1
Sets, Relations and Binary Operations

Set

Set is a collection of well defined objects which are distinct from each other. Sets are usually denoted by capital letters $A, B, C, \ldots$ and elements are usually denoted by small letters $a, b, c, \ldots$.

If a is an element of a set A, then we write $a \in A$ and say a belongs to A or a is in A or a is a member of A. If a does not belongs to A, we write $a \notin A$.

Standard Notations

N	: A set of natural numbers.
W	: A set of whole numbers.
Z	: A set of integers.
Z^+ / Z^-	: A set of all positive/negative integers.
Q	: A set of all rational numbers.
Q^+ / Q^-	: A set of all positive/negative rational numbers.
R	: A set of real numbers.
R^+ / R^-	: A set of all positive/negative real numbers.
C	: A set of all complex numbers.

Methods for Describing a Set

(i) **Roster/Listing Method/Tabular Form** In this method, a set is described by listing element, separated by commas, within braces.

e.g. $A = \{a, e, i, o, u\}$

(ii) **Set Builder/Rule Method** In this method, we write down a property or rule which gives us all the elements of the set by that rule.

e.g. $A = \{x : x$ is a vowel of English alphabets$\}$

Types of Sets

(i) **Finite Set** A set containing finite number of elements or no element.

(ii) **Cardinal Number of a Finite Set** The number of elements in a given finite set is called cardinal number of finite set, denoted by $n\ (A)$.

(iii) **Infinite Set** A set containing infinite number of elements.

(iv) **Empty/Null/Void Set** A set containing no element, it is denoted by ϕ or $\{\ \}$.

(v) **Singleton Set** A set containing a single element.

(vi) **Equal Sets** Two sets A and B are said to be equal, if every element of A is a member of B and every element of B is a member of A and we write $A = B$.

(vii) **Equivalent Sets** Two sets are said to be equivalent, if they have same number of elements.

If $n(A) = n(B)$, then A and B are equivalent sets.

(viii) **Subset and Superset** Let A and B be two sets. If every element of A is an element of B, then A is called subset of B and B is called superset of A.

Written as $A \subseteq B$ or $B \supseteq A$

(ix) **Proper Subset** If A is a subset of B and $A \neq B$, then A is called proper subset of B and we write $A \subset B$.

(x) **Universal Set** (U) A set consisting of all possible elements which occurs under consideration is called a universal set.

(xi) **Comparable Sets** Two sets A and B are comparable, if $A \subseteq B$ or $B \subseteq A$.

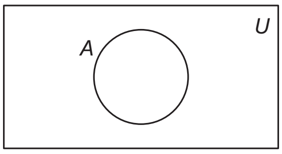
(xii) **Non-Comparable Sets** For two sets A and B, if neither $A \subseteq B$ nor $B \subseteq A$, then A and B are called non-comparable sets.

(xiii) **Power Set** The set formed by all the subsets of a given set A, is called power set of A, denoted by $P(A)$.

(xiv) **Disjoint Sets** Two sets A and B are called disjoint, if $A \cap B = \phi$. i.e. they do not have any common element.

Venn Diagram

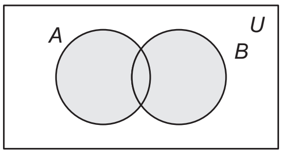
In a Venn diagram, the universal set is represented by a rectangular region and a set is represented by circle or a closed geometrical figure inside the universal set.



Operations on Sets

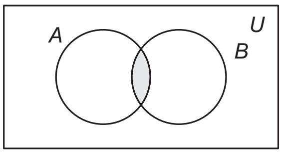
1. **Union of Sets**

The union of two sets A and B, denoted by $A \cup B$, is the set of all those elements, each one of which is either in A or in B or both in A and B.



2. **Intersection of Sets**

The intersection of two sets A and B, denoted by $A \cap B$, is the set of all those elements which are common to both A and B.

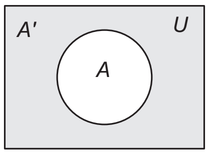


If $A_1, A_2, \ldots, A_n$ is a finite family of sets, then their intersection is denoted by

$$\bigcap_{i=1}^{n} A_i \text{ or } A_1 \cap A_2 \cap \ldots \cap A_n$$

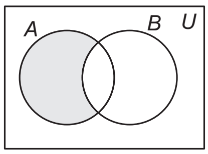
3. **Complement of a Set**

If A is a set with U as universal set, then complement of a set A, denoted by A' or A^c is the set $U - A$.



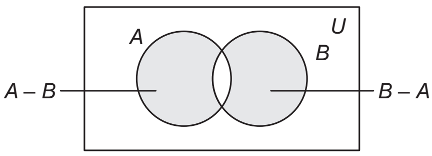
4. **Difference of Sets**

For two sets A and B, the difference $A - B$ is the set of all those elements of A which do not belong to B.



5. **Symmetric Difference**

For two sets A and B, symmetric difference is the set $(A - B) \cup (B - A)$ denoted by $A \Delta B$.



Laws of Algebra of Sets

For three sets A, B and C

(i) **Idempotent Law**

(a) $A \cup A = A$ (b) $A \cap A = A$

(ii) **Identity Law**

(a) $A \cup \phi = A$ (b) $A \cap U = A$

(iii) **Commutative Law**

(a) $A \cup B = B \cup A$ (b) $A \cap B = B \cap A$

(iv) **Associative Law**

(a) $(A \cup B) \cup C = A \cup (B \cup C)$

(b) $A \cap (B \cap C) = (A \cap B) \cap C$

(v) **Distributive Law**

(a) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

(b) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

(vi) **De-Morgan's Law**

(a) $(A \cup B)' = A' \cap B'$

(b) $(A \cap B)' = A' \cup B'$

(vii) (a) $A - (B \cap C) = (A - B) \cup (A - C)$

(b) $A - (B \cup C) = (A - B) \cap (A - C)$

(viii) (a) $A - B = A \cap B'$ (b) $B - A = B \cap A'$

(c) $A - B = A \Leftrightarrow A \cap B = \phi$ (d) $(A - B) \cup B = A \cup B$

(e) $(A - B) \cap B = \phi$ (f) $A \cap B \subseteq A$ and $A \cap B \subseteq B$

(g) $A \cup (A \cap B) = A$ (h) $A \cap (A \cup B) = A$

(ix) (a) $(A - B) \cup (B - A) = (A \cup B) - (A \cap B)$

(b) $A \cap (B - C) = (A \cap B) - (A \cap C)$

(c) $A \cap (B \Delta C) = (A \cap B) \Delta (A \cap C)$

(d) $(A \cap B) \cup (A - B) = A$

(e) $A \cup (B - A) = (A \cup B)$

(x) (a) $U' = \phi$

(b) $\phi' = U$

(c) $(A')' = A$

(d) $A \cap A' = \phi$

(e) $A \cup A' = U$

(f) $A \subseteq B \Leftrightarrow B' \subseteq A'$

Important Points to be Remembered

(i) Every set is a subset of itself i.e. $A \subseteq A$, for any set A.
(ii) Empty set ϕ is a subset of every set i.e. $\phi \subset A$, for any set A.
(iii) For any set A and its universal set U, $A \subseteq U$
(iv) If $A = \phi$, then power set has only one element
i.e. $$n(P(A)) = 1$$
(v) Power set of any set is always a non-empty set.
(vi) Suppose $A = \{1, 2\}$, then
$$P(A) = \{\{1\}, \{2\}, \{1, 2\}, \phi\}$$
(a) $A \notin P(A)$
(b) $\{A\} \in P(A)$
(vii) If a set A has n elements, then $P(A)$ or subset of A has 2^n elements.
(viii) Equal sets are always equivalent but equivalent sets may not be equal.
(ix) The set $\{\phi\}$ is not a null set. It is a set containing one element ϕ.

Results on Number of Elements in Sets

(i) $n(A \cup B) = n(A) + n(B) - n(A \cap B)$
(ii) $n(A \cup B) = n(A) + n(B)$, if A and B are disjoint sets.
(iii) $n(A - B) = n(A) - n(A \cap B)$
(iv) $n(B - A) = n(B) - n(A \cap B)$
(v) $n(A \Delta B) = n(A) + n(B) - 2n(A \cap B)$
(vi) $n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(A \cap C) + n(A \cap B \cap C)$
(vii) n (number of elements in exactly two of the sets A, B, C) $= n(A \cap B) + n(B \cap C) + n(C \cap A) - 3n(A \cap B \cap C)$
(viii) n (number of elements in exactly one of the sets A, B, C) $= n(A) + n(B) + n(C) - 2n(A \cap B) - 2n(B \cap C) - 2n(A \cap C) + 3n(A \cap B \cap C)$
(ix) $n(A' \cup B') = n(A \cap B)' = n(U) - n(A \cap B)$
(x) $n(A' \cap B') = n(A \cup B)' = n(U) - n(A \cup B)$

Ordered Pair

An ordered pair consists of two objects or elements in a given fixed order.

Equality of Ordered Pairs Two ordered pairs (a_1, b_1) and (a_2, b_2) are equal, iff $a_1 = a_2$ and $b_1 = b_2$.

Cartesian Product of Sets

For two sets A and B (non-empty sets), the set of all ordered pairs (a, b) such that $a \in A$ and $b \in B$ is called Cartesian product of the sets A and B, denoted by $A \times B$.

$$A \times B = \{(a, b) : a \in A \text{ and } b \in B\}$$

If there are three sets A, B, C and $a \in A$, $b \in B$ and $c \in C$, then we form an ordered triplet (a, b, c). The set of all ordered triplets (a, b, c) is called the cartesian product of these sets A, B and C.

i.e. $$A \times B \times C = \{(a, b, c) : a \in A, b \in B, c \in C\}$$

Properties of Cartesian Product

For three sets A, B and C

(i) $n(A \times B) = n(A)\, n(B)$

(ii) $A \times B = \phi$, if either A or B is an empty set.

(iii) $A \times (B \cup C) = (A \times B) \cup (A \times C)$

(iv) $A \times (B \cap C) = (A \times B) \cap (A \times C)$

(v) $A \times (B - C) = (A \times B) - (A \times C)$

(vi) $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$

(vii) If $A \subseteq B$ and $C \subseteq D$, then $(A \times C) \subseteq (B \times D)$

(viii) If $A \subseteq B$, then $A \times A \subseteq (A \times B) \cap (B \times A)$

(ix) $A \times B = B \times A \Leftrightarrow A = B$

(x) If either A or B is an infinite set, then $A \times B$ is an infinite set.

(xi) $A \times (B' \cup C')' = (A \times B) \cap (A \times C)$

(xii) $A \times (B' \cap C')' = (A \times B) \cup (A \times C)$

(xiii) If A and B be any two non-empty sets having n elements in common, then $A \times B$ and $B \times A$ have n^2 elements in common.

(xiv) If $A \neq B$, then $A \times B \neq B \times A$

(xv) If $A = B$, then $A \times B = B \times A$

(xvi) If $A \subseteq B$, then $A \times C \subseteq B \times C$ for any set C.

Relation

If A and B are two non-empty sets, then a relation R from A to B is a subset of $A \times B$.

If $R \subseteq A \times B$ and $(a, b) \in R$, then we say that a is related to b by the relation R, written as aRb.

Domain and Range of a Relation

Let R be a relation from a set A to set B. Then, set of all first components or coordinates of the ordered pairs belonging to R is called the domain of R, while the set of all second components or coordinates of the ordered pairs belonging to R is called the range of R.

Thus, domain of $R = \{a : (a, b) \in R\}$ and range of $R = \{b : (a, b) \in R\}$

Types of Relation

(i) **Void Relation** As $\phi \subset A \times A$, for any set A, so ϕ is a relation on A, called the empty or void relation.

(ii) **Universal Relation** Since, $A \times A \subseteq A \times A$, so $A \times A$ is a relation on A, called the universal relation.

(iii) **Identity Relation** The relation $I_A = \{(a, a) : a \in A\}$ is called the identity relation on A.

(iv) **Reflexive Relation** A relation R is said to be reflexive relation, if every element of A is related to itself.

Thus, $(a, a) \in R, \forall\ a \in A \Rightarrow R$ is reflexive.

(v) **Symmetric Relation** A relation R is said to be symmetric relation, iff

$$(a, b) \in R \Rightarrow (b, a) \in R, \forall\ a, b \in A$$

i.e. $$a\ R\ b \Rightarrow bRa, \forall\ a, b \in A$$

$\Rightarrow R$ is symmetric.

(vi) **Anti-Symmetric Relation** A relation R is said to be anti-symmetric relation, iff

$(a, b) \in R$ and $(b, a) \in R \Rightarrow a = b, \forall\ a, b \in A$

(vii) **Transitive Relation** A relation R is said to be transitive relation, iff $(a, b) \in R$ and $(b, c) \in R$

$\Rightarrow$ $$(a, c) \in R, \forall\ a, b, c \in A$$

(viii) **Equivalence Relation** A relation R is said to be an equivalence relation, if it is simultaneously reflexive, symmetric and transitive on A.

(ix) **Partial Order Relation** A relation R is said to be a partial order relation, if it is simultaneously reflexive, symmetric and anti-symmetric on A.

(x) **Total Order Relation** A relation R on a set A is said to be a total order relation on A, if R is a partial order relation on A.

Inverse Relation

If A and B are two non-empty sets and R be a relation from A to B, such that $R = \{(a, b) : a \in A, b \in B\}$, then the inverse of R, denoted by R^{-1}, is a relation from B to A and is defined by

$$R^{-1} = \{(b, a) : (a, b) \in R\}$$

Equivalence Classes of an Equivalence Relation

Let R be equivalence relation in A $(\neq \phi)$. Let $a \in A$.

Then, the equivalence class of a denoted by $[a]$ or $\{\overline{a}\}$ is defined as the set of all those points of A which are related to a under the relation R.

Composition of Relation

Let R and S be two relations from sets A to B and B to C respectively, then we can define relation SoR from A to C such that $(a, c) \in SoR \Leftrightarrow \exists\, b \in B$ such that $(a, b) \in R$ and $(b, c) \in S$.

This relation SoR is called the composition of R and S.

(i) $RoS \neq SoR$

(ii) $(SoR)^{-1} = R^{-1}oS^{-1}$ known as reversal rule.

Congruence Modulo *m*

Let m be an arbitrary but fixed integer. Two integers a and b are said to be congruence modulo m, if $a - b$ is divisible by m and we write $a \equiv b$ (mod m).

i.e. $a \equiv b \pmod{m} \Leftrightarrow a - b$ is divisible by m.

Important Results on Relation

(i) If R and S are two equivalence relations on a set A, then $R \cap S$ is also on equivalence relation on A.

(ii) The union of two equivalence relations on a set is not necessarily an equivalence relation on the set.

(iii) If R is an equivalence relation on a set A, then R^{-1} is also an equivalence relation on A.

(iv) If a set A has n elements, then number of reflexive relations from A to A is $2^{n^2 - n}$.

(v) Let A and B be two non-empty finite sets consisting of m and n elements, respectively. Then, $A \times B$ consists of mn ordered pairs. So, the total number of relations from A to B is 2^{nm}.

Binary Operations

Let S be a non-empty set. A function f from $S \times S$ to S is called a binary operation on S i.e. $f : S \times S \to S$ is a binary operation on set S.

Closure Property

An operation * on a non-empty set S is said to satisfy the closure property, if

$$a \in S, b \in S \Rightarrow a * b \in S, \forall\ a,\ b \in S$$

Also, in this case we say that S is closed for *.

An operation * on a non-empty set S, satisfying the closure property is known as a binary operation.

Properties

(i) Generally binary operations are represented by the symbols * , ⊕, ... etc., instead of letters figure etc.

(ii) Addition is a binary operation on each one of the sets N, Z, Q, R and C of natural numbers, integers, rationals, real and complex numbers, respectively. While addition on the set S of all irrationals is not a binary operation.

(iii) Multiplication is a binary operation on each one of the sets N, Z, Q, R and C of natural numbers, integers, rationals, real and complex numbers, respectively. While multiplication on the set S of all irrationals is not a binary operation.

(iv) Subtraction is a binary operation on each one of the sets Z, Q, R and C of integers, rationals, real and complex numbers, respectively. While subtraction on the set of natural numbers is not a binary operation.

(v) Let S be a non-empty set and $P(S)$ be its power set. Then, the union, intersection and difference of sets, on $P(S)$ is a binary operation.

(vi) Division is not a binary operation on any of the sets N, Z, Q, R and C. However, it is not a binary operation on the sets of all non-zero rational (real or complex) numbers.

(vii) Exponential operation $(a, b) \to a^b$ is a binary operation on set N of natural numbers while it is not a binary operation on set Z of integers.

Types of Binary Operations

(i) **Associative Law** A binary operation * on a non-empty set S is said to be associative, if $(a * b) * c = a * (b * c), \forall\ a, b, c \in S$.

Let R be the set of real numbers, then addition and multiplication on R satisfies the associative law.

(ii) **Commutative Law** A binary operation * on a non-empty set S is said to be commutative, if

$$a * b = b * a, \forall\ a, b \in S.$$

Addition and multiplication are commutative binary operations on Z but subtraction not a commutative binary operation, since

$$2 - 3 \neq 3 - 2\,.$$

Union and intersection are commutative binary operations on the power set $P(S)$ of all subsets of set S. But difference of sets is not a commutative binary operation on $P(S)$.

(iii) **Distributive Law** Let * and o be two binary operations on a non-empty sets. We say that * is distributed over o., if

$a * (b \; o \; c) = (a * b) \; o \; (a * c), \forall\ a, b, c \in S$ also called (left distribution) and $(b \; o \; c) * a = (b * a) \; o \; (c * a), \forall\ a, b, c \in S$ also called (right distribution).

Let R be the set of all real numbers, then multiplication distributes addition on R.

Since, $a \cdot (b + c) = a \cdot b + a \cdot c, \forall\ a, b, c \in R$.

(iv) **Identity Element** Let * be a binary operation on a non-empty set S. An element $e \in S$, if it exist such that

$$a * e = e * a = a, \forall\ a \in S.$$

is called an identity elements of S, with respect to *.

For addition on R, zero is the identity elements in R.

Since, $a + 0 = 0 + a = a, \forall\ a \in R$

For multiplication on R, 1 is the identity element in R.
Since, $a \times 1 = 1 \times a = a, \forall\ a \in R$

Let $P\ (S)$ be the power set of a non-empty set S. Then, ϕ is the identity element for union on $P\ (S)$ as

$$A \cup \phi = \phi \cup A = A, \forall\ A \in P(S)$$

Also, S is the identity element for intersection on $P(S)$.

Since, $A \cap S = A \cap S = A, \forall\ A \in P(S)$.

For addition on N the identity element does not exist. But for multiplication on N the identity element is 1.

(v) **Inverse of an Element** Let * be a binary operation on a non-empty set S and let e be the identity element.

Let $a \in S$ we say that a^{-1} is invertible, if there exists an element $b \in S$ such that $a * b = b * a = e$

Also, in this case, b is called the inverse of a and we write, $a^{-1} = b$

Addition on N has no identity element and accordingly N has no invertible element.

Multiplication on N has 1 as the identity element and no element other than 1 is invertible.

Let S be a finite set containing n elements.

Then, the total number of binary operations on S in n^{n^2}.

Let S be a finite set containing n elements.

Then, the total number of commutative binary operation on S is $n^{\frac{n(n+1)}{2}}$.