

▷ **3: Complex
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Euler's Equation: $e^{i\theta} = \cos \theta + i \sin \theta$

[see RHB 3.3]

Hence: $\cos \theta = \frac{e^{i\theta} + e^{-i\theta}}{2} = \frac{1}{2}e^{i\theta} + \frac{1}{2}e^{-i\theta}$

$$\sin \theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} = -\frac{1}{2}ie^{i\theta} + \frac{1}{2}ie^{-i\theta}$$

Most maths becomes simpler if you use $e^{i\theta}$ instead of $\cos \theta$ and $\sin \theta$

The **Complex Fourier Series** is the Fourier Series but written using $e^{i\theta}$

Examples where using $e^{i\theta}$ makes things simpler:

Using $e^{i\theta}$	Using $\cos \theta$ and $\sin \theta$
$e^{i(\theta+\phi)} = e^{i\theta} e^{i\phi}$	$\cos(\theta + \phi) = \cos \theta \cos \phi - \sin \theta \sin \phi$
$e^{i\theta} e^{i\phi} = e^{i(\theta+\phi)}$	$\cos \theta \cos \phi = \frac{1}{2} \cos(\theta + \phi) + \frac{1}{2} \cos(\theta - \phi)$
$\frac{d}{d\theta} e^{i\theta} = ie^{i\theta}$	$\frac{d}{d\theta} \cos \theta = -\sin \theta$

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Fourier Series: $u(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos 2\pi n F t + b_n \sin 2\pi n F t)$

Substitute: $\cos \theta = \frac{1}{2}e^{i\theta} + \frac{1}{2}e^{-i\theta}$ and $\sin \theta = -\frac{1}{2}ie^{i\theta} + \frac{1}{2}ie^{-i\theta}$

$$\begin{aligned} u(t) &= \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \left(\frac{1}{2}e^{i\theta} + \frac{1}{2}e^{-i\theta} \right) + b_n \left(-\frac{1}{2}ie^{i\theta} + \frac{1}{2}ie^{-i\theta} \right) \right) \\ &= \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(\left(\frac{1}{2}a_n - \frac{1}{2}ib_n \right) e^{i2\pi n F t} \right) \quad [\theta = 2\pi n F t] \\ &\quad + \sum_{n=1}^{\infty} \left(\left(\frac{1}{2}a_n + \frac{1}{2}ib_n \right) e^{-i2\pi n F t} \right) \end{aligned}$$

$$= \sum_{n=-\infty}^{\infty} U_n e^{i2\pi n F t}$$

where

$$[b_0 \triangleq 0]$$

$$U_n = \begin{cases} \frac{1}{2}a_n - \frac{1}{2}ib_n & n \geq 1 \\ \frac{1}{2}a_0 & n = 0 \\ \frac{1}{2}a_{|n|} + \frac{1}{2}ib_{|n|} & n \leq -1 \end{cases} \Leftrightarrow U_{\pm n} = \frac{1}{2} (a_{|n|} \mp ib_{|n|})$$

The U_n are normally complex except for U_0 and satisfy $U_n = U_{-n}^*$

Complex Fourier Series: $u(t) = \sum_{n=-\infty}^{\infty} U_n e^{i2\pi n F t}$ [simpler ☺]

Averaging Complex Exponentials

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If $x(t)$ has period $\frac{T}{n}$ for some integer n (i.e. frequency $\frac{n}{T} = nF$):

$$\langle x(t) \rangle \triangleq \frac{1}{T} \int_{t=0}^T x(t) dt$$

This is the average over an integer number of cycles.

For a complex exponential:

$$\begin{aligned} \langle e^{i2\pi n F t} \rangle &= \langle \cos(2\pi n F t) + i \sin(2\pi n F t) \rangle \\ &= \langle \cos(2\pi n F t) \rangle + i \langle \sin(2\pi n F t) \rangle \\ &= \begin{cases} 1 + 0i & n = 0 \\ 0 + 0i & n \neq 0 \end{cases} \end{aligned}$$

Hence:

$$\langle e^{i2\pi n F t} \rangle = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases}$$



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Complex Fourier Series: $u(t) = \sum_{n=-\infty}^{\infty} U_n e^{i2\pi n F t}$

To find the coefficient, U_n , we multiply by something that makes all the terms involving the other coefficients average to zero.

$$\begin{aligned}\langle u(t) e^{-i2\pi n F t} \rangle &= \langle \sum_{r=-\infty}^{\infty} U_r e^{i2\pi r F t} e^{-i2\pi n F t} \rangle \\ &= \langle \sum_{r=-\infty}^{\infty} U_r e^{i2\pi(r-n) F t} \rangle \\ &= \sum_{r=-\infty}^{\infty} U_r \langle e^{i2\pi(r-n) F t} \rangle\end{aligned}$$

All terms in the sum are zero, except for the one when $n = r$ which equals U_n :

$$U_n = \langle u(t) e^{-i2\pi n F t} \rangle$$



This shows that the Fourier series coefficients are **unique**: you cannot have two different sets of coefficients that result in the same function $u(t)$.

Note the sign of the exponent: “+” in the Fourier Series but “−” for Fourier Analysis (in order to cancel out the “+”).

Fourier Series $\leftrightarrow$ Complex Fourier Series

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$$\begin{aligned} u(t) &= \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos 2\pi n F t + b_n \sin 2\pi n F t) \\ &= \sum_{n=-\infty}^{\infty} U_n e^{i 2\pi n F t} \end{aligned}$$

We can easily convert between the two forms.

Fourier Coefficients $\rightarrow$ Complex Fourier Coefficients:

$$U_{\pm n} = \frac{1}{2} (a_{|n|} \mp i b_{|n|}) \quad [U_n = U_{-n}^*]$$

Complex Fourier Coefficients $\rightarrow$ Fourier Coefficients:

$$a_n = U_n + U_{-n} = 2\Re(U_n) \quad [\Re = \text{"real part"}]$$

$$b_n = i(U_n - U_{-n}) = -2\Im(U_n) \quad [\Im = \text{"imaginary part"}]$$

The formula for a_n works even for $n = 0$.

[Complex functions of time]

In these lectures, we are assuming that $u(t)$ is a periodic real-valued function of time. In this case we can represent $u(t)$ using either the Fourier Series or the Complex Fourier Series:

$$u(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos 2\pi n F t + b_n \sin 2\pi n F t) = \sum_{n=-\infty}^{\infty} U_n e^{i2\pi n F t}$$

We have seen that the U_n coefficients are complex-valued and that U_n and U_{-n} are complex conjugates so that we can write $U_{-n} = U_n^*$.

In fact, the complex Fourier series can also be used when $u(t)$ is a complex-valued function of time (this is sometimes useful in the fields of communications and signal processing). In this case, it is still true that $u(t) = \sum_{n=-\infty}^{\infty} U_n e^{i2\pi n F t}$, but now U_n and U_{-n} are completely independent and normally $U_{-n} \neq U_n^*$.

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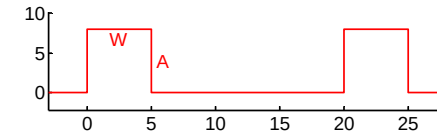
$T = 20$, width $W = \frac{T}{4}$, height $A = 8$

Method 1:

$$U_{\pm n} = \frac{1}{2}a_n \mp i\frac{1}{2}b_n$$

Method 2:

$$\begin{aligned} U_n &= \langle u(t)e^{-i2\pi nFt} \rangle \\ &= \frac{1}{T} \int_0^T u(t)e^{-i2\pi nFt} dt \\ &= \frac{1}{T} \int_0^W Ae^{-i2\pi nFt} dt \\ &= \frac{A}{-i2\pi nFT} [e^{-i2\pi nFt}]_0^W \\ &= \frac{A}{i2\pi n} (1 - e^{-i2\pi nFW}) \\ &= \frac{Ae^{-i\pi nFW}}{i2\pi n} (e^{i\pi nFW} - e^{-i\pi nFW}) \\ &= \frac{Ae^{-i\pi nFW}}{n\pi} \sin(n\pi FW) \\ &= \frac{8}{n\pi} \sin\left(\frac{n\pi}{4}\right) e^{-i\frac{n\pi}{4}} \end{aligned}$$



n	a_n	b_n	U_n
-6			$i\frac{8}{6\pi}$
-5			$\frac{4}{5\pi} + i\frac{4}{5\pi}$
-4			0
-3			$\frac{-4}{3\pi} + i\frac{4}{3\pi}$
-2			$i\frac{8}{2\pi}$
-1			$\frac{4}{\pi} + i\frac{4}{\pi}$
0	4		2
1	$\frac{8}{\pi}$	$\frac{8}{\pi}$	$\frac{4}{\pi} + i\frac{-4}{\pi}$
2	0	$\frac{16}{2\pi}$	$i\frac{-8}{2\pi}$
3	$\frac{-8}{3\pi}$	$\frac{8}{3\pi}$	$\frac{-4}{3\pi} + i\frac{-4}{3\pi}$
4	0	0	0
5	$\frac{8}{5\pi}$	$\frac{8}{5\pi}$	$\frac{4}{5\pi} + i\frac{-4}{5\pi}$
6	0	$\frac{16}{6\pi}$	$i\frac{-8}{6\pi}$

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Complex Fourier Series: $u(t) = \sum_{n=-\infty}^{\infty} U_n e^{i2\pi n F t}$

If $v(t)$ is the same as $u(t)$ but delayed by a time τ : $v(t) = u(t - \tau)$

$$\begin{aligned} v(t) &= \sum_{n=-\infty}^{\infty} U_n e^{i2\pi n F (t-\tau)} = \sum_{n=-\infty}^{\infty} (U_n e^{-i2\pi n F \tau}) e^{i2\pi n F t} \\ &= \sum_{n=-\infty}^{\infty} V_n e^{i2\pi n F t} \end{aligned}$$

where $V_n = U_n e^{-i2\pi n F \tau}$

Example:

$$u(t) = 6 \cos(2\pi F t)$$

Fourier: $a_1 = 6, b_1 = 0$

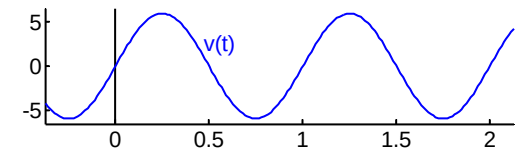
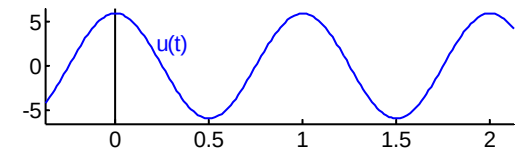
Complex: $U_{\pm 1} = \frac{1}{2} a_1 \mp \frac{1}{2} i b_1 = 3$

$$v(t) = 6 \sin(2\pi F t) = u(t - \tau)$$

Time delay: $\tau = \frac{T}{4} \Rightarrow F\tau = \frac{1}{4}$

Complex: $V_1 = U_1 e^{-i\frac{\pi}{2}} = -3i$

$$V_{-1} = U_{-1} e^{i\frac{\pi}{2}} = +3i$$



Note: If $u(t)$ is a sine wave, U_1 equals half the corresponding phasor.

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$$(1) \ u(t) \text{ real-valued} \Leftrightarrow U_n \text{ conjugate symmetric} [U_n = U_{-n}^*]$$

$$(2) \ u(t) \text{ even} [u(t) = u(-t)] \Leftrightarrow U_n \text{ even} [U_n = U_{-n}]$$

$$(3) \ u(t) \text{ odd} [u(t) = -u(-t)] \Leftrightarrow U_n \text{ odd} [U_n = -U_{-n}]$$

$$(1)+(2) \ u(t) \text{ real \& even} \Leftrightarrow U_n \text{ real \& even} [U_n = U_{-n}^* = U_{-n}]$$

$$(1)+(3) \ u(t) \text{ real \& odd} \Leftrightarrow U_n \text{ imaginary \& odd} [U_n = U_{-n}^* = -U_{-n}]$$

Proof of (2): $u(t)$ even $\Rightarrow U_n$ even

$$U_{-n} = \frac{1}{T} \int_0^T u(t) e^{-i2\pi(-n)Ft} dt$$

$$= \frac{1}{T} \int_{x=0}^{-T} u(-x) e^{-i2\pi n F x} (-dx) \quad [\text{substitute } x = -t]$$

$$= \frac{1}{T} \int_{x=-T}^0 u(-x) e^{-i2\pi n F x} dx \quad [\text{reverse the limits}]$$

$$= \frac{1}{T} \int_{x=-T}^0 u(x) e^{-i2\pi n F x} dx = U_n \quad [\text{even: } u(-x) = u(x)]$$

Proof of (3): $u(t)$ odd $\Rightarrow U_n$ odd

Same as before, except for the last line:

$$= \frac{1}{T} \int_{x=-T}^0 -u(x) e^{-i2\pi n F x} dx = -U_n \quad [\text{odd: } u(-x) = -u(x)]$$

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A waveform, $u(t)$, is **anti-periodic** if $u(t + \frac{1}{2}T) = -u(t)$.

If $u(t)$ is anti-periodic then $U_n = 0$ for n even.

Proof:

Define $v(t) = u(t + \frac{T}{2})$, then

$$(1) v(t) = -u(t) \Rightarrow V_n = -U_n$$

$$(2) v(t) \text{ equals } u(t) \text{ but delayed by } -\frac{T}{2}$$

$$\Rightarrow V_n = U_n e^{i2\pi n F \frac{T}{2}} = U_n e^{in\pi} = \begin{cases} U_n & n \text{ even} \\ -U_n & n \text{ odd} \end{cases}$$

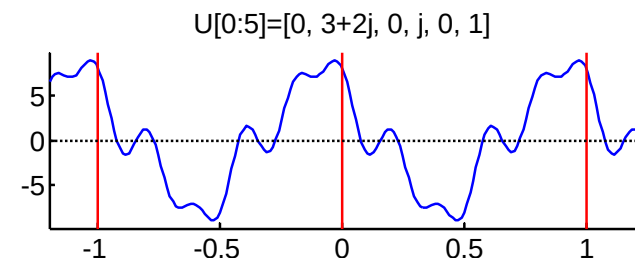
Hence for n even: $V_n = -U_n = U_n \Rightarrow U_n = 0$

Example:

$$U_{0:5} = [0, 3 + 2i, 0, i, 0, 1]$$

Odd harmonics only $\Leftrightarrow$

Second half of each period is the negative of the first half.



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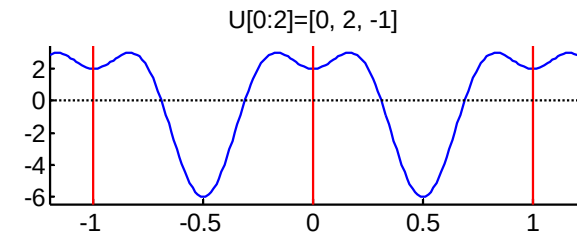
Examples

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All these examples assume that $u(t)$ is real-valued $\Leftrightarrow U_{-n} = U_n^*$.

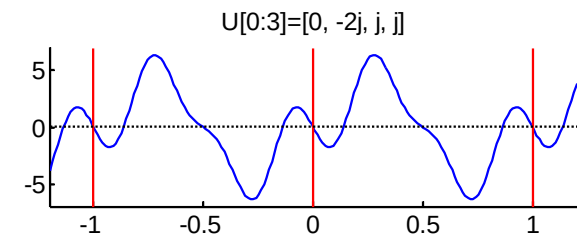
(1) Even $u(t) \Leftrightarrow$ real U_n

$$U_{0:2} = [0, 2, -1]$$



(2) Odd $u(t) \Leftrightarrow$ imaginary U_n

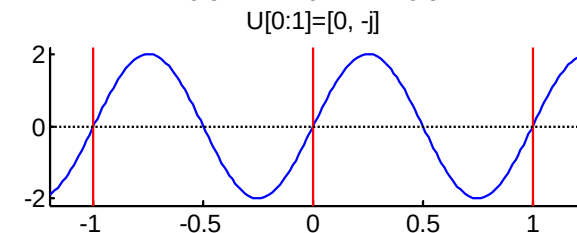
$$U_{0:3} = [0, -2i, i, i]$$



(3) Anti-periodic $u(t)$

$\Leftrightarrow$ odd harmonics only

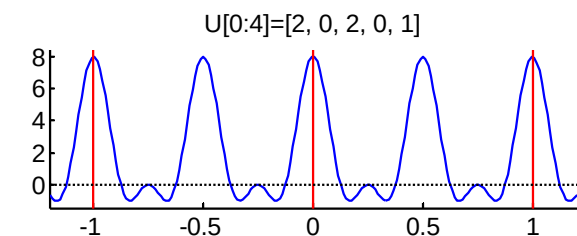
$$U_{0:1} = [0, -i]$$



(4) Even harmonics only

$\Leftrightarrow$ period is really $\frac{1}{2}T$

$$U_{0:4} = [2, 0, 2, 0, 1]$$



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- **Fourier Series:**

$$u(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos 2\pi n F t + b_n \sin 2\pi n F t)$$

- **Complex Fourier Series:** $u(t) = \sum_{n=-\infty}^{\infty} U_n e^{i2\pi n F t}$

- $U_n = \langle u(t) e^{-i2\pi n F t} \rangle \triangleq \frac{1}{T} \int_0^T u(t) e^{-i2\pi n F t} dt$

- Since $u(t)$ is real-valued, $U_n = U_{-n}^*$

- FS $\rightarrow$ CFS: $U_{\pm n} = \frac{1}{2} a_{|n|} \mp i \frac{1}{2} b_{|n|}$

- CFS $\rightarrow$ FS: $a_n = U_n + U_{-n}$

$$b_n = i(U_n - U_{-n})$$

- $u(t)$ **real and even** $\Leftrightarrow u(-t) = u(t)$

$$\Leftrightarrow U_n \text{ is real-valued and even} \Leftrightarrow b_n = 0 \quad \forall n$$

- $u(t)$ **real and odd** $\Leftrightarrow u(-t) = -u(t)$

$$\Leftrightarrow U_n \text{ is purely imaginary and odd} \Leftrightarrow a_n = 0 \quad \forall n$$

- $u(t)$ **anti-periodic** $\Leftrightarrow u(t + \frac{T}{2}) = -u(t)$

$$\Leftrightarrow \text{odd harmonics only} \Leftrightarrow a_{2n} = b_{2n} = U_{2n} = 0 \quad \forall n$$

For further details see RHB 12.3 and 12.7.