 07

Linear Systems in 3-D

The Global Positioning System (GPS) gives locations any where on earth by using the three coordinates of latitude, longitude, and elevation.

Any point in a three-dimensional coordinate space can be represented using an ordered triple of the form (x, y, z). The *z*-axis is another axis that extends out perpendicularly from the origin on our 2 dimensional coordinate plane.



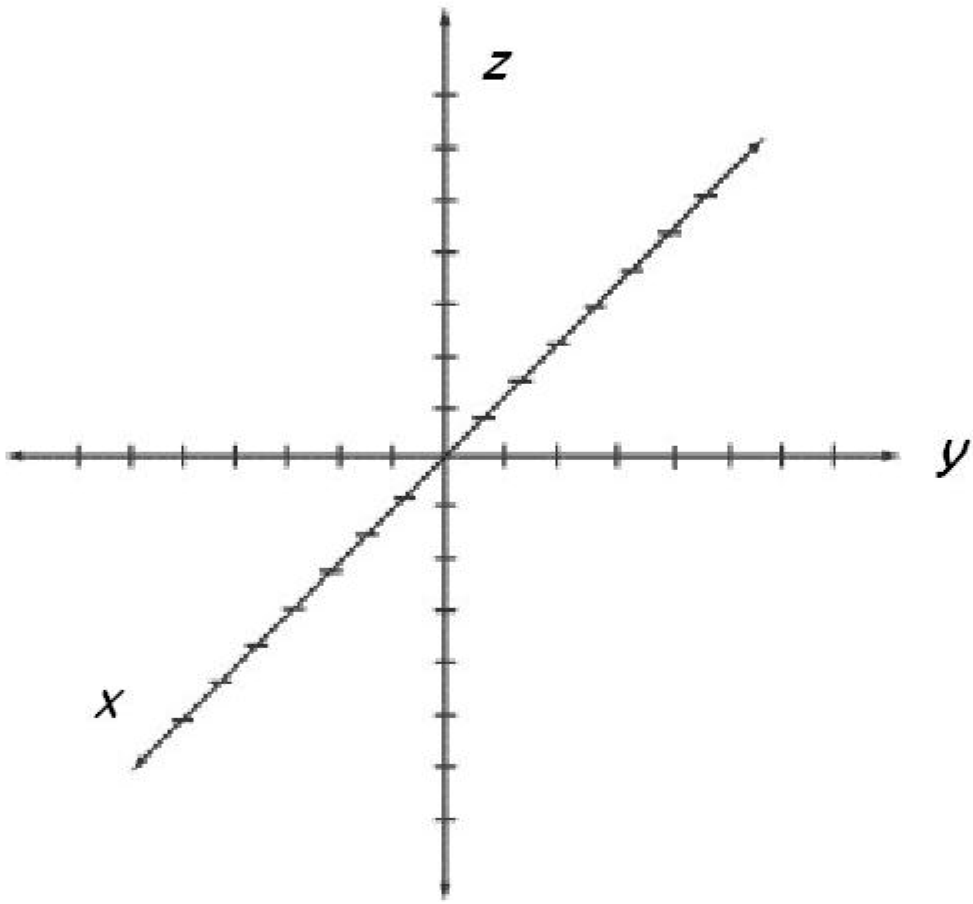


Let's try plotting a few points in this new 3-D space.

1. $(4,6,-4)$

2. $(3,-,2,5)$

3. $(-5,-5,0)$



Recall that the graph of a linear equation in two dimensions is a straight line. In 3-D space, the graph of a linear equation is a plane. Because a plane is defined by three points, you can graph linear equations in 3-D by finding the three intercepts.

Example:

Graph the linear equation $4x + 8y + z = 2$

Find x-intercept:
Let $y, z = 0$
$4x = 2$

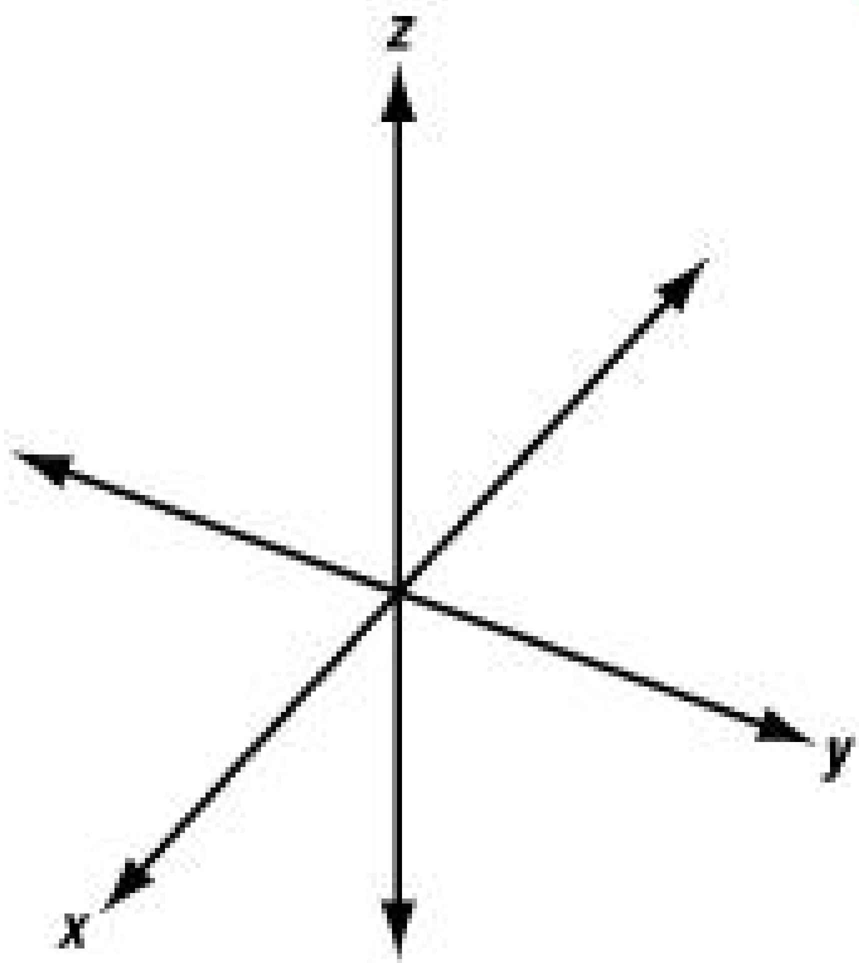
$x = 1/2$

Find y-intercept:
Let $x, z = 0$
$8y = 2$

$y = 1/4$

Find z-intercept:
Let $x, y = 0$
$z = 2$

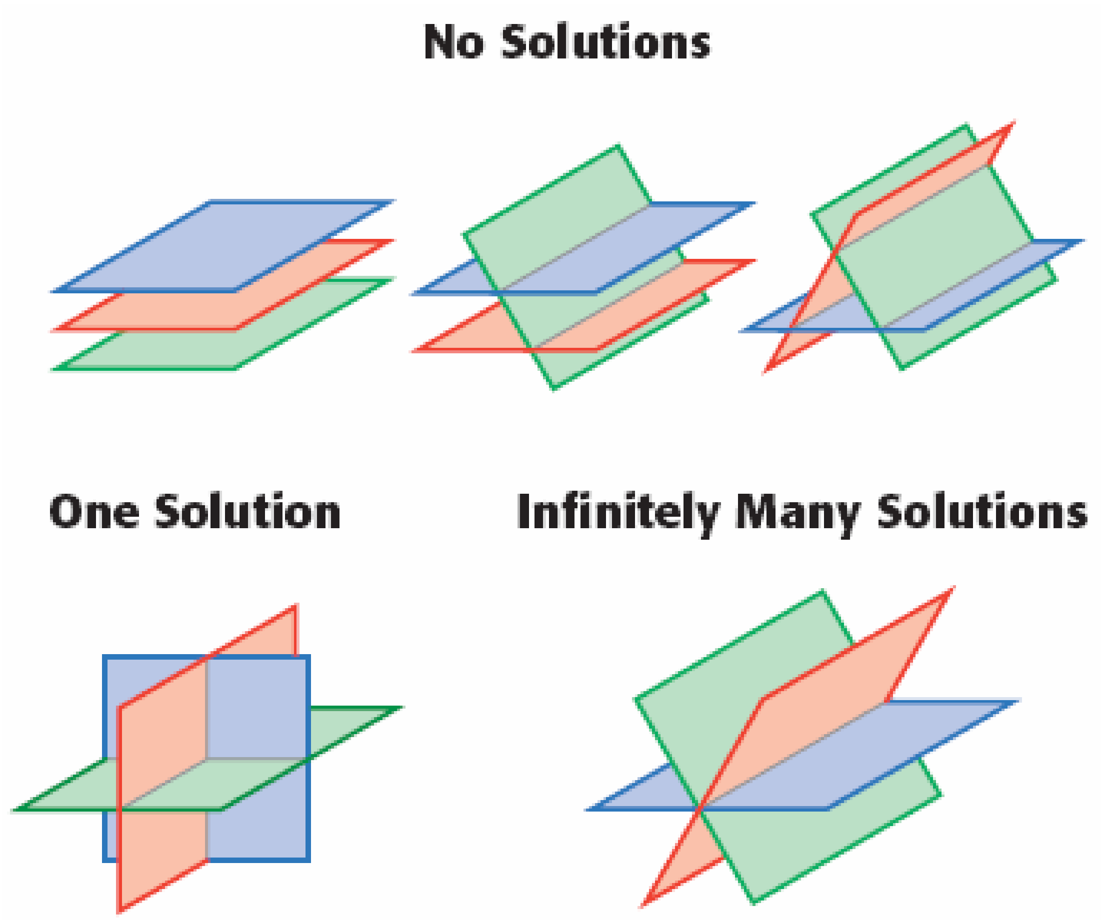
The three points make a triangle which represents PART of the plane. The actual plane can be thought to rest on the triangle. (See Image 1 at end of document.)



Just as we solved systems of equations with linear equations in two dimensions, so we will also do with linear equations in three dimensions.

Remember that in order to find unique solutions to any system of equations, we need as many equations as we have unknowns. Therefore, systems in 3-D will involve 3 equations!!

Just as the point of intersection of two LINES gave us the solution to a linear system in two dimensions, the point (x, y, z) of intersection of three PLANES gives us our solution to a linear system in three dimensions. There many be No, One, or Infinitely many solutions.



To algebraically solve a 3-by-3 system of linear equations, we will use elimination.

We first reduce the 3-by-3 system to a 2-by-2 system, then solve like we did previously.

Example:

$$\begin{cases} x + y - 2z = -11 \\ 2x - y + 4z = 15 \\ 3x + y - 6z = -35 \end{cases}$$

Consider the following system: $\begin{cases} x + y - 2z = -11 \\ 2x - y + 4z = 15 \\ 3x + y - 6z = -35 \end{cases}$

$$\begin{array}{r} x + y - 2z = -11 \\ 2x - y + 4z = 15 \\ \hline 3x \qquad + 2z = 4 \end{array} \qquad \begin{array}{r} 2x - y + 4z = 15 \\ 3x + y - 6z = -35 \\ \hline 5x \qquad - 2z = -20 \end{array}$$

$$\begin{array}{r} 3x + 2z = 4 \\ 5x - 2z = -20 \\ \hline 8x \qquad = -16 \\ x = -2 \end{array}$$

$$\begin{array}{r} 3x + 2z = 4 \\ 3(-2) + 2z = 4 \\ z = 5 \end{array} \longrightarrow \begin{array}{r} x + y - 2z = -11 \\ -2 + y - 2(5) = -11 \\ y = 1 \end{array}$$

The solution of the system is (−2, 5, 1).

Example:

$$\begin{cases} 2x - 4 = y - 3z \\ x + y = 2z - 7 \\ 3x + 2y + 7 = z \end{cases}$$

$$\begin{cases} 2x - y + 3z = 4 & ① \\ x + y - 2z = -7 & ② \\ 3x + 2y - z = -7 & ③ \end{cases}$$

Step 1 Eliminate one variable.

The coefficients of y are opposites in the first two equations.

$$\begin{array}{lr} ① & 2x - y + 3z = 4 \\ ② & + \;\; x + y - 2z = -7 \\ \hline & 3x \quad\quad + z = -3 \end{array}$$

Add equations ① and ②.

Use equations ② and ③ to create a second equation in x and z.

$$② \; 2(x + y - 2z = -7) \rightarrow 2x + 2y - 4z = -14 \; ④$$

$$\begin{array}{lr} ③ & -\;(3x + 2y - \;\; z = -7) \\ \hline & -x \quad\quad - 3z = -7 \\ & x + 3z = 7 \end{array}$$

Multiply equation ② by 2. Subtract equation ④ and ③.

Multiply by –1.

You now have a 2-by-2 system. $\begin{cases} 3x + z = -3 \; \star \\ x + 3z = 7 \end{cases}$

Step 2 Eliminate another variable.

$$\star \; 3(3x + z = -3) \rightarrow 9x + 3z = -9$$

$$\begin{array}{r} -\;(x + 3z = 7) \\ \hline 8x \quad\quad = -16 \\ x = -2 \end{array}$$

Multiply equation ⋆ by 3 and subtract.

Solve for x.

Step 3 Use one of the equations in the 2-by-2 system to solve for z.

$x + 3z = 7$

$-2 + 3z = 7$ *Substitute –2 for x.*

$3z = 9$ *Add 2 to both sides.*

$z = 3$ *Divide both sides by 3.*

Step 4 Substitute for x and z in one of the original equations to solve for y.

② $x + y - 2z = -7$

$-2 + y - 6 = -7$ *Substitute –2 for x and 3 for z, then simplify.*

$y - 8 = -7$

$y = 1$ *Solve for y.*

Déjà RE-Vu

The WALMAY (*we all love math and yogurt*) Yogurt company makes three yogurt blends: Vanilla-Chocolate, Strawberry-Vanilla, and Chocolate-Strawberry.

The Vanilla-Chocolate blend, *V*, requires 2 quarts of vanilla yogurt and 2 quarts of chocolate yogurt per gallon.

The Strawberry-Vanilla blend, *S*, requires 3 quarts of Vanilla and 1 quart of strawberry yogurt per gallon.

The Chocolate-Strawberry, *C*, requires 3 quarts of chocolate yogurt and 1 quart of strawberry yogurt per gallon.

Each day the company has 800 quarts of vanilla yogurt, 650 quarts of chocolate yogurt, and 350 quarts of strawberry yogurt available for the mixes.

How many gallons of each blend should it make each day if it wants to use up all the supplies?

The information is best arranged in a table:

Flavors	Amt needed	Amt needed	Amt needed	

	for V (VC mix) Quarts/gallon	for S (SV mix) Quarts/gallon	for C (CS mix) Quarts/gallon	Total quarts available
Vanilla	2	3	0	800
Chocolate	2	0	3	650
Strawberry	0	1	1	350

Once we have our information organized, it is much easier to write our system of equations. Let's explicitly define our variables:

Let V = number of gallons of vanilla-chocolate mix
Let S = number of gallons of strawberry-vanilla mix
Let C = number of gallons of chocolate-strawberry mix

$$\begin{cases} 2V + 3S + 0C = 800 \\ 2V + 0S + 3C = 650 \\ 0V + 1S + 1C = 350 \end{cases}$$

The solution is
100 gallons of V, vanilla-chocolate mixture
200 gallons of S, strawberry-vanilla mixture
150 gallons of C, chocolate-strawberry mixture

Math is Power!!

References:

http://go.hrw.com

Image 1:
Graph of linear equation $4x + 8y + z = 2$